THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL
PATRICK DU VAL
HOWARD EVES
F. A. FICKEN
J. W. GIVENS
L. M. KELLY
H. D. LARSEN
IVAN NIVEN
EDITH R. SCHNECKENBURGER
GEORGE SPRINGER
E. P. STARKE
G. B. THOMAS
E. P. VANCE

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME 59

1952

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND SEATTLE, WASH.

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 59 NUMBER 1

CONTENTS

Partial Orderings and Moore-Smith Limits . . . E. J. McSHANE 1
On the Rectangular Hyperbola . . . . . J. R. MUSSELMAN 11
L'Hospital's Rule . . . . . A. E. TAYLOR 20
Perspective and Orthologic Triangles and Tetrahedrons . . . . . VICTOR THÉBAULT 24
Mathematical Notes . . . . H. D. RUDERMAN, H. S. THURSTON 29
Classroom Notes . . . . . J. J. HART, J. J. CORLISS 33
Elementary Problems and Solutions 41
Advanced Problems and Solutions 45
Recent Publications 50
Clubs and Allied Activities 54
News and Notices 57
The Mathematical Association of America 63
New Members 64
February Meeting of the Louisiana-Mississippi Section 66
April Meeting of the Texas Section 69
May Meeting of the Wisconsin Section 70
Calendar of Future Meetings 72

JANUARY 1952

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	J. W. GIVENS	GEORGE SPRINGER
PATRICK DU VAL	L. M. KELLY	E. P. STARKE
HOWARD EVES	H. D. LARSEN	G. B. THOMAS
F. A. FICKEN	IVAN NIVEN	E. P. VANCE

EDITH R. SCHNECKENBURGER

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, SAUNDERS MACLANE, University of Chicago
Honorary President, W. D. CAIRNS, Oberlin College
First Vice-President, F. L. GRIFFIN, Reed College
Second Vice-President, JEWELL H. BUSHEY, Hunter College
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo
Editor, C. B. ALLENDOERFER, University of Washington

Additional Members of the Board of Governors: E. B. ALLEN, C. F. BARR, T. B. BICKERSTAFF, J. C. BRIXEY, R. H. CAMERON, G. R. CLEMENTS, T. F. COPE, H. S. M. COXETER, W. M. DAVIS, J. M. EARL, G. M. EWING, L. R. FORD, TOMLINSON FORT, J. S. FRAME, A. E. GAULT, R. E. GILMAN, E. H. HANSON, M. R. HESTENES, AUGHTUM S. HOWARD, C. G. JAEGER, B. W. JONES, M. S. KNEBELMAN, R. E. LANGER, D. H. LEHMER, C. C. MACDUFFEE, W. E. MILNE, H. P. PETTIT, J. C. POLLEY, G. E. RAYNOR, E. B. ROESSLER, R. G. SANGER, F. H. STEEN, EARL WALDEN, MARIE J. WEISS, F. B. WILEY

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

adopt another formulation which makes no difference when we are discussing the reals, but makes just the difference we want in other cases. We say that X is Dedekind-complete* if every non-empty subset S of X which *is directed by* $\succ$ and has an upper bound also has a supremum, and every non-empty subset S of X which is directed by $\succ$ and has a lower bound also has an infimum. Thus for example, the set of circular regions of the plane $(x-x_0)^2+(y-y_0)^2 \leqq r^2$, $r \geqq 0$ is Dedekind-complete if $\succ$ means $\supset$, "contains"; but these circles do not form a lattice.

If $f(a)$, a in A is a net of real numbers, the statement that the limit of $f(a)$ is k can be thus phrased in terms of order: For every real number $m<k$, it is eventually true that $f(a)>m$; and for every real number $n>k$, it is eventually true that $f(a)<n$. This suggests the following definition of convergence in partially ordered sets. If $f(a)$, a in A is a net of points of a partially ordered set X, it is convergent if there exist sets M and N in X, directed by $\succ$ and $\prec$ respectively, such that the supremum of M is the same as the infimum of N, and for every m in M and every n in N it is eventually true that $n \succ f(a) \succ m$. It is easy to show that the limit, if it exists, is unique. From the definition of limit we can proceed to the definition of continuity and to the study of continuous functions. Just where this leads us cannot yet be stated. I have done some studying of partially ordered sets and continuous functions on them, and expect to publish the results soon. But much remains to be done.

ON THE RECTANGULAR HYPERBOLA

J. R. MUSSELMAN, Western Reserve University

1. Introduction. By proper choice of axes and unit of scale any rectangular hyperbola may be expressed by the equation $xy=1$, or, parametrically, by $x=t$, $y=t^{-1}$. Choose any three points A_1, A_2, A_3 on the hyperbola such that no side of the triangle is parallel to an asymptote, and designate the parameters of these points by t_1, t_2, t_3, respectively.

If the altitude of the triangle drawn from A_1 to the side A_2A_3, produced if necessary, makes acute angles α and β with the asymptotes of the hyperbola, then the line drawn through A_1 which makes acute angles β and α, respectively, with the asymptotes is called the antiparallel to the altitude with respect to the asymptotes. For brevity in this article, such a line will be termed a slant-line. Thus the antiparallel to the altitude through A_i with respect to the asymptotes of the hyperbola will be called a slant-line and designated by the symbol H_i, $(i=1, 2, 3)$.

* [Added in proof.] Since writing this, I have learned that this concept, with isotone sequences in place of directed sets, has been thought of earlier; see M. H. Stone, *A general theory of spectra*, I, Proc. Nat. Acad. Sci. 26 (1940), p. 281, and S. Bochner, *Completely monotone functions*, Duke Math. Jour. 9 (1942), p. 520.

A second type of line is one that forms with a given line an isosceles triangle with respect to an asymptote of the hyperbola. Such a pair of lines shall be called *isosceles lines*, and it is evident that a pair of lines isosceles with respect to one asymptote is isosceles with respect to the other.

2. The point H′. The equation of the side A_2A_3 is $x+t_2t_3y=t_2+t_3$; the equation of the altitude through A_1 is $t_2t_3x-y=s_3-t_1^{-1}$; the equation of the slant-line H_1 is $x-t_2t_3y=t_1-t_2t_3t_1^{-1}$. Note that A_2A_3 and H_1 are isosceles lines. Now the point H' whose coordinates are $(s_1, s_2s_3^{-1})$, where the s_i are the elementary symmetric functions of t_1, t_2, and t_3, lies on the line H_1, since its coordinates satisfy the equation of H_1. Moreover, since the coordinates of H' are symmetric in the parameters t_i, $(i=1, 2, 3)$, it follows that the three slant-lines H_1, H_2, and H_3 are concurrent at H'.

The line H_1 cuts A_2A_3 at the point C_1 whose x-coördinate is $\frac{1}{2}(s_1-t_2t_3t_1^{-1})$; it cuts the hyperbola at B_1 with x-coördinate $-t_2t_3t_1^{-1}$. It is obvious that C_1 is the midpoint of $H'B_1$, whence we have the theorem that *the images of H' in the sides of the triangle $A_1A_2A_3$ measured along the slant-lines of $A_1A_2A_3$ lie on the hyperbola.*

The triangle $A_1A_2A_3$ has a centroid G and a center O of its circumscribed circle. Locate a point H on the line OG such that $OH=3OG$. The lines joining H to the vertices A_i are the altitudes of the triangle $A_1A_2A_3$ and thus H is its orthocenter and lies on the hyperbola [1].

Let us determine a point K on the line joining C, the center of the hyperbola, to G, such that $CK=3CG$. The coördinates of this point K turn out to be $(s_1, s_2s_3^{-1})$ or those of H'. The lines joining H' to the vertices A_i are the slant-lines H_i, and we shall call H' the slant-center of the triangle $A_1A_2A_3$. Moreover H' lies on the circumcircle of $A_1A_2A_3$.

3. The line of slant-images. The slant-line of any point T on the hyperbola, with parameter T, drawn to the side A_2A_3, has the equation $x-t_2t_3y=T-t_2t_3T^{-1}$. The image of T in A_2A_3 measured along this line—let us call it the slant-image of T in A_2A_3—has for its x-coördinate $x=t_2+t_3-t_2t_3T^{-1}$. The slant-image of T in the side A_3A_1 has for its x-coördinate the value $x=t_3+t_1-t_3t_1T^{-1}$. The equation of the line joining these two slant-images is

$$Tx - s_3y = Ts_1 - s_2. \tag{1}$$

From symmetry it is evident that the slant-image of T in the side A_1A_2 likewise lies on this line. We note that the coördinates of the point H' satisfy equation (1). Hence, *the three slant-images of any point on the hyperbola as to $A_1A_2A_3$ lie on a straight line. As T runs over the hyperbola, we obtain a pencil of lines on H'.*

Given any line on H', to find the point T on the hyperbola for which it is the line of slant-images, we may proceed as follows: draw through any vertex A_i a line parallel to the given line which intersects the hyperbola at a second point V_i, the further intersection of the slant-line through V_i, parallel to H_i,

will cut the hyperbola at the point T.

The diametrically opposite point on the hyperbola to the point T has the coördinates $(-T, -T^{-1})$. The equation of its line of slant-images is

$$Tx + s_3y = Ts_1 + s_2. \tag{2}$$

Hence *the slant-image lines of any two diametrically opposite points on the hyperbola are isosceles lines.*

The lines (1) and (2) cut the side A_2A_3 in two points whose midpoint D_1 is found to have the x-coördinate $(s_3+s_1T^2)(T^2-t_1^2)^{-1}$. Similarly the lines (1) and (2) cut the side A_3A_1 in two points whose midpoint D_2 has the x-coördinate $(s_3+s_1T^2)(T^2-t_2^2)^{-1}$. Now the line joining D_1 to D_2 has the equation

$$T^2(s_2 + T^2)x + s_3(s_3 + s_1T^2)y = (s_2 + T^2)(s_3 + s_1T^2). \tag{3}$$

From symmetry, the point D_3, the midpoint of the segment cut on A_1A_2 by lines (1) and (2), must lie on this line. Hence, *the midpoints of the segments cut on the sides of the triangle $A_1A_2A_3$ by the lines of slant-images of any two diametrically opposite points of the hyperbola lie on a line.*

The line joining H' to D_1 has the equation

$$T^2x + t_1s_3y = s_1T^2 + t_3s_2.$$

Now the line isosceles to $H'D_1$ through A_1 is

$$T^2x - t_1s_3y = T^2t_1 - s_3. \tag{4}$$

This equation obviously is satisfied by $x = -s_3T^{-2}$, $y = -T^2s_3^{-1}$, the coördinates of a point R on the hyperbola. Since the coördinates of R are symmetric in t_1, t_2 and t_3 we can state that *the lines through A_i isosceles to $H'D_i$ meet at a point R on the hyperbola.*

4. The nine-point hyperbola. It has been shown that the points where the slant-lines H_i cut the sides A_jA_k, namely, the points C_i, the midpoints E_i of the sides of $A_1A_2A_3$, and the midpoints of the segments A_iH', all lie on a rectangular hyperbola whose asymptotes are parallel to the asymptotes of the given hyperbola and whose transverse axis is one half as long [2]. In our notation the equation of this nine-point hyperbola is

$$(2x - s_1)(2s_3y - s_2) = s_3. \tag{5}$$

The center of this hyperbola is $\frac{1}{2}(s_1, s_2s_3^{-1})$, a point N' lying on the line joining the center C of the given hyperbola to the point H' such that $CN' = N'H'$. The centroid G of the triangle $A_1A_2A_3$ lies also on this line since $2CG = GH'$. Thus there exists another line, analogous to the Euler line connected with a triangle, on which lie four harmonic points, namely, C the center of the given hyperbola, G the centroid of $A_1A_2A_3$, N' the center of the nine-point hyperbola, and H' the meet of the slant-lines of $A_1A_2A_3$.

The line joining C_2C_3 has the equation

$$2t_2t_3(x + t_1^2y) = t_2t_3s_1 + t_1s_2 - t_1t_2^2 - t_1t_3^2.$$

This line is parallel to the tangent to the hyperbola at A_1 and also parallel to the tangent to the nine-point hyperbola at E_1, the midpoint of the side A_2A_3.

5. Lines related to a diameter. The equation of the line joining the points with x-coördinates T and $-T$ is given by

$$x - T^2y = 0. \tag{6}$$

The line through A_1 isosceles to (6) is

$$x + T^2y = t_1 + T^2t_1^{-1}.$$

This cuts the diameter (6) at F_1 whose x-coördinate is $\frac{1}{2}(t_1+T^2t_1^{-1})$. The equation of the slant-line through F_1, parallel to H_1, is

$$x - t_2t_3y = \tfrac{1}{2}(t_1 - t_2t_3t_1^{-1} + T^2t_1^{-1} - s_3T^{-2}). \tag{7}$$

The equation of the slant-line through F_2, parallel to H_2, is

$$x - t_3t_1y = \tfrac{1}{2}(t_2 - t_3t_1t_2^{-1} + T^2t_2^{-1} - s_3T^{-2}). \tag{8}$$

The lines (7) and (8) meet at the point whose x-coördinate is

$$x = \tfrac{1}{2}(s_1 - s_3T^{-2}). \tag{9}$$

Through this point must pass likewise the slant-line through F_3 parallel to H_3. Hence, *given any diameter of the hyperbola, the lines through A_i isosceles to the diameter meet the diameter in three points F_i. The slant-lines through F_i, parallel to H_i, meet at a point on the nine-point hyperbola, the midpoint of $H'R$.*

The line through A_1, parallel to the diameter (6) of the hyperbola, cuts the hyperbola again at L_1 whose x-coördinate is $-T^2t_1^{-1}$. The diameter (6) cuts the side A_2A_3 at Q_1 whose x-coördinate is $T^2(t_2+t_3)(T^2+t_2t_3)^{-1}$. The equation of L_1Q_1 is

$$t_1(s_1T^2 + s_3)x - T^4(s_2 + T^2)y = T^2(t_2 + t_3)(t_1^2 - T^2). \tag{10}$$

The second intersection of L_1Q_1 with the hyperbola is the point whose x-coördinate is

$$x = \frac{T^2(s_2 + T^2)}{s_3 + s_1T^2}. \tag{11}$$

Hence, we can state that *if the parallels through A_i to any diameter of the hyperbola cut the hyperbola again at points L_i, and if the diameter cuts the sides of the triangle $A_1A_2A_3$ at points Q_i, the three lines L_iQ_i meet at a point on the hyperbola whose line of slant-images is isosceles to the line* (3).

The equation of the slant-line through L_1, parallel to H_1, is

$$x - t_2t_3y = -T^2t_1^{-1} + s_3T^{-2}.$$

The equation of the slant-line through L_2, parallel to H_2, is

$$x - t_3t_1y = -T^2t_2^{-1} + s_3T^{-2}.$$

These two lines intersect at $R'(s_3T^{-2}, T^2s_3^{-1})$, a point on the hyperbola diagonally opposite to point R, mentioned in Section 3. Naturally the slant-line through L_3, parallel to H_3, is on R'. Hence, *the three slant-lines through L_i parallel to H_i meet at a point R' on the hyperbola.*

The following theorem is stated without proof: *The parallel through H' to A_1R' meets the side A_2A_3 at D_1. Hence the three points in which parallels through H' to A_iR' meet the sides A_jA_k are collinear and lie on the line* (3).

6. The lines isosceles to any line. The equation of any line in the plane not parallel to any asymptote may be written as

$$ax + by = c, \qquad a = b \neq 0. \tag{12}$$

The isosceles line to (12) through A_1 cuts (12) at the point with coördinates $1/2a[at_1 - bt_1^{-1} + c]$, $1/2b[bt_1^{-1} - at_1 + c]$. The equation of the slant-line through this point parallel to H_1 is

$$x - t_2t_3y = \frac{1}{2a}[at_1 - bt_1^{-1} + c] - \frac{1}{2b}[bt_2t_3t_1^{-1} - as_3 + t_2t_3c].$$

On this line lies the point $1/2(s_1 + ca^{-1} + ab^{-1}s_3)$, $1/2(s_2s_3^{-1} + b^{-1}c + a^{-1}bs_3^{-1})$. Hence, *given the points N_i where the lines isosceles to $ax+by=c$ through the vertices A_i cut the given line $ax+by=c$, the three slant-lines through N_i parallel to H_i meet in a point.* For the diameter $x - T^2y = 0$ this point is (9), a point on the nine-point hyperbola.

7. Sets of four points. The four points A_1, A_2, A_3 and their orthocenter $H(-s_3^{-1}, -s_3)$ which lie on the hyperbola are a set of orthic points, *i.e.*, any point is the orthocenter of the triangle formed by the other three points. All four triangles have the same nine-point circle [3].

The four points A_1, A_2, A_3, and H' which lie on the circumcircle of A_1, A_2, A_3 are a set of slant points, *i.e.*, any point is the slant-center of the triangle formed by the other three points. All four triangles have the same nine-point hyperbola.

The four points A_1, A_2, A_3 and S which lie on both the circumcircle and the hyperbola have as orthocenters for the four triangles formed from these points, the points S_i, and H. It is known [4] that these points lie on the given hyperbola, diametrically opposite to A_i and S, and also on a circle with center at

$$x = -\tfrac{1}{2}(s_1 + s_3^{-1}),$$

and with radius equal to the radius of the circumcircle. It follows immediately

that the orthocenters of the four triangles formed from the points S_i and H are the original points A_i and S.

The four points A_1, A_2, A_3 and S have as slant-centers, for the four triangles formed from these points, the points S_i' and H'. These four points lie on the circumcircle diametrically opposite to A_i and S, and also on a hyperbola, whose equation is

$$(x - s_1 - s_3^{-1})(y - s_2 s_3^{-1} - s_3) = 1.$$

The center of this hyperbola C' lies on the line CO, such that $CO = OC'$. It follows at once that the slant-centers of the four triangles formed from the points S_i' and H' are the points A_i and S. Consequently, *the four points $A_1A_2A_3$, S, the intersections of the circumcircle and the given hyperbola, have as orthocenters a set of four points whose orthocenters again are the original points; likewise they have as slant-centers a set of four points whose slant-centers are the original points.*

8. The pedal line of the hyperbola. The line through any point T of the hyperbola parallel to the slant-line H_1 cuts the side A_2A_3 in a point whose x-coördinate is given by $2x = t_1 + t_2 + T - t_1t_2T^{-1}$. Three such points, each on a side of the triangle $A_1A_2A_3$, lie on the line whose equation is

$$Tx - s_3y = \tfrac{1}{2}(T^2 + s_1T - s_2 - s_3T^{-1}). \tag{13}$$

We shall call this line the pedal line of the point T. The pedal line of the diametrically opposite point to T on the hyperbola is

$$Tx + s_3y = \tfrac{1}{2}(-T^2 + s_1T + s_2 - s_3T^{-1}). \tag{14}$$

These lines intersect at $x = \frac{1}{2}(s_1 - s_3T^{-2})$, the coördinate of a point on the nine-point hyperbola which is the midpoint of the slant-center H' and the point R of Section 3. The equation of the lines joining R to T and to T', the diametrically opposite point of T on the hyperbola are, respectively,

$$Tx - s_3y = T^2 - s_3T^{-1},$$

and

$$Tx + s_3y = -T^2 - s_3T^{-1}.$$

Hence, the *angle between the pedal lines of any two diametrically opposite points T and T' of the hyperbola equals the angle TRT' at the point R on the hyperbola.*

Given any four points A_1, A_2, A_3, A_4 of the hyperbola, the four nine-point hyperbolas of the four triangles formed from the four points have a common point. Thus we see that the equation of the nine-point hyperbola of $A_1A_2A_3$, namely,

$$(2x - s_1)(2s_3y - s_2) = s_3,$$

is satisfied by the coördinates of the point $(\frac{1}{2}\sigma_1, \frac{1}{2}\sigma_3\sigma_4^{-1})$, where the σ_i are symmetric functions of the four t_1, t_2, t_3, t_4. From symmetry this point must lie on the other three nine-point hyperbolas.

The pedal line of the point A_4 as to the triangle $A_1A_2A_3$ is

$$t_4x - s_3y = \tfrac{1}{2}(t_4^2 + s_1t_4 - s_2 - s_3t_4^{-1}), \tag{15}$$

which is also on the point $(\frac{1}{2}\sigma_1, \frac{1}{2}\sigma_3\sigma_4^{-1})$. This point is the midpoint of the join of A_i to the slant-center of the other three points. Hence, *the four pedal lines of A_i as to the triangle $A_jA_kA_l$ are concurrent.* Given *any four points A_1, A_2, A_3, A_4 on the hyperbola, from which we can form four triangles. Any point T of the hyperbola will determine four pedal lines, one for each triangle, the lines isosceles to each of these lines from T meets each line in four collinear points,* for the pedal line of T as to $A_1A_2A_3$ is

$$Tx - s_3y = \tfrac{1}{2}(T^2 + s_1T - s_2 - s_3T^{-1}).$$

The line through T, isosceles to this line is

$$Tx + s_3y = T^2 + s_3T^{-1}.$$

Their point of intersection has for x-coördinate

$$4x = 3T + s_1 - s_2T^{-1} + s_3T^{-2}.$$

Four such points lie on the line

$$T^2x + \sigma_4y = \tfrac{1}{4}(3T^3 + \sigma_1T^2 - \sigma_2T + \sigma_3 + \sigma_4T^{-1}),$$

where the σ_i refer to symmetric functions of t_1, t_2, t_3, t_4. The pedal lines of any two points T_1 and T_2 with respect to the triangle $A_1A_2A_3$ meet at the point M whose x-coördinate is given by

$$2x = T_1 + T_2 + s_1 + s_3T_1^{-1}T_2^{-1}. \tag{16}$$

Now the pedal line of the point $T_3(s_3T_1^{-1}T_2^{-1}, T_1T_2s_2^{-1})$ is also on the point (16), and since T_3 is on the hyperbola we see that for the triangle $A_1A_2A_3$ there are three points T_1, T_2, T_3 whose pedal lines are concurrent. *The point of concurrence is midway between the slant-centers of the two triangles.* To locate T_3, given T_1, T_2 and the triangle $A_1A_2A_3$, we construct the slant-center H' of $A_1A_2A_3$. The pedal lines of T_1 and T_2 as to $A_1A_2A_3$ intersect at M. Produce $H'M = MY$, then Y is the slant-center of T_1, T_2 and T_3. Also the pedal line of T_i and the join of T_jT_k are isosceles lines.

Given, *any four points $A_1A_2A_3A_4$ on the hyperbola, which will form four triangles. Points T_1 and T_2 on the hyperbola will each have four pedal lines with respect to each triangle; each pair will intersect in a point, thus giving rise to four points which lie on a line.* We saw that the pedal lines of any two points T_1 and T as to the triangle $A_1A_2A_3$ met at

$$2x = T_1 + T_2 + s_1 + s_3T_1^{-1}T_2^{-1}.$$

The four such points arising from the four triangles satisfy the equation

$$(17)\quad 2(T_1T_2x + \sigma_4 y) = T_1T_2\sigma_1 + \sigma_3 + T_1^2T_2 + T_1T_2^2 + \sigma_4(T_1 + T_2)T_1^{-1}T_2^{-1},$$

where the σ_i are symmetric functions of t_1, t_2, t_3, t_4. For any third point T_3 on the hyperbola, with reference to $A_1A_2A_3A_4$, we shall have three such lines as (17), taking the points T_i by pairs. The three lines meet at a point M' whose x-coördinate is given by

$$2x = \sigma_1 + T_1 + T_2 + T_3 - \sigma_4 T_1^{-1}T_2^{-1}T_3^{-1}.$$

Since this is symmetric in the t_i and T_i, we have the theorem that *the three lines* (17) *for* T_1T_2, T_2T_3 *and* T_3T_1 *as to* $A_1A_2A_3A_4$ *are concurrent.*

9. The eighteen-point hyperbola. If hyperbolas* be drawn with centers at C_i, the feet of the slant-lines, and passing through C, the center of the hyperbola, they cut the corresponding sides of triangle $A_1A_2A_3$ in six points lying on a hyperbola whose center is at H'.

The equation of the hyperbola with center at C_3 and on C is

$$(2t_3x - t_3s_1 + t_1t_2)(2s_3y - s_2 + t_3^2) = (s_2 - t_3^2)(t_3s_1 - t_1t_2).$$

This cuts the side A_1A_2 in two points whose coördinates are given by

$$(18)\qquad 2t_3x = t_3s_1 - t_1t_2 \pm a^{1/2}, \qquad 2s_3y = s_2 - t_3^2 \mp a^{1/2},$$

where

$$a = (s_2 - t_3^2)(t_1t_2 - t_3s_1).$$

The two points (18) lie on the hyperbola

$$(19)\qquad 2(x - s_1)(s_3y - s_2) = s_1s_2 + s_3.$$

Since this equation is symmetric in $t_i (i = 1, 2, 3)$, it follows that the points cut on the sides A_2A_3 and A_3A_1 by hyperbolas with centers at C_1 and C_2, respectively, and on C, likewise lie on (19).

Secondly, if hyperbolas be drawn with centers at the vertices A_i of the triangle $A_1A_2A_3$, and with transverse axis equal to the given hyperbola, they cut the lines joining the midpoints of the sides A_iA_j and A_iA_k, respectively, in six points lying on (19).

The equation of the hyperbola, with transverse axis equal to the given hyperbola and with center at A_2, is

$$(x - t_2)(t_2y - 1) = t_2.$$

* The word hyperbola, in this section, refers to a rectangular hyperbola with asymptotes parallel to those of the given hyperbola.

The equation of the line joining the midpoints of A_2A_3 and A_2A_1 is

$$2(t_2x + s_3y) = s_2 + t_2^2.$$

The coördinates of the two points of intersection of this line and hyperbola are given by

$$(20) \qquad 4t_2x = s_2 + 3t_2^2 - 2t_1t_3 \mp b^{1/2}, \qquad 4s_3y = s_2 - t_2^2 + 2t_1t_3 \pm b^{1/2}$$

where

$$b = (s_2 - t_2^2 + 2t_1t_3)^2 - 8t_1t_3s_2 - 8t_2s_3.$$

By substituting the values given in (20) in equation (19) one can show (19) is satisfied.

Thirdly, if hyperbolas be drawn with centers at the midpoints of A_iH' and passing through C, they cut the lines parallel to A_jA_k, and on these midpoints, in six points lying on (19). This follows from the fact that the hyperbola,

$$(2x - s_1 - t_3)(2s_3y - s_2 - t_1t_2) = (s_1 + t_3)(s_2 + t_1t_2),$$

meets the line,

$$t_3x + s_3y = s_2 + t_3^2,$$

in the two points given by

$$(21) \qquad 2t_3x = t_3s_1 + t_3^2 \pm c^{1/2}, \qquad 2s_3y = 2s_2 + t_3^2 - t_3s_1 \mp c^{1/2},$$

where

$$c = t_3^2s_1^2 - t_3^4 - 2t_3s_1s_2 - 2t_3^2s_2.$$

The coördinates of the points (21) will satisfy equation (19). Hence this hyperbola passes through eighteen points, two on each side of triangle $A_1A_2A_3$, two on each side of the lines joining the midpoints of the sides of $A_1A_2A_3$, and two on each line parallel to the sides of $A_1A_2A_3$ and on the midpoints of A_iH'.

References

1. S. L. Loney, The Elements of Coordinate Geometry, Second Edition, 1904, p. 300.
2. E. F. Allen, This MONTHLY, vol. 48, 1941, p. 677.
3. R. A. Johnson, Modern Geometry, 1929, p. 197.
4. J. Casey, A Treatise on the Analytical Geometry of the Point, Line, Circle and Conic Sections, second edition, 1893, p. 291.

L'HOSPITAL'S RULE

A. E. TAYLOR, University of California, Los Angeles

1. Prefatory remarks. In this article I have assembled some material pertaining to l'Hospital's rule which it seem to me may be of interest to teachers of analysis. The immediate stimulus for the writing of the paper was my discovery of a new proof which has the virtue of disposing completely of the rule in brief compass, without any need for tedious changes of variable or the treatment of a variety of special instances. The material of §3 is not new, but is probably unfamiliar to many readers of the MONTHLY.

2. The unified theorem and its proof. Let I be an open interval of the x-axis. We understand that I may be either bounded or unbounded. Let c be one of the extremities of I. If I is bounded, c is a real number, but if I is unbounded, c may be $+\infty$ or $-\infty$. L'Hospital's rule is concerned with two functions $f(x)$, $g(x)$, which are real-valued and defined on I. The first additional hypotheses of f and g are the following:

H_0: *The derivatives $f'(x)$, $g'(x)$ are defined (as finite limits) on I. Furthermore, $g(x)$ and $g'(x)$ are never zero on I.*

There are two cases of l'Hospital's rule, depending on which of the two following hypotheses are fulfilled.

H_1: $$\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0.$$

H_2: $$\lim_{x\to c} |g(x)| = \infty.$$

We are now ready to state the unified theorem.

THEOREM I. *Let A denote either a real number or one of the symbols $+\infty$, $-\infty$. Suppose that f, g satisfy hypothesis H_0, either one of the two hypotheses H_1, H_2, and the further hypothesis*

H_3: $$\lim_{x\to c} \frac{f'(x)}{g'(x)} = A.$$

Then we can conclude that

$$\lim_{x\to c} \frac{f(x)}{g(x)} = A. \tag{2.1}$$

Theorem I (l'Hospital's rule) is obviously a corollary of the following more general proposition:

THEOREM II. *Let f, g satisfy H_0 and either H_1 or H_2. Then*

$$\underline{\lim_{x\to c}} \frac{f'(x)}{g'(x)} \leqq \underline{\lim_{x\to c}} \frac{f(x)}{g(x)} \leqq \overline{\lim_{x\to c}} \frac{f(x)}{g(x)} \leqq \overline{\lim_{x\to c}} \frac{f'(x)}{g'(x)}. \tag{2.2}$$

For the proof of Theorem II we start with Cauchy's formula (the extended law of the mean)

$$\frac{f(x) - f(y)}{g(x) - g(y)} = \frac{f'(X)}{g'(X)}, \tag{2.3}$$

where x and y are any distinct points of I, and X is some point between x and y; that $g(x) \neq g(y)$ follows from H_0. Let x be in the interval I, and define

$$m(x) = \text{g.l.b.} \frac{f'(\xi)}{g'(\xi)}, \qquad M(x) = \text{l.u.b.} \frac{f'(\xi)}{g'(\xi)} \tag{2.4}$$

for ξ between x and c. There are of course the possibilities that $m(x) = -\infty$ and that $M(x) = +\infty$. We take y between x and c; then, by (2.3) and (2.4) we may write

$$m(x) \leqq \frac{\dfrac{f(x)}{g(x)} - \dfrac{f(y)}{g(x)}}{1 - \dfrac{g(y)}{g(x)}} \leqq M(x) \tag{2.5}$$

and

$$m(x) \leqq \frac{\dfrac{f(y)}{g(y)} - \dfrac{f(x)}{g(y)}}{1 - \dfrac{g(x)}{g(y)}} \leqq M(x). \tag{2.6}$$

Now let $y \to c$. If H_1 holds, we use (2.5) and obtain

$$m(x) \leqq \frac{f(x)}{g(x)} \leqq M(x).$$

If H_2 holds, we use (2.6) and obtain

$$m(x) \leqq \underline{\lim_{x \to c}} \frac{f(y)}{g(y)} \leqq \overline{\lim_{x \to c}} \frac{f(y)}{g(y)} \leqq M(x).$$

We then let $x \to c$ and the desired conclusion (2.2) follows in either case.

Comment on the proof. The Polish mathematician T. Wazewski has given interesting proofs of Theorems I and II, using as new ingredient a lemma about pairs of sequences of real numbers [**4** and **5**]. Wazewski's arguments suggested to me the principle of the proof given here. My arrangement of the proof has been somewhat shortened and improved by the suggestions of the referee.

3. Other proofs. Fritz Lettenmeyer [1] has devised a wholly different method of proof which can be applied to all cases of l'Hospital's rule. The method does not make use of Cauchy's formula at all. This proof does not seem to be widely known. I shall give the proof of Theorem I with hypothesis H_2 on the assumption that A is finite and $c = +\infty$.

The derivative $g'(x)$ must be of constant sign, since its never vanishes; this is a consequence of Darboux's theorem (see [3], p. 637). We may then assume $g'(x) > 0$; otherwise we consider the ratio of $-f(x)$ to $-g(x)$. It follows from H_2 that $\lim_{x\to+\infty} g(x) = +\infty$. With $\epsilon > 0$ given we have

$$A - \epsilon < \frac{f'(x)}{g'(x)} < A + \epsilon$$

for all sufficiently large values of x. Then

$$(A - \epsilon)g'(x) < f'(x) < (A + \epsilon)g'(x),$$

whence it follows that $f(x) - (A-\epsilon)g(x)$ is monotone increasing and $f(x) - (A+\epsilon)g(x)$ is monotone decreasing. These same conclusions hold with ϵ replaced by $\epsilon/2$. Now

$$f(x) - \left(A - \frac{\epsilon}{2}\right)g(x) < f(x) - (A - \epsilon)g(x). \tag{3.1}$$

We infer that

$$\lim_{x\to+\infty} [f(x) - (A - \epsilon)g(x)] = +\infty, \tag{3.2}$$

for if this is not so, the monotone function $f(x) - (A-\epsilon)g(x)$ has a finite limit, and hence, by (3.1), so has the function $f(x) - (A-\epsilon/2)g(x)$. But then the difference of these functions, namely $\frac{1}{2}\epsilon\, g(x)$, also has a finite limit. This contradicts H_2, however, so that (3.2) holds. In the same way we prove that

$$\lim_{x\to+\infty} [f(x) - (A + \epsilon)g(x)] = -\infty. \tag{3.3}$$

Therefore, for all sufficiently large values of x,

$$f(x) - (A + \epsilon)g(x) < 0 < f(x) - (A - \epsilon)g(x),$$

or

$$A - \epsilon < \frac{f(x)}{g(x)} < A + \epsilon.$$

Thus (2.1) is proved.

We leave the proof when $A = \pm\infty$ to the reader. This method of proof can readily be adapted to use with the hypothesis H_1. It would be worth while to cast this method into a form where the consideration of various cases is obviated.

Finally, another group of proofs of l'Hospital's rule is obtained if we take advantage of the fact that $g(x)$ is a monotone function and hence possesses a continuous inverse function. If we let G denote the inverse function, so that $x=G(y)$ is equivalent to $y=g(x)$, and define

$$F(y) = f(G(y)), \tag{3.4}$$

then

$$\frac{f(x)}{g(x)} = \frac{F(y)}{y}, \qquad F'(y) = \frac{f'(x)}{g'(x)}. \tag{3.5}$$

If $b=\lim_{x\to c} g(x)$, G is defined in an open interval J with extremity $y=b$. For the proof under hypothesis H_1 with c finite we may apply the ordinary law of the mean to $F(y)$ (see e.g. [2], p. 177).

We shall use this method to give an alternative proof of Theorem II under hypothesis H_2.

LEMMA. *Let J be an unbounded open interval having one extremity b (where $b=+\infty$ or $-\infty$). Let $F(y)$ be defined and admit a finite derivative $F'(y)$ at each point of J. Then*

$$\overline{\lim_{y\to b}} \frac{F(y)}{y} \leqq \overline{\lim_{y\to b}} F'(y) \tag{3.6}$$

and

$$\underline{\lim_{y\to b}} F'(y) \leqq \underline{\lim_{y\to b}} \frac{F(y)}{y}. \tag{3.7}$$

Under the change of variable $y=g(x)$, $x\to c$ is equivalent to $y\to b$. By virtue of equations (3.5) it is clear that the Lemma is equivalent to Theorem II with hypothesis H_2.

For definiteness assume $b=+\infty$. We may assume

$$B = \overline{\lim_{y\to b}} F'(y) < +\infty,$$

since otherwise there is nothing to prove in (3.6). If $\epsilon>0$ we have

$$F'(y) < B + \frac{\epsilon}{2} < B + \epsilon$$

for all sufficiently large values of y. By reasoning exactly like that in Lettenmeyer's proof given earlier in this section we conclude that

$$\frac{F(y)}{y} < B + \epsilon$$

when y is sufficiently large. This proves (3.6) for $b=+\infty$. A similar proof may be given for (3.7) when $b=+\infty$. To prove (3.7) when $b=-\infty$ we may take the change of variable

$$t=-y, \qquad \Phi(t)=F(-t).$$

Then

$$\overline{\lim_{t\to+\infty}}\ \Phi'(t) = \overline{\lim_{y\to-\infty}}\ \{-F'(y)\} = -\underline{\lim_{y\to-\infty}}\ F'(y),$$

$$\overline{\lim_{t\to+\infty}}\ \frac{\Phi(t)}{t} = -\underline{\lim_{y\to-\infty}}\ \frac{F(y)}{y},$$

so that the inequality

$$\underline{\lim_{y\to-\infty}}\ F'(y) \leqq \underline{\lim_{y\to-\infty}}\ \frac{F(y)}{y}$$

follows from what has already been proved. Likewise we may deduce (3.6) when $b=-\infty$ from (3.7) when $b=+\infty$. This completes the proof of the lemma.

References

1. Fritz Lettenmeyer, Über die sogenannte Hospitalsche Regel, Journal für die reine und angewandte Mathematik, vol. 174, pp. 246–247 (1936).

2. G. E. F. Sherwood and A. E. Taylor, Calculus (revised) New York, 1946.

3. A. E. Taylor, Derivatives in the Calculus, this MONTHLY, vol. 49, pp. 631–642 (1942).

4. T. Wazewski, Quelques démonstrations uniformes pour tous les cas du théorème de l'Hospital. Generalisations, Prace Mat. Fiz. vol. 47, pp. 117–128 (1949).

5. T. Wazewski, Une démonstration uniforme du théorème généralisé de l'Hospital, Ann. Soc. Polon. Math. vol. 22, pp. 161–168 (1949).

PERSPECTIVE AND ORTHOLOGIC TRIANGLES AND TETRAHEDRONS*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. Introduction. In 1894 P. Sondat published a fundamental theorem concerning two perspective and orthologic triangles [1]. The corresponding theorem for two perspective and orthologic tetrahedrons was stated several years later [2]. These two theorems, which have not been presented to the readers of the MONTHLY previously, have many applications. It is proposed to give an elementary proof of the Sondat theorem in such a manner that it may be generalized, and to cite some interesting particular cases. The word bilogic intro-

* Translated by W. E. Byrne, Virginia Military Institute, (with additional information supplied by M. Thébault in letters dated 13 February and 13 April 1950).

duced by Neuberg [3] will be used to designate triangles (tetrahedrons) which are perspective and orthologic.

2. Sondat's theorem. *If two triangles $T\equiv ABC$, $T_1\equiv A_1B_1C_1$ are bilogic, the center of perspective P and the orthologic centers Q and Q_1 are on the same line perpendicular to the axis of perspectivity d.*

Proof. By definition of orthologic triangles the point Q is the point common to the perpendiculars drawn through A_1, B_1, C_1 to the sides BC, CA, AB of T, and Q_1 is the point common to the perpendiculars drawn through A, B, C to the sides B_1C_1, C_1A_1, A_1B_1 of T_1. Let us designate by B' and C' the intersections of CA and C_1A_1, AB and A_1B_1, which are on d, and write

$$\overrightarrow{A_1P} = k_a\overrightarrow{A_1A}, \qquad \overrightarrow{B_1P} = k_b\overrightarrow{B_1B}, \qquad \overrightarrow{C_1P} = k_c\overrightarrow{C_1C},$$
$$\overline{PA}^2 - \overline{QA}^2 = l_a, \qquad \overline{PB}^2 - \overline{QB}^2 = l_b, \qquad \overline{PC}^2 - \overline{QC}^2 = l_c.$$

If it can be proved that

$$\overline{B'P}^2 - \overline{B'Q}^2 = \overline{C'P}^2 - \overline{C'Q}^2, \tag{1}$$

then it will follow that PQ is perpendicular to d. The fact that

$$B'C/B'A = A_1P\cdot C_1C/A_1A\cdot C_1P = k_a/k_c$$

shows that B' is the centroid of C and A with weighting coefficients k_c and $-k_a$. The theorem of Leibnitz gives

$$\overline{PC}^2\cdot k_c - \overline{PA}^2\cdot k_a = \overline{PB'}^2(k_c - k_a) + \overline{B'C}^2\cdot k_c - \overline{B'A}^2\cdot k_a$$
$$\overline{QC}^2\cdot k_c - \overline{QA}^2\cdot k_a = \overline{QB'}^2(k_c - k_a) + \overline{B'C}^2\cdot k_c - \overline{B'A}^2\cdot k_a.$$

Subtracting and dividing by k_c-k_a, we have

$$\overline{PB'}^2 - \overline{QB'}^2 = (l_ck_c - l_ak_a)/(k_c - k_a).$$

Relation (1) is equivalent to

$$k_bk_c\cdot(l_b - l_c) + k_ck_a\cdot(l_c - l_a) + k_ak_b\cdot(l_a - l_b) = 0. \tag{2}$$

Since BC is perpendicular to A_1Q,

$$\overline{BA_1}^2 - \overline{CA_1}^2 = \overline{BQ}^2 - \overline{CQ}^2. \tag{3}$$

A_1 is the centroid of A, P with weighting coefficients k_a, -1; hence by the theorem of Leibnitz,

$$\overline{BA}^2\cdot k_a - \overline{BP}^2 = \overline{BA_1}^2\cdot(k_a - 1) + \overline{A_1A}^2\cdot k_a - \overline{A_1P}^2,$$
$$\overline{CA}^2\cdot k_a - \overline{CP}^2 = \overline{CA_1}^2\cdot(k_a - 1) + \overline{A_1A}^2\cdot k_a - \overline{A_1P}^2.$$

By subtraction we obtain, using (3),

$$\overline{BA}^2 - \overline{CA}^2 + \overline{CQ}^2 - \overline{BQ}^2 = (l_a - l_c)/k_a,$$

and by analogy

$$\overline{CB}^2 - \overline{AB}^2 + \overline{AQ}^2 - \overline{CQ}^2 = (l_c - l_a)/k_b$$
$$\overline{AC}^2 - \overline{BC}^2 + \overline{BQ}^2 - \overline{AQ}^2 = (l_a - l_b)/k_c.$$

Addition of the above equations gives us (2) and hence (1), so we conclude that PQ is perpendicular to d. By similar reasoning with the points P and Q_1 we conclude that P, Q, Q_1 are collinear.

COROLLARY. *The conics $ABCPQ_1$ and $A_1B_1C_1PQ$ are equilateral hyperbolas.*

3. Theorem. *If two tetrahedrons $T \equiv ABCD$, $T_1 \equiv A_1B_1C_1D_1$ are bilogic, the center of perspective P and the orthologic centers Q and Q_1 are on a straight line perpendicular to the plane (π) of perspectivity of the two tetrahedrons.*

Proof. Let Q designate the point common to the perpendiculars drawn through A_1, B_1, C_1, D_1 to the faces BCD, CDA, DAB, ABC of T, and Q_1 the point common to the perpendiculars drawn through A, B, C, D to the faces $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$ of T_1. If B' is the intersection of the edges CA, C_1A_1, and C' is the intersection of the edges AB, A_1B_1, we may write

$$\overrightarrow{A_1P} = k_a\overrightarrow{A_1A}, \qquad \overrightarrow{B_1P} = k_b\overrightarrow{B_1B}, \qquad \overrightarrow{C_1P} = k_c\overrightarrow{C_1C},$$
$$\overline{PA}^2 - \overline{QA}^2 = l_a, \qquad \overline{PB}^2 - QB^2 = l_b, \qquad \overline{PC}^2 - \overline{QC}^2 = l_c.$$

It will be sufficient to prove that

$$\overline{B'P}^2 - \overline{B'Q}^2 = \overline{C'P}^2 - \overline{C'Q}^2$$

to show that PQ is perpendicular to (π). The demonstration is identical with that carried through for Sondat's theorem.

a). *Another proof.* The plane of perspectivity (π) contains the points

$$B' = (AB, A_1B_1), \qquad C' = (AC, A_1C_1), \qquad D' = (AD, A_1D_1).$$

The altitudes h_1, h_2, h_3, h_4 of the tetrahedron $A_1B'C'D'$ are rulings of the same system of an equilateral hyperboloid (H'). The plane ABQ_1B', which is perpendicular to C_1D_1, contains the altitude h_2. Hence the line AQ_1 meets h_2 and likewise h_3, h_4. AQ_1 is then a ruling of the second system of (H'); it meets h_1 also. Thus the plane AA_1Ph_1, which projects the line APA_1 orthogonally on (π), contains the line AQ_1; P, Q_1, Q are in the plane AA_1Ph_1. Likewise the planes projecting BPB_1, CPC_1, DPD_1 have the same property. It follows that P, Q_1, Q are on a line perpendicular to (π).

b). *Remarks.* The pair of tetrahedrons T, T_1 determine a tetrahedron T_2 of vertices $A_2 = (QA_1, Q_1A)$, $B_2 = (QB_1, Q_1B)$, $C_2 = (QC_1, Q_1C)$, $D_2 = (QD_1, Q_1D)$. Q_1 is the perspective center of T and T_2. Their plane of perspectivity is (π), since the perspective triangles Q_1AB, QA_1B_1 show that the points (AB, A_1B_1),

(QA_1, Q_1A), (QB_1, Q_1B) are collinear. T and T_2 are orthologic; Q is the orthologic center of T and P is the orthologic center of T_2, since the intersection B_2C_2 of the planes Q_1BC, QB_1C_1 is perpendicular to plane ADA_1D_1. Hence the line AA_1P is perpendicular to B_2C_2 and likewise to C_2D_2, D_2B_2. Consequently the perpendicular from A to the face $B_2C_2D_2$ of T_2 contains the point P. This system of tetrahedrons has the following properties [4]:

The tetrahedrons T, T_1, T_2 are bilogic in pairs; the corresponding orthologic centers are Q, Q_1, P. The orthologic center of one of these tetrahedrons is the perspective center of the two others. The three perspectivities have the same perspective plane. Any two of the tetrahedrons T, T_1, T_2 determine the third.

c). THEOREM. *The line $\Delta = PQQ_1$ is a ruling of the second system of the hyperboloids (H) and (H_1) of the altitudes of T and T_1.*

Proof. If A'' is the foot of the altitude AA'' of T, QA_1 is parallel to AA''. The figure $AA''QA_1$ is plane and the line PQ meets AA'' as well as the other altitudes of T. Hence PQ is a ruling of the second system of (H). By permuting A and A_1, T and T_1, it follows that PQ is also a ruling of the second system of (H_1).

4. Polar reciprocation with respect to a quadric. The poles A_1, B_1, C_1, D_1 of the faces BCD, CDA, DAB, ABC of a tetrahedron $T \equiv ABCD$ with respect to a quadric (Q) are the vertices of a tetrahedron T_1. T and T_1 are polar reciprocals with respect to (Q).

CHASLES'S THEOREM. *Two polar reciprocal tetrahedrons T and T_1 with respect to a quadric (Q) are hyperbolic* [5].

COROLLARY 1. *The four lines of intersection of the faces of a tetrahedron T with the polar planes of the opposite vertices with respect to a quadric form a hyperbolic set.*

COROLLARY 2. *If the tetrahedrons T and T_1 are homologic, then T and the tetrahedron formed by the polar planes of the vertices of T with respect to a quadric are also homologic.*

If the quadric is a sphere we have the

THEOREM: *Two tetrahedrons T, T_1 conjugate with respect to a sphere (S) are orthologic and hyperbolic.*

This theorem follows from the fact that the vertices A_1, B_1, C_1, D_1 of T_1 are on the axes of the circles BCD, CDA, DAB, ABC, and the point Q is the center S of the sphere (S).

THEOREM. *If the center S of (S) is on the hyperboloid (H) determined by the altitudes of T, and also on the hyperboloid (H_1) determined by the altitudes of T_1,*

then the lines Δ_a, Δ_b, Δ_c, Δ_d *of intersection of the planes of faces* BCD *and* $B_1C_1D_1$, CDA *and* $C_1D_1A_1$, DAB *and* $D_1A_1B_1$, ABC *and* $A_1B_1C_1$ *are on a hyperbolic paraboloid.*

Proof. Through S there pass rulings G and G_1 of the second system of rulings of (H) and (H_1), respectively. G and G_1 coincide since otherwise they would determine a plane (P) passing through S. As G meets the altitude AA'' of T, G and G_1 would be in the plane $SAA'' \equiv SAA_1$, and by the same argument G and G_1 would also be in the planes SBB_1, SCC_1, SDD_1. This contradicts the assumption that T is a non-degenerate tetrahedron. Hence G and G_1 coincide, and (H), (H_1) have a common ruling passing through S. Furthermore G is in the planes SAA_1, SBB_1, SCC_1, SDD_1, and is perpendicular to Δ_a, Δ_b, Δ_c, Δ_d, which belong to a hyperbolic paraboloid whose director plane is perpendicular to G.

If $(S) \equiv (O)$ is the circumsphere of T, Q and Q_1 coincide with O.

THEOREM. *If the tetrahedrons* T, T_1 *are bilogic,* T *is involutory** [6].

The line which joins the perspective center to the common orthologic center O is a ruling of the hyperboloids (H), (H_1). O is on (H).

COROLLARY 1. *If* T *and its tangential tetrahedron* t *are bilogic,* T *is isodynamic and conversely* [7].

In this particular case the lines of intersection of the corresponding faces of T and t are coplanar and conversely [8].

COROLLARY 2. *In an isodynamic tetrahedron the second Lemoine point* L *is on the hyperboloid* (H) *and conversely.*

L coincides with the perspective center of T and t.

References

1. P. Sondat, L'intermédiaire des mathématiciens, 1894, p. 10 [question 38, solved by Sollerstinsky, p. 94].
2. V. Thébault, Bulletin de l'Académie royale de Belgique, 1921, p. 57.
3. J. Neuberg, Mathesis, 1922, p. 163.
4. C. Servais, Bulletin de l'Académie royale de Belgique, 1923, p. 49.
5. H. F. Baker, Principles of Geometry, vol. 3, p. 41, Ex. 7.
6. J. Neuberg, Mémoire sur le tétraèdre, 1884, p. 58.
7. N. A. Court, Modern Pure Solid Geometry, p. 281.
8. N. A. Court, ibid., p. 281 and V. Thébault, this MONTHLY, 1949, p. 637, advanced problem 4368.

* The term *involutory tetrahedron* used in connection with [6] is the translation of *tétraèdre involutif* used by Neuberg to refer to the fact that the polar line of an edge with respect to the circumsphere meets the respectively opposite edge of the tetrahedron. The term has been accepted by French writers in this field.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

TWO NEW INEQUALITIES

H. D. RUDERMAN, New York City

If $p_j \geqq 0$ $(j=1, 2, \cdots, n)$, then the familiar inequality between geometric and arithmetic means yields the relation

$$np_1p_2 \cdots p_n \leqq p_1^n + p_2^n + \cdots + p_n^n. \tag{1}$$

Now suppose that $p_1 \geqq p_2 \geqq \cdots \geqq p_n$ and consider the square arrays

$$\begin{array}{llll} p_1 & p_2 & \cdots & p_n \\ p_2 & p_3 & \cdots & p_1 \\ \cdots & \cdots & \cdots & \cdots \\ p_n & p_1 & \cdots & p_{n-1} \end{array} \qquad \begin{array}{lll} p_1 & p_2 & \cdots & p_n \\ p_1 & p_2 & \cdots & p_n \\ \cdots & \cdots & \cdots & \cdots \\ p_1 & p_2 & \cdots & p_n \end{array} \tag{2}$$

If we denote the array on the left by a_{ij} and that on the right by a'_{ij}, then inequality (1) states that

$$\sum_{j=1}^{n} \prod_{i=1}^{n} a_{ij} \leqq \sum_{j=1}^{n} \prod_{i=1}^{n} a'_{ij}. \tag{3}$$

More generally, consider the rectangular array a_{ij} $(i=1, \cdots, m;$ $j=1, \cdots, n)$ of non-negative numbers: $a_{ij} \geqq 0$. Let a'_{ij} denote the array obtained by rearranging each row in non-increasing order: $a'_{i1} \geqq \cdots \geqq a'_{in}$ $(i=1, \cdots, m)$. Then our main result is the

THEOREM:

$$\sum_{j=1}^{n} \prod_{i=1}^{m} a_{ij} \leqq \sum_{j=1}^{n} \prod_{i=1}^{m} a'_{ij} \tag{I}$$

and

$$\prod_{j=1}^{n} \sum_{i=1}^{m} a_{ij} \geqq \prod_{j=1}^{n} \sum_{i=1}^{m} a'_{ij}. \tag{II}$$

Observe that (II) is, in a sense, dual to (I). For the arrays considered in (2), (II) specializes to essentially the same inequality as (1), now taking the form

$$n^n p_1 \cdots p_n \leqq (p_1 + \cdots + p_n)^n. \tag{4}$$

The case $m=2$ of assertion (I) is treated by Hardy, Littlewood, and Polya (*Inequalities*, p. 261, Theorem 368 and footnote a).

Since the order in which the terms occur within a row of the array a_{ij} is given quite arbitrarily, the theorem really makes an assertion about maxima and minima obtainable by arbitrary rearrangements within the rows of the fixed array a'_{ij}. In order to deal with permutations of the elements within the various rows, we introduce a convenient notation. By a permutation function u, defined on the integers $1, 2, \cdots, n$, we shall mean a mapping of the integers $1, 2, \cdots, n$ onto themselves. Thus, ui $(i=1, \cdots, n)$ is one of the integers $1, \cdots, n$ and u simply rearranges the ordered sequence $1, 2, \cdots, n$. In the place of "permutation function" we shall use simply the word "permutation".

The theorem may now be stated in more convenient form.

THEOREM. *Let sequences* $a_1, \cdots, a_m$ *be defined as follows*:

$$a_i: \quad a_{i1} \geqq a_{i2} \geqq \cdots \geqq a_{in} \geqq 0 \qquad \text{for } i = 1, \cdots, m.$$

Let $x_2, x_3, \cdots, x_m$ *be* $m-1$ *permutations on* $1, \cdots, n$. *Let* x *be the ordered set* $(x_2, x_3, \cdots, x_m)$. *Let* e *be the identity permutation. Finally, let*

$$x(a_1, \cdots, a_m) = \sum_{j=1}^{n} a_{1j}a_{2x_2j}a_{3x_3j} \cdots a_{mx_mj}.$$

Then for every choice of x,

I. $$x(a_1, \cdots, a_m) \leqq e(a_1, \cdots, a_m) = \sum_{j=1}^{n} a_{1j}a_{2j} \cdots a_{mj}$$

II. $$\prod_{i=1}^{n} (a_{1i} + a_{2x_2i} + \cdots + a_{mx_mi}) \geqq \prod_{i=1}^{n} (a_{1i} + \cdots + a_{mi}).$$

Proof of I: The theorem is trivially true for $m=1$. The proof will proceed by an induction on m. Assume that the theorem is valid for all sequences of length $\leqq n$ which number $\leqq m-1$. We seek to prove validity for m sequences all of length n.

To Prove I it will suffice to prove

$$d = e(a_1, \cdots, a_m) - x(a_1, \cdots, a_m) \geqq 0.$$

If we let $b_j = a_{2j} \cdots a_{mj} - a_{2x_2j} \cdots a_{mx_mj}$, we may write

$$d = \sum_{j=1}^{n} a_{1j}b_j.$$

We need the following lemma.

LEMMA 1: $T = b_1 + b_2 + \cdots + b_s \geqq 0$ *for* $s = 1, \cdots, n$.

This lemma is immediately true for $s=1$. Let v_p be any permutation on $1, \cdots, s$ such that $v_p j = x_p j$ if $x_p j \leqq s$ for $j=1, \cdots, s$ and for $p=2, \cdots, m$. Then

$$a_{2v_2 j} \cdots a_{mv_m j} \geqq a_{2x_2 j} \cdots a_{mx_m j} \qquad \text{for } j = 1, \cdots, s.$$

We may now write

$$T = \sum_{j=1}^{s} a_{2j}a_{3j} \cdots a_{mj} - \sum_{j=1}^{s} a_{2x_2 j}a_{3x_3 j} \cdots a_{mx_m j}$$

$$\geqq \sum_{j=1}^{s} a_{2j}a_{3j} \cdots a_{mj} - \sum_{j=1}^{s} a_{2v_2 j}a_{3v_3 j} \cdots a_{mv_m j}.$$

As v_p for $p=2, \cdots, m$ is a permutation on $1, \cdots, s$ we may use our induction hypothesis for sequences of length $s \leqq n$ which number $m-1$. This tells us that the right member of the last inequality $\geqq 0$. Hence $T \geqq 0$ and the lemma is proved.

We return now to the proof of I. We have $a_{11}b_1 \geqq 0$, and from Lemma 1, $a_{11}b_1 + a_{12}b_2 \geqq a_{12}(b_1+b_2) \geqq 0$. We use an induction again. Assume for a fixed $1 \leqq r < n$

$$a_{11}b_1 + \cdots + a_{1r}b_r \geqq a_{1r}(b_1 + \cdots + b_r) \geqq 0.$$

$$\begin{aligned} a_{11}b_1 + \cdots + a_{1r+1}b_{r+1} &\geqq a_{1r}(b_1 + \cdots + b_r) + a_{1r+1}b_{r+1} \\ &\geqq a_{1r+1}(b_1 + \cdots + b_{r+1}) \geqq 0. \end{aligned}$$

Hence, $d = a_{11}b_1 + \cdots + a_{1n}b_n \geqq a_{1n}(b_1 + \cdots + b_n) \geqq 0$. This concludes the proof of I.

Part I of the theorem remains valid if a single negative number appears among all the a_{ij}. However, if two or more negative numbers appear when $m \geqq 3$, there are counterexamples to show that the theorem is then false.

Proof of II: The proof makes use of the following simple lemma.

LEMMA 2. *If* $A+B=C+D$ *and* $|A-B| > |C-D|$, *then* $AB < CD$. *For*, $(A+B)^2 - (A-B)^2 < (C+D)^2 - (C-D)^2$.

Let the minimum be attained for the ordered set of permutations $u = (u_2, \cdots, u_m)$. Then

$$P = \min_x \left\{ \prod_{i=1}^{n} (a_{1i} + a_{2x_2 i} + \cdots + a_{mx_m i}) \right\} = \prod_{i=1}^{n} (a_{1i} + a_{2u_2 i} + \cdots + a_{mu_m i}).$$

We shall show that u_i may be replaced by e without changing the value of P, for $i=2, \cdots, m$.

Suppose $u_i = e$ for $i=2, \cdots, g-1$, but $u_g \neq e$. We shall show that u_g may be replaced by e without changing P. Once this is shown, then $m-g+1$ changes of this kind will give the result we are trying to prove. Let $u_g i = i$ for $i=1, \cdots,$

$k-1$, but $u_g k=h\neq k$. In fact, $h>k$. Let $u_g w=k$. Now consider the kth and wth factors of P, represented by K and W respectively. Then

$$K = (a_{1k} + a_{2k} + \cdots + a_{g-1,k} + a_{gh} + a_{g+1,u_{g+1}k} + \cdots + a_{m,u_m k}),$$

and

$$W = (a_{1w} + a_{2w} + \cdots + a_{g-1,w} + a_{gk} + a_{g+1,u_{g+1}w} + \cdots + a_{m,u_m w}).$$

For P to be minimum, no interchange of terms in K and W which have the same first subscript can decrease KW. From our lemma, this will be the case when $|K-W|$ is a maximum. There are two possibilities to be considered.

Case A: $a_{gh}=a_{gk}$. Replace u_g by the permutation u_g' defined by $u_g' i=u_g i$ for all i except for $i=k$, w. $u_g' k=k$, and $u_g' w=h$.

Case B: $a_{gh}\neq a_{gk}$. Then from the hypothesis we must have $a_{gk}>a_{gh}$. If $K\geqq W$, then interchanging a_{gk} and a_{gh} will increase the difference $K-W$ and so decrease the product KW and hence, also P. Thus, if P is minimum, we must have $K<W$. Let R stand for the sum of the first $g-1$ terms in K; and T, for the sum of the first $g-1$ terms in W. Let $S=K-R$ and $U=W-T$. Thus, $K=R+S$ and $W=T+U$. As $w>k$, we must have $a_{ik}\geqq a_{iw}$ for $i=1, \cdots, g-1$. Hence, $R\geqq T$, so $S<U$. Let $K'=R+U$ and $W'=T+S$. If $R>T$ then $K'-W'$ $=|(R+U)-(T+S)|>|(R+S)-(T+U)|=|K-W|$. Hence, using K' and W' in place of K and W decreases KW, and so, also P, which is impossible. So we must have in this case $R=T$, and then $K'-W'=|K-W|$. Using K' and W' in place of K and W does not change the value of P but it does replace a_{gh} by a_{gk}. Successive changes of the type indicated in Case A and the permissible type in Case B will replace u_g by e without affecting P. This completes the proof of part II of the theorem.

A CORRECTION

H. S. THURSTON, University of Alabama

In the note "A Simplified Technique for a Tschirnhaus Transformation" (this MONTHLY, vol. 58, pp. 483–484) the last paragraph needs to be completely revised. The first sentence of this paragraph should read: "On the other hand, if $B\neq 0$, and η is found to satisfy an equation of degree three but not one of degree two, we must conclude that $f(x)$ is not irreducible." The remainder of the paragraph not only contains factual errors, but, since $B\neq 0$, is now irrelevant. The following example illustrates the point in question. Let $f(x)=x^4-9x^2+x+3$ and $\eta=1+\rho^2$. Then $B\neq 0$, η satisfies the irreducible equation x^3-12x^2+27x $-17=0$, and $f(x)$ is reducible being identical with $(x+3)(x^3-3x^2+1)$.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

A CORRECTION FOR THE TRAPEZOIDAL RULE

J. J. HART, Redstone Arsenal, Huntsville, Alabama

One approach to the problem of numerical integration is to approximate the integrand with an interpolation formula which is then integrated. We use this procedure to supply a correction to the well-known trapezoidal rule.

Consider

$$\int_{x_0}^{x_1} f(x)dx.$$

We require that $f(x)$ and its first three derivatives be continuous in the interval $x_0 \leqq x \leqq x_1$. We use a linear interpolation formula with correction to approximate $f(x)$.* Let

$$f(x) = I + C + R$$

where

$$I = \frac{(x_1 - x)f(x_0) + (x - x_0)f(x_1)}{(x_1 - x_0)}, \tag{1}$$

the linear interpolation formula, and where

$$C = \frac{[f'(x_0) - f'(x_1)]}{2(x_1 - x_0)}(x_1 - x)(x - x_0), \tag{2}$$

the correction.

Now,

$$\int_{x_0}^{x_1} f(x)dx = \int_{x_0}^{x_1} Idx + \int_{x_0}^{x_1} Cdx + E_1 \tag{3}$$

where

$$E_1 = \int_{x_0}^{x_1} Rdx. \tag{4}$$

When the integration in (3) is carried out, we get the formula

* A new interpolation formula, P. M. Hummel and C. L. Seebeck, Jr., This MONTHLY, vol. 58, No. 6, 1951, pp. 383–389.

$$\int_{x_0}^{x_1} f(x)dx = \frac{1}{2}[f(x_0) + f(x_1)](x_1 - x_0)$$

(5)
$$+ \frac{1}{12}[f'(x_0) - f'(x_1)](x_1 - x_0)^2 + E_1.$$

The first term is the area of a trapezoid and the second term is a correction for it.

To determine E_1, which is the error due to using the first two terms of (5) as a formula, we have

(6)
$$E_1 = \int_{x_0}^{x_1} R dx = \int_{x_0}^{x_1} \frac{(x_1 - x)(x - x_0)}{12(x_1 - x_0)}[f'''(\theta_2)(x_1 - x)^2 - f'''(\theta_1)(x - x_0)^2]dx$$

where $x_0 \leqq \theta_1 \leqq x$ and $x \leqq \theta_2 \leqq x_1$. We write (6) in the form

(7)
$$E_1 = \frac{1}{12(x_1 - x_0)}\left[\int_{x_0}^{x_1} (x_1 - x)^3(x - x_0)f'''(\theta_2)dx - \int_{x_0}^{x_1} (x_1 - x)(x - x_0)^3 f'''(\theta_1)dx\right].$$

It is clear that $(x_1-x)^3(x-x_0)$ and $(x_1-x)(x-x_0)^3$ do not change sign in the interval $x_0 \leqq x \leqq x_1$. We use the first mean value theorem for integrals and get

$$E_1 \leqq \frac{1}{12(x_1 - x_0)}\left[f'''(\xi_2)\int_{x_0}^{x_1} (x_1 - x)^3(x - x_0)dx - f'''(\xi_1)\int_{x_0}^{x_1} (x_1 - x)(x - x_0)^3 dx\right]$$

or

(8)
$$E_1 \leqq \frac{(x_1 - x_0)^4}{240}[f'''(\xi_2) - f'''(\xi_1)]$$

with ξ_1 and ξ_2 between x_0 and x_1. If we now let

$$f(x_0) = y_0, \qquad f'(x_0) = y_0', \cdots,$$

we may write the formula in the more compact form

(9)
$$\int_{x_0}^{x_1} f(x)dx = \frac{1}{2}[y_0 + y_1](x_1 - x_0) + \frac{1}{12}[y_0' - y_1'](x_1 - x_0)^2 + E_1$$

where E_1 is given by (8).

Now, suppose we consider

$$\int_{x_0}^{x_n} f(x)dx \tag{10}$$

where the interval (x_0, x_n) is subdivided by the points

$$x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n.$$

We may integrate over each subinterval (x_{i-1}, x_i), $i=1, 2, \cdots, n$ using (9) on the subinterval (x_0, x_1) and similar formulas for each succeeding subinterval. If we add the results of the n integrations, we get

$$\begin{aligned}\int_{x_0}^{x_n} f(x)dx = {} & \frac{1}{2}\left[(y_0 + y_1)(x_1 - x_0) + (y_1 + y_2)(x_2 - x_1) + \cdots\right. \\ & \left. + (y_{n-1} + y_n)(x_n - x_{n-1})\right] \\ & + \frac{1}{12}\left[(y_0' - y_1')(x_1 - x_0)^2 + (y_1' - y_2')(x_2 - x_1)^2 + \cdots\right. \\ & \left. + (y_{n-1}' - y_n')(x_n - x_{n-1})^2\right] + E\end{aligned} \tag{11}$$

where

$$E = \sum_{i=1}^{n} E_i.$$

This formula may be simplified by subdividing the interval (x_0, x_n) into equal subintervals. Let

$$x_i - x_{i-1} = h, \qquad i = 1, 2, \cdots, n$$

and (11) becomes

$$\begin{aligned}\int_{x_0}^{x_n} f(x)dx = {} & \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right] \\ & + \frac{h^2}{12}\left[y_0' - y_n'\right] + E\end{aligned} \tag{12}$$

where now

$$E \leqq \frac{h^4}{240}\left[f'''(\xi_2) - f'''(\xi_1) + f'''(\xi_4) - f'''(\xi_3) + \cdots + f'''(\xi_{2n}) - f'''(\xi_{2n-1})\right] \tag{13}$$

with ξ_1 and ξ_2 between x_0 and x_1, ξ_3 and ξ_4 between x_1 and x_2 $\cdots$. The formula (12) will be recognized as the familiar trapezoidal rule with the second term an added correction. To get a simpler expression for E, we may replace the terms in

(13) in brackets by $n(M-m)$ where M and m are maximum and minimum $|f'''(x)|$ in the interval $x_0 \leqq x \leqq x_n$ respectively. We now have

$$|E| \leqq \frac{nh^4}{240}(M - m). \tag{14}$$

We may use (14) to compute an upper bound of error committed in using the first two terms of (12) as a formula.

We will compare the results of the formula with those given by several other well-known formulas. We evaluate

$$\int_4^{5.2} \log x \, dx$$

which is 1.82784744, correct to the number of places stated. We let $h=0.2$, $n=6$, $f(x)=\log x$ and $f'(x)=1/x$. We have

$$\begin{aligned} y_0 &= 1.38629436 \qquad y_0' = 0.25 \\ y_1 &= 1.43508453 \\ y_2 &= 1.48160454 \\ y_3 &= 1.52605630 \\ y_4 &= 1.56861592 \\ y_5 &= 1.60943791 \\ y_6 &= 1.64865863 \qquad y_6' = 0.192307692. \end{aligned}$$

The results of this and other formulas appear listed below with their differences from the correct value:

	Result	Error
Trapezoidal	1.82765514	19230×10^{-8}
Simpson's Rule	1.82784726	18×10^{-8}
Weddle's Rule	1.82784741	3×10^{-8}
Trapezoidal rule with correction	1.82784745	1×10^{-8}

Another example is of interest. We evaluate

$$\int_0^1 e^{-x^2} dx,$$

an integral which cannot be obtained in closed form. We let $h=0.2$ and $n=5$ and get

$$\int_0^1 e^{-x^2} dx = 0.744613$$

where $|E| \leqq 0.00005$, using (14).

Certain advantages of the formula should be noted:

1. The formula is simple, requiring that the derivatives be evaluated only at the end points if equal subintervals are chosen.

2. There are no restrictions on the number of subintervals which may be used, whereas Simpson's rule requires an even number and Weddle's rule a multiple of six.

3. Increased accuracy.

VOLUMES OF REVOLUTION

J. J. CORLISS, University of Illinois, Chicago, Illinois

Assume the parametric equations of a curve to be

$$x = x(t), \qquad y = y(t) \tag{1}$$

where $x(t)$ and $y(t)$ are single-valued, differentiable functions of t for

$$t_0 \leqq t \leqq t_1 \tag{2}$$

such that

$$x(t_0) = x(t_1), \qquad y(t_0) = y(t_1), \tag{3}$$

where t_1 is the least value of $t > t_0$ for which (3) holds.

Let G be the graph of that portion of (1) which satisfies (2) and (3).

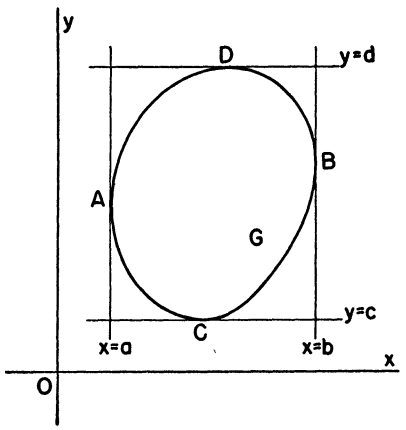


FIG. 1

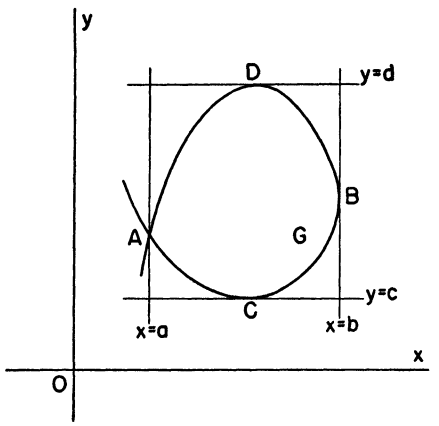


FIG. 2

Graphically, G is a continuous closed curve which does not cut itself. It may be the entire curve (1) (see Figure 1) or a closed portion of (1) (see Figure 2).

Assume that as t increases from t_0 to t_1, the point $P(x, y)$ traces G in the positive direction, where the positive direction is defined as that in which a person walking around the curve has the enclosed area on his left. Let us first

examine the case in which a line parallel to a coordinate axis intersects G in no more than two distinct points.

Let G be within the rectangle determined by the lines $x=a$, $x=b$, $y=c$, $y=d$ (Figures 1, 2), where each of these lines is tangent to G, or passes through a multiple point of (1) but not cutting G in a second distinct point. That is, the lines are through the maximal and minimal values of $x(t)$ and of $y(t)$ in the interval (2).

On arc ACB we may take $y=f_1(x)$ and on arc ADB, $y=f_2(x)$, where $f_1(x)$ and $f_2(x)$ are single-valued, continuous functions of x.

If the x-axis does not cross G, then the volume generated by revolving the area enclosed by G about the x-axis is given by the formula

$$
\begin{aligned}
V &= \pi \int_a^b f_2^2(x)dx - \pi \int_a^b f_1^2(x)dx \\
&= -\pi \int_b^a f_2^2(x)dx - \pi \int_a^b f_1^2(x)dx \qquad (4) \\
&= -\pi \int_G f^2(x)dx.
\end{aligned}
$$

In the last integral, $f(x)=f_1(x)$ if $P(x, y)$ is on arc ACB and $f(x)=f_2(x)$ if $P(x, y)$ is on arc ADB. The letter G at the lower right of the integral sign means that the integral is to be taken around G in the positive direction.

By hypothesis, for points on G

$$
\begin{aligned}
y &= f(x) = y(t) \\
x &= x(t)
\end{aligned}
$$

whence

$$dx = x'(t)dt.$$

We may then write the last integral in terms of t and dt as

$$V = -\pi \int_{t_0}^{t_1} y^2(t)x'(t)dt. \qquad (5)$$

We may also write

$$
\begin{aligned}
V &= 2\pi \int_c^d g_2(y)ydy - 2\pi \int_c^d g_1(y)ydy \\
&= 2\pi \int_c^d g_2(y)ydy + 2\pi \int_d^c g_1(y)ydy \\
&= 2\pi \int_G g(y)ydy
\end{aligned}
$$

or, in terms of t and dt

$$V = 2\pi \int_{t_0}^{t_1} x(t)y(t)y'(t)dt.* \tag{6}$$

Similar formulas can be written for rotation about Oy.

If as t increases from t_0 to t_1, the point $P(x, y)$ traces G in the negative direction, the above integrals give the negative of V.

The restriction that no line parallel to a coordinate axis cuts G in more than two distinct points insures that G can be separated into two arcs each defined by a single-valued function. This restriction can be removed if, by drawing a finite number of lines, the area enclosed by G can be divided into non-overlapping sub-areas whose boundaries satisfy the previous conditions.

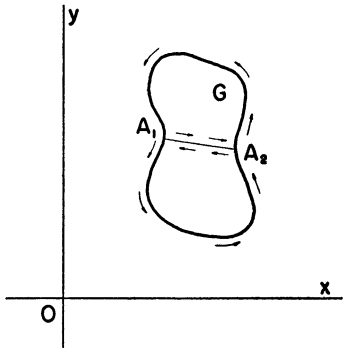


FIG. 3

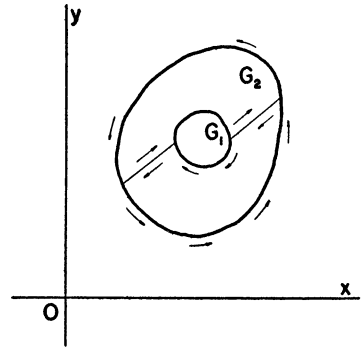


FIG. 4

Figure 3 suggests the method. The integrals taken along A_1A_2 are taken in opposite directions and hence add to zero. The sum of the integrals taken (in the positive direction) around the boundary of each sub-area gives the volume generated by G, and hence the integral taken around G (in a positive direction) gives the volume generated by G.

In Figure 4, the area is bounded by two curves G_1, G_2. By drawing lines as indicated in the figure, the region is divided into sub-areas satisfying the previous conditions. Integrals may be taken around boundaries of sub-areas as indicated.

Calculus texts frequently imply that such volumes must be expressed as the sum of two or more integrals. This paper shows that it is unnecessary to do this, if the boundaries meet the condition given above.

The theory may be applied to regions bounded by suitable arcs of two or more distinct curves. The integrals (5) and (6) in such cases are to be replaced by a sum of integrals, one for each arc, with limits chosen so that P describes the entire boundary in the positive direction. In particular, if the axis of revolution forms a part of the boundary, the limits may be chosen so that P describes

* It may be noted that formula (6) can be derived from (5) by integration by parts.

(in the positive direction) that portion of the boundary which is distinct from the axis of revolution.

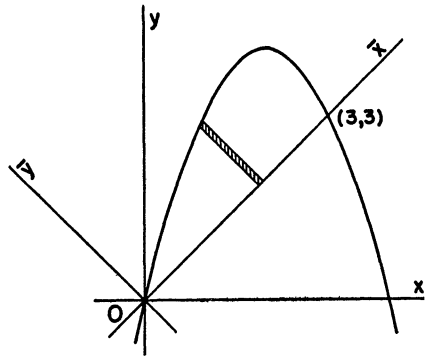


FIG. 5

EXAMPLE: Find the volume generated by rotating about $y=x$ the area bounded by $y=4x-x^2$ and $y=x$.

Referring $y=4x-x^2$ to $y=x$ and $y=-x$ as axes (which we will call $\bar{x}$, $\bar{y}$ axes) we have

$$\bar{x} = \frac{1}{\sqrt{2}}(5x - x^2)$$

$$\bar{y} = \frac{1}{\sqrt{2}}(3x - x^2)$$

as parametric equations of $y=4x-x^2$ in $\bar{x}$, $\bar{y}$ coordinates with x as parameter.

Employing formula (5), and noting that

$$y(t) = \bar{y} = \frac{1}{\sqrt{2}}(3x - x^2)$$

$$x'(t)dt = d\bar{x} = \frac{1}{\sqrt{2}}(5 - 2x)dx$$

we have

$$V = -\pi \int_3^0 \left[\frac{1}{\sqrt{2}}(3x - x^2)\right]^2 \frac{1}{\sqrt{2}}(5 - 2x)dx = \frac{81\sqrt{2}\pi}{20}.$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, N. Y. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 996. *Proposed by Josef Andersson, Vaxholm, Sweden*

If a, b, c and r, r_a, r_b, r_c denote the measures of the sides and of the radii of the inscribed and escribed circles of a triangle, show that

$$abc/r \geqq a^3/r_a + b^3/r_b + c^3/r_c.$$

E 997. *Proposed by David Ellis, University of Florida*

Let x, y, z, n be positive integers such that $n>1$ and $x^n+y^n=z^n$. Show that

$$z^{n-1} < \min(x^n, y^n) < \max(x^n, y^n) < z^n, \tag{1}$$

$$1/z < \min((x/y)^n, (y/x)^n) < \max((x/y)^n, (y/x)^n) < z. \tag{2}$$

E 998. *Proposed by Leo Moser, Texas Technological College*

Show that the largest area A that can be enclosed by two fixed lines, inclined at an angle $\theta \leqq \pi$, and n straight line segments of unit length is given by

$$A = (n/4)\cot(\theta/2n).$$

E 999. *Proposed by L. G. Johnson, General Motors Corporation, Detroit*

Derive general expressions for the coordinates of the center and for the lengths of the semi-axes of the ellipse formed by the plane $Ax+By+Cz+D=0$ cutting the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

E 1000. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB of a triangle ABC, A', B', C' the feet of the corresponding altitudes, and G the centroid of the triangle. Show that the centroid of triangle $A'B'C'$ and the reflections A_2, B_2, C_2 of the centroids of triangles $AB'C'$, $BC'A'$, $CA'B'$ in the lines B_1C_1, C_1A_1, A_1B_1 are concyclic on a circle with center at G, and that triangles ABC and $A_2B_2C_2$ are inversely similar.

SOLUTIONS

A Variation of the Snow Plow Problem

E 963 [1951, 260]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

It had started snowing before noon and three snow plows set out at noon, 1 o'clock, and 2 o'clock, respectively, along the same path. If at some later time they all came together simultaneously, find the time of meeting and also the time it started snowing.

Solution by L. A. Ringenberg, Eastern Illinois State College. Let n denote the number of hours before noon that it started to snow and let x, y, z denote the distances that the first, second, and third snow plow traveled by t hours past noon. Assume that it snows at a constant rate and that each plow removes snow at the same constant rate. Then, if length units are properly chosen, we have

$$dt/dx = t + n, \qquad x = 0 \quad \text{when} \quad t = 0,$$

with solution

$$t = e^x n - n. \tag{1}$$

Also

$$dt/dy = t - (e^y n - n), \qquad y = 0 \quad \text{when} \quad t = 1,$$

with solution

$$t = e^y(n + 1 - ny) - n. \tag{2}$$

Also

$$dt/dz = t - [e^z(n + 1 - nz) - n], \qquad z = 0 \quad \text{when} \quad t = 2,$$

with solution

$$t = e^z(n + 2 - nz - z + nz^2/2) - n. \tag{3}$$

Let d denote the common value of x, y, and z when the plows meet and let T denote the value of t when the meeting occurs. It follows from (1), (2), and (3) that

$$(T + n)/e^d = n = n + 1 - nd = n + 2 - nd - d + nd^2/2.$$

Solving we get $n = 1/2$ and $T = 3.195$. Therefore it began to snow at 11:30 and the plows met at about 3:12.

Also solved by Peter Bender, J. H. Braun, W. D. Serbyn, Peter Treuenfels, R. E. Wild, and the proposer.

Star-Polygons

E 967 [1951, 338]. *Proposed by L. A. Ringenberg, Eastern Illinois State College.*

Let $P_1, P_2, \cdots, P_n (n>4)$ be n points equally spaced and arranged in order of subscripts on a circle of radius 1, and let S_n be the area of the star formed by the line segments $P_1P_3, P_2P_4, \cdots, P_{n-1}P_1, P_nP_2$. (a) Find a formula for S_n. (b) Find the smallest positive integer n such that $nS_n > (n-1)\pi$. (c) Show that if k is any positive integer greater than 1, then $n^kS_n < (n^k-1)\pi$ for $n=5, 6, 7, \cdots$.

Solution by J. H. Braun, Illinois Institute of Technology. (a) The area of the regular polygon of n sides and radius 1 is $(n/2) \sin (2\pi/n)$. To obtain the area of the star we must subtract the combined area of the isosceles triangles (one in each sector of the polygon), each triangle having vertex angle $(n-2)\pi/n$ and base $2 \sin (\pi/n)$. The area of each isosceles triangle is $\sin^2(\pi/n) \tan (\pi/n)$, whence, by simple trigonometry, we find

$$S_n = n \tan (\pi/n) \cos (2\pi/n). \tag{1}$$

(b) Expanding the expression for S_n we find

$$S_n = \pi - 5\pi^3/3n^2 + \epsilon, \qquad 2\pi^5/15n^4 > \epsilon > 0, \qquad n > 4. \tag{2}$$

Let N be the required minimal integer. Then

$$\pi - 5\pi^3/3N^2 + \epsilon > \pi - \pi/N,$$

or

$$5\pi^2/3N - \epsilon' < 1, \qquad \epsilon' = N\epsilon/\pi.$$

Disregarding ϵ' for the moment we find $N=17$. From (1) we find

$$S_{16} \doteq 2.9296 \not> (1 - 1/16)\pi \doteq 2.9452,$$

whence ϵ' is not sufficiently large to invalidate the solution $N=17$.

(c) From (2) it follows that

$$S_n < \pi[1 - 5\pi^2/3n^2].$$

But

$$\pi[1 - 5\pi^2/3n^2] < \pi(1 - 1/n^2) \leqq \pi(1 - 1/n^k), \qquad k \geqq 2, n > 4.$$

Therefore

$$n^kS_n < (n^k - 1)\pi, \qquad k \geqq 2, n > 4.$$

Also solved by George Millman, Margaret Olmsted, J. A. Tierney, and the proposer.

A Definite Integral

E 968 [1951, 338]. *Proposed by R. M. Redheffer, University of California.*

If $f(s)=\int_{-\pi}^{\pi} e^{\cos x}\cos(\sin x)\cos sx\,dx$, find $\sum_1^\infty f(n)$.

Solution by M. R. Spiegel, Rensselaer Polytechnic Institute. Setting $z=e^{ix}$ in the Maclaurin expansion for e^z we obtain

$$e^{\cos x+i\sin x}=\sum_{n=0}^{\infty}(\cos nx+i\sin nx)/n!.$$

Since the series is uniformly convergent for all x we may multiply both sides by $\cos sx$ and integrate from $-\pi$ to π, obtaining

$$\int_{-\pi}^{\pi} e^{\cos x+i\sin x}\cos sx\,dx=\pi/s!, \qquad s=1,2,\cdots.$$

Taking the real part of the integral we find $f(s)=\pi/s!$, whence

$$\sum_{n=1}^{\infty} f(n)=\pi(e-1).$$

Also solved by F. Bagemihl and W. Seidel, W. Fulks, G. R. Johnston, P. G. Kirmser, M. S. Klamkin, A. E. Livingston, Gordon Raisbeck, Edgar Reich, L. A. Ringenberg, W. D. Serbyn, O. E. Stanaitis, R. E. Wild, and the proposer.

Klamkin called attention to Bierans de Haan Table No. 277.

On the Infinitude of Twin Primes

E 969 [1951, 338]. *Proposed by Solomon Golomb, The Johns Hopkins University.*

Show that a necessary and sufficient condition that there be infinitely many twin primes is that there be infinitely many numbers n not of the form $6ab\pm a\pm b$.

Solution by Azriel Rosenfeld, Columbia University. If p and $p+2$ are primes, then, with the single exception of $p=3$, we must have $p=6n-1$ and $p+2=6n+1$. On the other hand, if a number of either of the forms $6n-1$ and $6n+1$ is composite, then each factor is of one or other of these forms, and we have

$$6n\pm 1=(6a\pm 1)(6b\pm 1)=6(6ab\pm a\pm b)\pm 1.$$

Thus $n=6ab\pm a\pm b$ if and only if either $6n-1$ or $6n+1$ is composite. The desired result now follows.

Also solved by the proposer.

A Parabola Inscribed in a Triangle

E 970 [1951, 338]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that in triangle ABC side BC bisects the segments intercepted by the

other two sides on the tangents to the inscribed parabola having for focus the point A_1 where the symmedian AA_1 intersects the circumcircle.

Solution by O. J. Ramler, Catholic University, Washington, D. C. Since the focus lies on a symmedian, the second focus must, by a fundamental property of foci of conics inscribed in a triangle, lie on the corresponding median. Since the conic is a parabola, the second focus lies at infinity in the direction of the median, and the median is therefore a diameter of the parabola. Let the sides AB and AC of triangle ABC circumscribing the parabola touch the curve in C' and B' respectively. Then the diameter of the parabola bisects $B'C'$, and $B'C'$ is parallel to the tangent to the curve at the point O where the diameter cuts the curve. The tangent BC to the parabola at O is therefore not only parallel to $B'C'$ but is bisected at O. The ideal line in the plane of ABC is tangent to the parabola. By an elementary theorem in the projective theory of conics any tangent cuts four fixed tangents of a conic in a range of points of constant cross ratio. Let any tangent of the parabola inscribed in the triangle ABC meet BC, CA, AB in L, M, N, respectively. Then the cross ratio $(NMLI)$, where I is the ideal point on the tangent, is constant. In particular, when NM assumes the position of BC, M will be at C, N at B, and L at O. But O is the midpoint of BC, whence the constant cross ratio $(NMLI)$ is -1, and L is the midpoint of NM.

Also solved by Joseph Langr.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4468. *Proposed by H. S. Shapiro, Massachusetts Institute of Technology*

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be convergent for $|z| < 1$, and further let $0 < |f(z)| \leqq 1$ for $|z| < 1$. Prove $|f'(0)| \leqq 2/e$.

4469. *Proposed by A. J. Coleman, University of Toronto*

Evaluate the integral

$$\int_0^\infty \frac{e^{-a^2x^2-b^2/x}}{x^2(1+x^2)}\,dx.$$

The dominant term when a is very much smaller than b is also desired. (The integral was met in a discussion of the diffusion of gamma rays.)

4470. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle ABC, three lines a', b', c' drawn through the vertices A, B, C determine by their intersections a triangle $A'B'C'$, and their isogonals a'', b'', c'' determine a triangle $A''B''C''$. (1) Show that the orthic triangles of $A'B'C'$ and $A''B''C''$ have equal perimeters. (2) If, further, a', b', c' are equally inclined to BC, CA, AB, show that the circles $A'B'C'$ and $A''B''C''$ are symmetric with respect to the line joining the symmedian point to the circumcenter.

4471. *Proposed by Ky Fan, University of Notre Dame*

Let $K(x, y)$ be a non-negative Lebesgue integrable function over the square $a \leqq x \leqq b$, $a \leqq y \leqq b$. Suppose that B is a positive constant such that $\int_a^b K(x, y)dy \leqq B$ for almost all x in $[a, b]$, and also $\int_a^b K(x, y)dx \leqq B$ for almost all y in $[a, b]$. If two finite-valued functions $f(x)$, $g(x)$ are both non-negative and non-increasing in $[a, b]$, prove that

$$\int_a^b \int_a^b K(x, y)f(x)g(y)dxdy \leqq B\int_a^b f(x)g(x)dx. \tag{1}$$

[For the particular case when $K(x, y)$ is constant, (1) is known as Tchebychef's inequality. See Hardy, Littlewood, and Pólya, *Inequalities*, Cambridge, 1934, p. 168, Theorem 236.]

4472. *Proposed by G. T. Williams, Elmont, New York*

If n is an odd integer >1, the curve $x^n+y^n=1$ has the line $x+y=0$ as asymptote, and the area in the second (or fourth) quadrant between the curve and its asymptote exists and is equal to the area in the first quadrant divided by $2\cos(\pi/n)$.

SOLUTIONS

Linear Combination of Vectors with Non-negative Coefficients

4395 [1950, 342]. *Proposed by David Gale, Princeton University*

Let S be a set of vectors (points) in n-space with the property that every vector in n-space can be expressed as a linear combination of vectors from S with non-negative coefficients. Show that one can find a subset R of S containing not more than $2n$ elements such that R has this same property.

Solution by the Proposer. Since S generates all of n-space it must contain some set of n linearly independent vectors, $v_1, \cdots, v_n$. We first observe that if we let $v_0 = -\sum_{i=1}^n v_i$, then every vector can be expressed as a non-negative

linear combination of the vectors $v_0, v_1, \cdots, v_n$. For let v be any vector in n-space. Then

$$v = \sum_{i=1}^{n} \alpha_i v_i.$$

If all α_i are non-negative there is nothing more to prove. If not, suppose α_1 is the smallest of the α_i. Then

$$v = \sum_{i=1}^{n} (\alpha_i - \alpha_1)v_i + (-\alpha_1)v_0$$

where again all coefficients are non-negative. To prove the desired result then we need only show that v_0 can be expressed as a non-negative linear combination of not more than n vectors from S. Now by assumption

$$v_0 = \sum_{i=1}^{k} \lambda_i u_i, \qquad \lambda_i \geqq 0,\ u_i \in S. \tag{1}$$

If $k \leqq n$ we are finished. If not, we can find n vectors from among the u_i, say the first n, such that

$$v_0 = \sum_{i=1}^{n} \beta_i u_i \tag{2}$$

and again we only worry about the case where some β_i is negative. Then let $\theta = \max(-\beta_i/\lambda_i)$. Then θ is positive and suppose it is $-\beta_1/\lambda_1$. Then from (1) and (2) we get

$$(1 + \theta)v_0 = \sum_{i=0}^{k} \gamma_i v_i, \qquad \gamma_i = \theta\lambda_i + \beta_i \geqq 0.$$

But $\gamma_1 = 0$, so that after dividing by $(1+\theta)$ we have expressed v_0 non-negatively in terms of $k-1$ vectors from S. By repetition of this argument, the result follows.

Also solved by L. M. Blumenthal and J. W. Gaddum (jointly), V. L. Klee, Jr., A. E. Taylor, and Albert Wilansky.

Editorial Note. Klee and Taylor point out the equivalence of the present result with the following known theorem: *Any point interior to the convex hull of a set E in an n-dimensional euclidean space is interior to the convex hull of some subset of E containing at most $2n$ points.* See Bull. Amer. Math. Soc., 53 (1947), 299–301.

The Lemoine Point of a Conic for an Inscribed Triangle

4398 [1950, 343]. *Proposed by Lucien Droussent, Laon, France*

The Lemoine point of a rectangular hyperbola for an inscribed triangle T

lies on the orthic axis of T, and the polar of this point with respect to the hyperbola is tangent to the conic which touches the sides of T at the feet of its altitudes and has the symmedian point of T for center. (The tangents of a conic at the vertices of an inscribed triangle T form a triangle which is perspective to T. The center of perspectivity is called the *Lemoine point* of the conic for the triangle T).

Solution by E. D. Camier, Berkenhead, Cheshire, England. If the triangle T, with vertices A, B, C, is taken as reference triangle in a system of trilinear coördinates, the equilateral hyperbola has the equation $\sum lyz=0$ where $\sum l \cos A=0$. The Lemoine point for T is then (l, m, n), so that the previous relation means that (l, m, n) lies on the orthic axis whose equation is $\sum x \cos A=0$.

The tangential equation of the conic inscribed in T and having (p, q, r) as center is $\sum a(-ap+bq+cr)\mu\nu=0$. Hence when the symmedian point (a, b, c) is the center, the equation is

$$\sum \mu\nu \cos A = 0. \tag{1}$$

The polar of (l, m, n) with respect to $\sum lyz=0$ is $\sum x/l=0$ and the condition that this line should touch the conic (1) is $\sum l \cos A=0$, which is satisfied by hypothesis. Finally, the conic (1) touches BC at $(0, \cos C, \cos B)$, which is the foot of the perpendicular from A to BC.

This result can be generalized. If the Lemoine point of a conic Γ for an inscribed triangle T lies on a fixed straight line L, the polar of this point with respect to Γ touches a second conic Γ', where Γ' is inscribed in T and has its center at the pole of L with respect to the conic which touches T at the midpoints of its sides.

Also solved by the Proposer.

Sequences of Greatest Integer Functions

4399 [1950, 343]. *Proposed by Ky Fan, University of Notre Dame*

Let $f(n)$ and $g(n)$ be two sequences of natural numbers defined by the following three conditions:

(1) $f(1)=1$.

(2) $g(n)=na-1-f(n)$, a being an integer >4.

(3) $f(n+1)$ is the least natural number distinct from the $2n$ numbers $f(1)$, $f(2), \cdots, f(n); g(1), g(2), \cdots, g(n)$.

Prove that there exist constants α and β such that

$$f(n) = [\alpha n], \qquad g(n) = [\beta n].$$

The brackets denote, as usual, the greatest integer function.

Solution by H. S. Zuckerman, University of Washington, Seattle. Let α, β be the roots of $x^2-ax+a=0$. Since $a>4$, α, β are real and we take $\alpha<\beta$. We then have $\alpha+\beta=a$, $\alpha\beta=a$, $1/\alpha+1/\beta=1$, $1<\alpha\leqq 2$, $2\leqq\beta$. Further α, β are irrational

since if one were rational both would be integers and we would have $\alpha=2, \beta=2$, $a=4$. We have immediately

$$[\alpha 1] = 1. \tag{1}$$

Furthermore, for $n \geqq 1$, we have

$$[\beta n] = [(a - \alpha)n] = na - 1 - [\alpha n]. \tag{2}$$

Now if $[\alpha n]=[\beta m]=k$ with n, m positive integers, then

$$\alpha n = k + \theta, \qquad \beta m = k + \phi, \qquad 0 < \theta < 1, 0 < \phi < 1,$$

and hence

$$n + m = k\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + \frac{\theta}{\alpha} + \frac{\phi}{\beta} = k + \frac{\theta}{\alpha} + \frac{\phi}{\beta}$$

which is impossible since $0<\theta/\alpha+\phi/\beta<1$. We therefore have

$$[\alpha n] \neq [\beta m] \qquad \text{for any integers } n, m > 0. \tag{3.1}$$

We have also

$$[\alpha(n + 1)] \geqq [\alpha n] + 1, \qquad [\beta(n + 1)] \geqq [\beta n] + 2 > [\alpha n] + 1. \tag{3.2}$$

Finally let $k>0$ be any integer and let $n=[(k+1)/\alpha]$. If $n>k/\alpha$ then $k<\alpha n$ $<\alpha(k+1)/\alpha=k+1$ and $[\alpha n]=k$. If $n<k/\alpha$ then

$$\beta(k - n) > \beta k - \frac{\beta}{\alpha} k = \beta k\left(1 - \frac{1}{\alpha}\right) = k,$$

$$\beta(k - n) < \beta k - \beta\left(\frac{k + 1}{\alpha} - 1\right) = k + 1,$$

from which we have $[\beta(k-n)]=k$. This with (3.1) and (3.2) shows that

(3.3) each $k = 1, 2, 3, \cdots$ appears once and only once in one or other of the two sequences $[\alpha n]$, $[\beta n]$.

Now (3.3) with (3.2) proves that

(3) $[\alpha(n + 1)]$ = the least natural number distinct from $[\alpha 1], [\alpha 2], \cdots, [\alpha n]; [\beta 1], [\beta 2], \cdots, [\beta n]$.

Formulas (1), (2) and (3) are just the definition of $f(n)$, $g(n)$ and, since the sequences so defined are unique, we have $f(n) = [\alpha n]$, $g(n) = [\beta n]$.

We remark that the only part of the proof which requires a to be an integer is (2). Therefore all the other results are valid for any irrational α, β with $1/\alpha$ $+1/\beta=1$. This result is problem 9, p. 98 in *Elementary Number Theory*, Uspensky and Heaslet.

Also solved by the Proposer.

RECENT PUBLICATIONS

EIDTED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, **80** *Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.*

Theory of Probability. By M. E. Munroe, McGraw-Hill Book Company, Inc., New York. 1951. 8+213 pages. $4.50.

The problem of presenting probability to mathematically immature undergraduates is a difficult one because measure theory, which is essential to a unified treatment, has not yet filtered down to the undergraduate curriculum. One way out of the difficulty is to confine attention to the discrete case in order to drive home fundamental ideas and modern methods in a simple context. (Examples are the recent texts by J. Neyman and by William Feller.) The more usual solution is to treat the discrete and continuous cases within the limits of the students' knowledge of calculus. Professor Munroe's book brings some welcome innovations to this approach. Probability is defined axiomatically. Discrete and continuous variables are treated side by side in order to bring out the analogy and even to suggest explicitly the general unified theory. Almost a third of the book is devoted to limit theorems in order to give the student "the true flavor of modern probability theory." Topics involving mathematics beyond elementary calculus are treated intuitively with frank acknowledgment of lack of rigor. The book is suitable for a one semester course whose purpose is to impart a speaking acquaintance with significant notions, techniques, and results of modern probability theory.

Space does not permit mentioning minor good and bad features, such as the very clear discussion of the chi-square distribution and the regrettable listing of "suggested procedures" as numbered theorems, but attention should be called to the weakness of the discussion of the meaning of probability preceding the axiomatic definition. The notion of "likelihood" is used without any indication of its meaning except that it appears to be an entity of which probability is the measure. Logical certainty is confused with unit probability on page 10, only to be correctly distinguished from it nine pages later. The notion of frequency is introduced, but probability is identified with "expected" frequency. It would be better to state explicitly the contending views on the foundations of probability instead of giving an eclectic presentation which leaves the student unaware of the existence of any consistent approach and unprepared to cope with the conceptual difficulties that arise in statistics.

KENNETH MAY
Carleton College

Ordinary Differential Equations. By Michael Golomb and Merrill Shanks. New York, McGraw-Hill, 1950. ix+356 pp. $3.75.

Differential Equations. Revised Third Edition. By H. B. Phillips. New York,

John Wiley and Sons, Inc., 1951. viii+149 pp. $3.00.

Differential Equations. By J. E. Powell and C. P. Wells. Boston, Ginn and Company, 1950. vi+205 pp. $3.00.

These three textbooks are designed for use in undergraduate courses on ordinary differential equations as offered at the high sophomore or low junior level of almost every institution that has an engineering or science major for a bachelors degree. The present brief review is not intended as a comparison of the books but rather as an indication of the features of each.

The books by Phillips and Powell-Wells are somewhat brief and would be fully used in a five quarter-hour or three semester-hour course. Either of these would seem suitable for a course that requires no more prerequisite than the usual first course in differential and integral caculus. The book by Golomb and Shanks contains sufficient material for more than a one semester course but selected omissions readily adapt it to such courses. This latter book includes a summary of the background material, above the elementary calculus level, needed for the text. The reviewer is of the opinion that best results would be obtained from use of this text in courses that are preceded by some advanced calculus.

Golomb and Shanks have written a book that bridges a serious gap in the textbook literature on differential equations. An institution that can include a course at the level of this book will prepare its students for further work with differential equations, or applications of such equations, in a much more substantial way than can be achieved through the usual, problem-solving courses now given in many colleges. The entire book is written with care for mathematical accuracy in statements of theory back of each operation presented. Much thought and ingenuity have been successfully used in bringing correct definitions, concepts, existence theorems, and procedures within reach of undergraduate students of reasonable ability. There are many sections of the book that merit commendation but space limitations on this review do not permit their enumeration. Among these are: the distinction between "a" and "the" when used in connection with "general solution" (page 23); the discussion of $y'=2\sqrt{y}$ on page 29; the caution on page 53 about use of the vague formula (showing integral signs without limits and using x ambiguously) frequently given for "solving" a first-order, linear equation; and the well-designed approach to Green's functions given on pages 259–270.

The Powell-Wells book is written with primary view to the needs of engineering students. It features a grouping of solution methods under such general headings as: (a) variables separable or reducible to variables separable; (b) exact equations or those reducible to exact equations by means of integrating factors; (c) solution by infinite series. Through such groupings the authors hope to correct an impression on the part of students that "a course in differential equations is nothing but a study of numerous and rather disconnected methods of solving very special types of equations." Two full chapters are devoted to applications of differential equations. Problems are of *practice*, *general*,

or *miscellaneous* type according as they illustrate the principles just preceding the exercise, require the student to seek a method of solution from the full body of preceding text, or include extensions beyond anything given in the text. The reviewer seriously questions the advisability of having the *practice* problems embody the principle which the authors state as follows: "In these lists we have included problems that cannot be worked by the methods of that section and often not by any method previously discussed. It is hoped that this plan will emphasize the importance of knowing where as well as how to apply a certain method." It is likely that an encounter with a problem of the type described in this quote would cause serious loss of confidence by students of the maturity level expected in a course from the text. The authors cite judgment factors which lead to omission of certain points of strict mathematical rigor. The reviewer is not unsympathetic to this view but feels that some opportunities were missed where, with no loss of teaching quality, more precise formulas and language could have been used. For example, limits might have been used on the integral signs that appear in many of the formulas given.

The book by Phillips is a third edition of a text that has had wide use since its first edition appeared in 1922. It is designed for, and well adapted to, the one-quarter or one-semester course that is almost standard in the undergraduate colleges of the country. The present edition contains new problems and several rewritten sections that reflect the experience gained from classroom use of earlier editions. The book has achieved a good balance between applied problems and problems of theoretic nature. Care has been exercised in handling points of mathematical rigor as is evidenced by the inclusion of section 10, page 18, where *integration between limits* is carefully noted.

The typography of the three books under review is uniformly good. The publisher of each is to be congratulated upon the careful work of his staff in the editing and composition of the book.

W. M. WHYBURN
University of North Carolina

NEW BOOKS RECEIVED

Vorlesungen über Differential und Integralrechnung, Zweiter Band, *Differentialrechnung auf dem Gebiete mehrerer Variablen*. By A. Ostrowski. Basel, Verlag Birkhäuser, 482 pp. Swiss fr. 67.

An Introduction to the Theory of Control in Mechanical Engineering. By R. H. MacMillan. Cambridge University Press, 1951. xiv+195 pp. $6.00.

Elementary Problems in Engineering. By H. W. Leach and G. C. Beakley. New York, the Macmillan Co., 1951. 14+269 pp. $3.50.

Mathematische Grundlagen der höheren Geodäsie und Kartographie. By R. König and K. H. Weise. Berlin, Göttingen, Heidelberg; Springer-Verlag; 1951. 18+522 pp. 49.60 German Marks.

Linear Computations. By P. S. Dwyer. New York, John Wiley and Sons, 1951. xi+344 pp. $6.50.

Index Numbers. By B. D. Mudgett. New York, John Wiley and Sons, 1951. x+135 pp. $3.00.

A Concise History of Astronomy. By Peter Doig. New York, Philosophical Library, 1949. xi+320 pp. $4.75.

Everyday Algebra, Intermediate Course. By William Betz and A. P. Windt. Boston, Ginn and Company, 1951. 566 pp. $2.48.

Practical Mathematics, Part IV, Trigonometry and Logarithms. Fifth Edition. By Claude Irwin and S. F. Bibb. New York, McGraw-Hill Book Co., 1951. xii+193 pp. $2.60.

Mathematics Essential for Elementary Statistics. Revised Edition. By Helen M. Walker. New York, Henry Holt and Co., 1951. xiii+382 pp.

Gewöhnliche Differentialgleichungen Sammlung Göschen. By Dr. Guido Hoheisel. Berlin, 1951. 129 pp. +13.

The Mathematics of Finance. By F. C. Smith, New York, Appleton-Century-Crofts, Inc. xii+356 pp.

The Teaching of Mathematics. By D. R. Davis. Cambridge, Addison-Wesley Press, Inc., 1951. xv+415. pp. $4.50.

Integral Transforms in Mathematical Physics (Methuen's Monographs). By C. J. Tranter. New York, John Wiley and Sons, Inc. ix+118 pp. $1.50.

Analytic Geometry and Calculus. By W. R. Longley, P. F. Smith and W. A. Wallace. Boston, Ginn and Company, 1951. ix+578 pp+xx. $5.00.

Machine Shop Mathematics. Second Edition. By Aaron Axelrod. New York, McGraw-Hill. xi+359 pp. $3.60.

Nomographic Charts. By C. A. Kulmann. New York, McGraw-Hill Book Co. xi+244 pp. $6.50.

The Physical Sciences, 3rd ed. By E. J. Cable, R. W. Getchell, and W. H. Kadesch. New York, Prentice-Hall, 1951. xvii+496 pp. $5.50.

Applied Mechanics for Engineers. By Sir Charles Inglis. Cambridge, University Press, 1951. xii+404 pp. $7.50.

College Algebra. Second Edition. By R. W. Brink. New York, Appleton-Century-Crofts, Inc. xvi+495 pp.

Intermediate Algebra. Second Edition. By R. W. Brink. New York, Appleton-Century-Crofts, Inc., 1951. xi+295 pp.

Statistische Methoden. By A. Linder. Switzerland, Verlag Birkhäuser, Basel, 1951. 238 pp. 30 Fr.

Trigonometry for Today. By Milton Brooks and A. C. Schock. New York, Harper and Brothers, 1951. ix+102 pp. $2.96.

Theory of Perfectly Plastic Solids. By William Prager and P. G. Hodge. New York, John Wiley and Sons, Inc., 1951. x+264 pp. $5.50.

Mathematics of Finance. By A. E. May. American Book Company, 1951. viii+264 pp.

Statistical Methodology Reviews. By O. K. Buros. New York, John Wiley and Sons, 1951. 457 pp. $7.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of special features, student papers, bibliographies of program topics, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB REPORTS, 1950–51

Pi Mu Epsilon, University of Kansas

The *Kansas Alpha* chapter of *Pi Mu Epsilon* held its annual spring banquet in April. The speaker, Prof. E. S. Robinson, gave an address on *Why some philosophers are interested in mathematics.*

The winter initiation and tea was held in January and in addition there were various business meetings during the year.

The officers elected for the year 1951–52 are: Director, Robert Fisher; Vice-Director, David Murcray; Secretary, James Larkin; Treasurer, Kathleen O'Donnell; Corresponding Secretary, Wealthy Babcock.

Mathematics Club, Case Institute of Technology

The *Mathematics Club* at the Case Institute of Technology held six meetings during the year 1950–51, at which the following talks were presented:

The calculus of variations, by J. T. Morse

The concept of order of magnitude in optics, by P. A. Wayman

The mathematical theory of games, by M. D. Morley

The Ritz method in the calculus of variations, by Dr. Robert Clark

Matrices and transformations, by J. T. Morse

The computer at Case for solving simultaneous equations, by C. A. Young.

The Officers were: President, J. T. Morse; Faculty Advisor, Prof. Max Morris.

Mathematics Section, Chamberlin Science Club, Beloit College

The program of the *Mathematics Section* of the *Chamberlin Science Club* of Beloit College for 1950–51 was:

A method for solving certain quartic equations, by Donald Walsh

Duplication of the cube, by Marilyn Marshall

Hyperbolic functions, by Margery Jenkins

Early arithmetic, by Richard Bening

Some interesting problems from Mathematical Snapshots, by Priscilla Kong

Find the digit represented by each letter, by Arthur Ahlgrim

The life of Isaac Newton, by Phyllis Willoughby

The beginnings of numbers, by Fern Lohrber.

The officers for the year 1951–52 are: President, Richard Bening; Vice-President and Program Chairman, Fern Lohrber; Secretary, Marilyn Marshall.

Kappa Mu Epsilon, Baldwin-Wallace College

Talks given before the *Ohio Gamma* chapter of *Kappa Mu Epsilon* during the year 1950–51 were:

Engineering education in England, by Prof. Phillip Clyne of the University of Chelmsford, England

The mathematics of democracy, by Prof. John Wilson

Some applications of the laws of chance, by Jack Hafferkamp

Fun with figures, by Kenneth Buser

Professional opportunities in mathematics, by Glenn Carter

The mathematics of life insurance, by Ellsworth Strock, Assistant Actuary of the Prudential Insurance Co. of America.

The officers elected for the year 1951–52 are: President, Ruth Oberer; Vice-President, Kenneth Buser; Secretary, Carol Schirmer; Treasurer, Donald Larsen; Corresponding Secretary and Faculty Advisor, Prof. Paul Annear.

Mathematics Club, Immaculate Heart College

The following papers were presented at regular meetings of the *Mathematics Club* of Immaculate Heart College of Los Angeles during 1950–51:

Careers for women in mathematics, by Romayne MacFarlane and Mary Lou Wagner

Works and life of Einstein, by Mary Parker and Rita Anerson

Inversion, by Lillian Hampton

History of linkages and their use in describing approximate and exact straight line motion, by Glenna Christie

Linkages which use the principle of inversion, by Rita Ricciardi.

At each meeting of the club a period was devoted to mathematical recreations followed by refreshments and a social hour.

An award for excellence in freshman mathematics was won by Lillian Hampton.

Officers for the year 1950–51 were: President, Mary Lou Wagner; Secretary-Treasurer, Romayne MacFarlane.

Pi Mu Epsilon, Oregon State College

The *Oregon Beta* Chapter of *Pi Mu Epsilon* reports that for the academic year 1950–51 six meetings were held including the annual initiation and banquet. The following papers were presented:

Linear Diophantine equations, by Philip Anselone

Arithmetic of the complex domain, by Robert Brown

Theory of runs, by Gene Thompson

Different sizes of infinities, by Dr. James Price.

An annual Mathematics Contest was inaugurated. Prize winners were: William Gribble, first prize; Richard Lee Adey, second prize; Marshall McMurran, third prize.

Following the untimely death of the second prize winner, the chapter decided to perpetuate his memory by designating the first prize as the *Richard Lee Adey Award.*

The initiation of new members and the annual banquet of the chapter were held in May. Forty-one new members were initiated.

Officers elected for 1951–52 are: Director, Patricia Pearson; Vice-Director, Dallas Banks; Secretary, Robert Brown; Treasurer, Prof. G. A. Williams.

Newtonian Society, Lehigh University

The *Newtonian Society* of Lehigh University is open to Sophomores and Juniors who have demonstrated outstanding ability in mathematics during their freshman year. During the academic year 1950–51, the society held seven meetings. The following programs were presented:

The International Congress of Mathematicians, by Prof. J. O. Chellevold

Magic squares, by Bruce Reinhart

Actuarial mathematics, by Prof. Frank Beale

The game of Nim, by Benjamin Kenny

Going to extremes—Greek style, by Prof. A. A. Wilansky

Elliptic integrals, by Hubert Snyder

A simple branching process, by Dr. S. Goldberg.

Officers elected are: President, Robert Moore; Vice-President, Fred Otter Jr.; Secretary, Donald Clapp; Faculty Advisor, Prof. J. O. Chellevold.

Pi Mu Epsilon, Northwestern University

The program of the *Northwestern University* chapter of *Pi Mu Epsilon* for the year 1950–51 was as follows:

Theorems of the universe, by Dr. W. S. Krogdahl of the Department of Astronomy

Interesting facts about numbers, by Dr. R. P. Boas

Philosophy and mathematics, by Dr. Douglas Morgan of the Department of Philosophy

Science and religion, by Dr. Edson Peck of the Department of Physics

Analogue computer, by Dr. Elliott Buell, as part of a tour through the Technological Institute.

First and second prizes in the annual mathematics contest were awarded to Franklin Peterson and Donald Malm, respectively.

The officers elected for 1951–52 are: President, Richard Goldberg; Vice-President, Franklin Peterson; Treasurer, David Whitehouse; Secretary, Grant Steffen.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

JOURNAL OF THE GRADUATE INSTITUTE FOR APPLIED MATHEMATICS

The Graduate Institute for Applied Mathematics announces the forthcoming publication of *The Journal of Rational Mechanics and Analysis*, edited by T. Y. Thomas and C. A. Truesdell, with the assistance of David Gilbarg, Vaclav Hlavaty, and Eberhard Hopf. The cooperation of twenty-five European and American specialists in mechanics and analysis has been secured. About 600 pages per year will be published, the first number to appear in January, 1952. The printing is done by the Waverly Press. *The Journal of Rational Mechanics and Analysis* nourishes mathematics with physical applications, aiming especially to close up the rift between "pure" and "applied" mathematics and to foster the discipline of mechanics as a deductive, mathematical science in the classical tradition. Its scope comprises those parts of pure mathematics or other theoretical sciences which contribute to mechanics; among the included fields are all branches of analysis, differential geometry, analytical dynamics, elasticity, fluid dynamics, plasticity, thermodynamics, relativity, and statistical mechanics. Engineering applications, numerical work, perturbation procedures, etc., are acceptable only as incidental illustration in a paper devoted to sound mathematical theory. Empirical and semi-empirical conjectures, book reviews, notices, etc., are excluded. Each paper must meet a standard of deductive rigor set by the best work in its field. Contributed papers must contain original research. The editors may occasionally invite an expository paper in a field where they perceive a genuine need.

English, French, German, and Italian are the languages of the Journal. A high expository level is desired, and papers written in an excessively condensed style will not be printed.

It is preferred that an author who desires to publish in the Journal send his manuscript to a member of the Editorial Board, although it is also permissible to send it to the Editors. Names and addresses of the members of the Editorial Board may be found in the Journal.

PERSONAL ITEMS

Mrs. Jewell H. Bushey, chairman of the Department of Mathematics of Hunter College, Dr. Mina S. Rees who is Director of Mathematical Science, Office of Naval Research, and Assistant Professor Edith R. Schneckenburger of the University of Buffalo were appointed to serve as delegates of the Association at a national conference on Women in the Defense Decade which was held in New York City on September 27–28, 1951.

Brother Damian Connelly of LaSalle College represented the Association at

the Centennial Program of St. Joseph's College in Philadelphia, Pennsylvania.

Professor J. S. Frame of Michigan State College was the delegate of the Association at the inauguration of President H. H. Hatcher of the University of Michigan on November 27, 1951.

Professor S. A. Jennings, University of British Columbia, represented the Association at the installation of Chancellor Sherwood Lett of the University of British Columbia on October 26, 1951.

Professor C. G. Latimer of Emory University was the representative of the Association at the inauguration of President W. M. Alston of Agnes Scott College on October 22–23, 1951.

Professor Saunders MacLane of the University of Chicago was the representative of the Association at the inauguration of Chancellor L. A. Kimpton of the University of Chicago on October 18, 1951.

Dr. C. V. Newsom, Associate Commissioner of Higher Education of the State of New York, was appointed to represent the Association at the inauguration of President R. R. De Marco of Finch Junior College, New York City, on November 2, 1951.

Dr. M. M. Slotnick of Humble Oil and Refining Company, Houston, Texas, represented the Association at the inauguration of President R. O. Lanier of Texas Southern University on November 4, 1951.

Professor Charles Wexler, Arizona State College, was appointed to represent the Association at the inauguration of President R. A. Harvill of the University of Arizona on November 16, 1951.

Professor C. R. Wylie, Jr., University of Utah, served as the delegate of the Association at the inauguration of President E. L. Wilkinson of Brigham Young University on October 8, 1951.

At the celebration of the Four Hundredth Anniversary of the National University of Mexico the degree of Doctor Honoris Causa was conferred on Professor Garrett Birkhoff of Harvard University, Professor Solomon Lefschetz of Princeton University and Professor Norbert Wiener, Massachusetts Institute of Technology.

Mr. J. B. Herreshoff IV of the University of California at Berkeley was the recipient of the William Lowell Putnam Scholarship at Harvard University for the Eleventh Annual Competition.

Dr. D. W. Snader, professor of mathematical education at the University of Illinois was honored at the Home Coming Celebration of Albright College with an Alumni Citation. Professor Snader was unable to attend the festivities because he was serving on the Institute for Educational Leadership Program in Hiroshima, Japan.

Beloit College makes the following announcements: Dr. H. H. Conwell has retired from his position as Dean of the College and Chairman of the Department of Mathematics but is continuing to teach in the Department; Professor R. C. Huffer has been promoted to the chairmanship.

Catholic University announces the following: Professor A. E. Landry has

retired with the title of Professor Emeritus; Dr. R. W. Moller has been promoted to an associate professorship.

At Florida State University: Dr. H. E. Taylor has been promoted to an assistant professorship; Dr. Nicholas Heerema, formerly graduate student and research associate at the University of Tennessee, has been appointed to an assistant professorship; Mr. H. H. Barnett, previously a graduate student at the University of Minnesota, has been appointed Acting Asssistant Professor; Miss Mary C. Heath, formerly graduate student at the University of Kentucky, has been appointed temporary instructor.

Johns Hopkins University announces: Assistant Professor F. I. Mautner of Pennsylvania State College has been appointed to an assistant professorship; Dr. J. M. Cook, formerly a graduate student, has been appointed to an instructorship.

Memphis State College reports the appointment of Mr. G. N. Wollan as assistant professor and the appointment of Miss Kathleen Cannon as instructor.

State University of Iowa announces the following: Associate Professor E. N. Oberg has been granted a leave of absence for the year 1951–52 to serve with North American Aviation, Incorporated, Downey, California; Assistant Professor Byron Cosby, Jr., has been promoted to an associate professorship.

At the University of Arizona: Dr. K. A. Fowler, who was an assistant at the University of Michigan, has been appointed to an assistant professorship; Mr. Robert Krans, a graduate student at the University, has been appointed to an assistantship.

The University of Florida makes the following announcements: Professor T. M. Simpson, who has been Head Professor of Mathematics, has retired with the title of Professor Emeritus; Professor F. W. Kokomoor, who has been serving as Professor of Mathematics and Chairman of Freshman Mathematics, has been promoted to the position of Head Professor of Mathematics and is continuing in the position of Chairman of Freshman Mathematics; Associate Professor C. B. Smith has been promoted to a professorship.

The University of Georgia announces that it has purchased the library of the American Mathematical Society. The library contains more than thirteen thousand volumes including many mathematical journals.

The University of Southern California reports the following: Associate Professor D. H. Hyers has been promoted to a professorship; Assistant Professors James Dugundji and A. L. Whiteman have been promoted to associate professorships; Dr. Milton Drandell, previously a teaching assistant at the University of California at Los Angeles, has been appointed Lecturer.

Mr. Walter Abramowitz has been appointed Research Fellow at Polytechnic Institute of Brooklyn.

Graduate Assistant R. P. Bacon of Miami University has accepted a position at Douglas Aircraft Company, El Segundo, California.

Mr. W. G. Bade, who has been a teaching assistant at the University of

California at Los Angeles, has been appointed to an instructorship at the University of California at Berkeley.

Associate Professor Frances E. Baker of Vassar College has been promoted to a professorship.

Mr. E. W. Banhagel, previously an instructor at Northwestern University, has accepted a position as mathematician in the Research Laboratories of Bendix Aviation Company, Detroit, Michigan.

Assistant Professor W. R. Baum of the University of Tennessee has been appointed to an assistant professorship at Syracuse University.

Assistant Professor S. Louise Beasley of Lindenwood College for Women has been promoted to an associate professorship.

Assistant Professor O. B. Bennett of Washington College has been promoted to an associate professorship.

Dr. W. D. Berg, previously visiting assistant professor at Kenyon College, has accepted a position as Senior Technical Assistant with Mutual Life Insurance Company, New York City.

Professor Emeritus H. E. Buchanan of Tulane University has been appointed Acting President of the College of the Ozarks.

Mr. Reginald Cobb, who has been a student at the University of Florida, has a position as statistician at the Naval Air Technical Training Center, Jacksonville, Florida.

Associate Professor G. M. Eddington of Tusculum College has been appointed Dean of Men.

Miss Jacqueline P. Evans has been appointed to an instructorship at Wellesley College.

Mr. Harold Glander, formerly an instructor at Superior State College, has been appointed to an instructorship at Carroll College.

Associate Professor H. G. Haefeli of Boston College has been promoted to a professorship.

Reverend C. F. Hartmann, previously associate dean of the College of Arts and Sciences of Loyola University, is now a member of the Department of Mathematics.

Acting Associate Professor Nola L. A. Haynes of the University of Missouri has been appointed to an associate professorship.

Mr. S. L. Hull of the University of Arkansas has been appointed Research Participant at the Oak Ridge National Laboratory, Tennessee.

Instructor A. R. Hyde of Trinity College, Connecticut, has accepted a position as mathematician at Emhart Manufacturing Company, Hartford, Connecticut.

Mr. T. L. Jordan, Jr., who has been a teaching fellow at Vanderbilt University, has been appointed to an associate professorship at Wofford College.

Mrs. Evelyn K. Kinney, head of the Department of Mathematics of Ferrum Junior College, Virginia, has been appointed to an associate professorship at Illinois Wesleyan University.

Mr. Blair Kinsman of St. John's College has a position as mathematician at the Chesapeake Bay Institute, Johns Hopkins University.

Mr. D. H. Kraft, formerly a student at Ohio State University, has a position as staff mathematician at Wright-Patterson Air Force Base, Dayton, Ohio.

Mr. S. B. Kramer, who has been a student at the Polytechnic Institute of Brooklyn, has been appointed Research Assistant in the Department of Aeronautical Engineering and Applied Mechanics of the Institute.

Assistant Professor L. L. Layton of James Millikin University has accepted a position as mathematician at the Naval Ordnance Laboratory, Silver Spring, Maryland.

Adjunct Professor T. H. Lee of the University of South Carolina is on leave of absence for the academic year 1951–52 and is engaged in graduate work at the University of North Carolina.

Mr. G. R. Lehner, formerly a student at Loyola University, has been appointed to a graduate assistantship at the University of Wisconsin.

Dr. J. A. Lewis, who has been a research physicist at Corning Glass Works, has accepted a position as a member of the technical staff of Bell Telephone Laboratories, Incorporated, Murray Hill, New Jersey.

Professor J. J. Livers of Montana State College is on leave of absence for the academic year 1951–52 and is employed by Boeing Aircraft Company, Seattle, Washington, as an educational consultant.

Mr. J. L. Maddox, Jr., who has been a teaching fellow at Alabama Polytechnic Institute, is now employed as an engineer by Philco Corporation, Philadelphia, Pennsylvania.

Instructor A. W. Mall of St. Lawrence University has a position as mathematician at the Frankford Arsenal, Philadelphia, Pennsylvania.

Lieutenant Colonel J. D. Matheson of the United States Army has been appointed to an assistant professorship in the United States Military Academy.

Instructor J. E. Maxfield of the University of Oregon has a position as mathematician at the Naval Ordnance Test Station, China Lake, California.

Associate Professor J. R. Mayor of the University of Wisconsin has been promoted to a professorship.

Mr. W. J. McClintock, previously a teaching fellow at the University of Detroit, is now Superintendent of Schools, Trenary, Michigan.

Graduate Assistant D. C. McCune of Purdue University has been appointed Instructor in Mathematics and Research Associate in the Statistical Laboratory of the University.

Mr. J. R. McPherson, previously a student at the University of North Dakota, is teaching now at Benson County Agricultural School, Maddock, North Dakota.

Associate Professor E. J. Mickle of Ohio State University has been promoted to a professorship.

Dr. G. W. Morgan has been promoted to an assistant professorship in the Graduate Division of Applied Mathematics of Brown University.

Mr. N. G. Myers, Jr., formerly a student at Gannon College, has accepted a position as statistician at Lord Manufacturing Company, Erie, Pennsylvania.

Mr. M. V. Neisius, formerly an engineer with Firestone Tire and Rubber Company, Atlanta, Georgia, has been appointed to an assistant professorship at Georgia Institute of Technology.

Miss Leola P. Odland has been appointed to a graduate assistantship at the University of Kansas.

Associate Professor E. G. Olds of Carnegie Institute of Technology has been promoted to a professorship.

Assistant Professor T. G. Ostrom of Montana State University has been promoted to an associate professorship.

Mr. W. D. Paxton, Jr., previously a student at Pomona College, has a position as insurance inspector with Retail Credit Company, San Bernardino, California.

Mrs. Patricia C. Pearson has been appointed to a teaching fellowship at Oregon State College.

Mr. G. M. Petersen, who has been a graduate student at the University of Toronto, has been appointed Lecturer at Brandon College, Manitoba, Canada.

Mr. V. J. Rathbun, formerly a graduate student at Catholic University, has been appointed to an instructorship at Mount St. John Normal, Dayton, Ohio.

Graduate Assistant Conrad Renneman, Jr., of the University of Nebraska is now Aeronautical Research Scientist with the National Advisory Committee for Aeronautics, Langley Field, Virginia.

Mr. Gordon Ritchie is teaching at Sharbot Lake High School, Ontario, Canada.

Associate Professor P. C. Rosenbloom of Syracuse University has been appointed to an associate professorship at the University of Minnesota.

Mr. H. L. Royden, formerly a graduate student at Harvard University, has been appointed to an assistant professorship at Stanford University.

Dr. Abraham Schwartz of City College of New York has been promoted to an assistant professorship.

Associate Professor Nathan Schwid of the University of Wyoming has been promoted to a professorship.

Associate Professor C. L. Seebeck, Jr., of the University of Alabama has been promoted to a professorship.

Assistant Professor R. E. Smith of Duquesne University has been promoted to an associate professorship.

Professor R. G. Sturm of the School of Civil Engineering of Purdue University has been appointed Director of Auburn Research Foundation, Alabama Polytechnic Institute.

Mr. Henry Tucker has been appointed to an assistant professorship at Kansas State College.

Mr. J. H. Wahab, formerly a teaching fellow at the University of North Carolina, has been appointed to an assistant professorship at Georgia Institute

of Technology.

Mr. J. R. Wesson, formerly a graduate fellow at Vanderbilt University, has been appointed to an assistant professorship at Birmingham-Southern College.

Mr. E. H. Weiss, formerly a mathematician and senior programmer at the Naval Proving Ground, has received an appointment as mathematical statistician at the Bureau of Census.

Mr. Dale Woods of North Dakota Agricultural College has been appointed to an instructorship at Texas Western College.

Chancellor R. L. Flowers of Duke University died on August 24, 1951; Dr. Flowers had served the University as Professor of Mathematics for forty-three years.

Assistant Professor Emeritus F. H. Hodge of Purdue University died on August 21, 1951. He was a charter member of the Association.

Assistant Professor Emeritus Ina E. Holyroyd of Kansas State College died on October 1, 1951.

Professor Emeritus Richard Morris of Rutgers University died on September 19, 1951. He was a charter member of the Association.

Professor A. V. Richardson of Bishop's University, Lennoxville, Quebec, died on June 25, 1951.

Professor C. E. Van Horn, head of the Department of Mathematics of Wartburg College, died on September 14, 1951. He had been a member of the Association for fifteen years.

Reverend Joseph Wilczewski, professor of mathematics at Marquette University, died on March 6, 1951.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE TWELFTH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The twelfth annual William Lowell Putnam Mathematical Competition will be held on Saturday, March 22, 1952. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of America and is open to undergraduate students in universities and colleges of the United States and Canada who have not received a degree. The examination will consist of two parts of three hours each. The questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics not involving techniques beyond the usual applications, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). Any college or university wishing to enter a team

or individual contestants may secure an application blank from Professor L. E. Bush, 112 Albertus Magnus Hall, College of St. Thomas, St. Paul 1, Minnesota, by a postcard request. All applications must be filed with the Director not later than March 1, 1952. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application. Fewer than three from one college or university may compete as individuals.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to this rule may be made by the Director in cases where it would entail unusual inconvenience to a contestant. Sealed copies of the examinations will be sent to the supervisor of the examination in time for the examination day and are not to be opened before the hour set.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $400, $300, $200, and $100, in the order of their rank. In addition, there will be prizes of $40, $30, $20 and $10 awarded to the members of these teams according to the rank of the team; a prize of $50 to each of the five highest contestants and a prize of $20 to each of the succeeding five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to several teams next in order after the four winning teams and to several individuals next in order after the ten individual winners. For further encouragement of the Competition, there will be awarded at Harvard University (or at Radcliffe College in the case of a woman) an annual $2000 William Lowell Putnam Prize Scholarship to one of the first five contestants, this to be available either immediately or on the completion of the student's undergraduate work.

Reports on the eleven previous competitions and examinations will be found in this MONTHLY for May, 1938, 1939, 1940, 1941, 1942, October, 1946, August-September, 1947, December, 1948, August-September, 1949, 1950, and 1951.

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following seventy-seven persons have been elected to membership on applications duly certified.

L. R. ABRAMSON, Student, Columbia University, New York, N. Y.

M. H. AHRENDT, M.A. (Wichita) Executive Secretary, National Council of Teachers of Mathematics, Washington, D. C.

DAVID ALBERT, B.S. (Connecticut) Production Supervisor, Talon, Inc., Hamden, Conn.

S. A. ANDERSON, M.A. (George Peabody) Asso. Professor, Memphis State College, Tenn.

LEON BANKOFF, D.D.S. (Southern California) 5410 Wilshire Blvd., Los Angeles, Calif.

F. L. BARTMAN, M.A. (Michigan) Research Engineer, Willow Run Research Center, Ypsilanti, Mich.

P. O. BELL, Ph.D. (California) Professor, University of Kansas, Lawrence, Kan.

J. M. BOSWELL, M.A. (Kentucky) President, Cumberland College, Williamsburg, Ky.

G. U. BRAUER, M.A. (Michigan) Grad. Student, University of Michigan, Ann Arbor, Mich.

G. L. BULLIS, M.A. (Wisconsin) State Teachers College, Platteville, Wis.

MRS. MARY L. CANTWELL, M.A. (Columbia) Teacher, St. Petersburg Junior College, Fla.

P. F. CAUFFMAN, M.A. (Lehigh) Head of Mathematics Department, Maryland State Teachers College, Salisbury, Md.

ABRAHAM CHARNES, Ph.D. (Illinois) Asso. Professor, Carnegie Institute of Technology, Pittsburgh, Pa.

HOWARD CHITWOOD, Student, Carson-Newman College, Jefferson City, Tenn.

P. O. CIOFFI, M.S. (New Mexico S. C.) Research Engineer, Massachusetts Institute of Technology, Cambridge, Mass.

R. M. CONKLING, M.S. (Bucknell) Grad. Student, University of Florida, Gainesville, Fla.

J. G. COX, M.S. (Alabama Poly.) Instr., Alabama Polytechnic Institute, Auburn, Ala.

C. E. DIESEN, M.A. (Minnesota) Dynamics Engineer, Bell Aircraft Corporation, Niagara Falls, N. Y.

F. F. DORSEY, L.L.B. (Boston U.) Retired Patent Attorney. 205 Garfield Place, South Orange, N. J.

MARY T. DUNLEAVY, A.M. (Columbia) Statistician, E. I. duPont de Nemours and Company, Wilmington, Del.

J. D. ESARY, A.B. (Whitman) Faculty, Chemical Corps School, Ft. McClellan, Ala.

D. T. FINKBEINER, II, Ph.D. (C.I.T.) Asso. Professor, Kenyon College, Gambier, Ohio

A. L. FISCH, M.S. (Wisconsin) Acting Instr., Ohio University, Athens, Ohio

H. D. FRIEDMAN, A.M. (Columbia) Grad. Student, Pennsylvania State College, Pa.

K. G. GETMAN, A.B. (Alfred) Assoc. Editor, John Wiley & Sons, New York, N. Y.

J. A. GIORDMAINE, Student, University of Toronto, Ontario

R. L. GLASS, M.Ed. (Peabody) Head of Mathematics Department, Texarkana College, Tex.

BETTY GROSSMAN, M.A. (Michigan) Teacher, Detroit Public Schools, Mich.

J. I. HARRIS, Student, University of Michigan, Ann Arbor, Mich.

C. L. HASSELL, JR., M.S. (Purdue) Staff Member, Sandia Corporation, Albuquerque, N. M.

MARY C. HEATH, M.S. (Kentucky) Instr., Florida State University, Tallahassee, Fla.

E. E. HECKROTH, B.A. (Buffalo) Computer, Bell Aircraft Corporation, Niagara Falls, N. Y.

E. L. HITCHCOCK, B.S. (Middle Tennessee) Compounder, Goodyear Tire and Rubber Company, Akron, Ohio

T. R. HORTON, M.S. (Florida) Asst. Commandant and Instr., Bolles School, Jacksonville, Fla.

MIRIAM O. HUGGINS, A.B. (Women's C., North Carolina) Teacher, Grainger High School, Kinston, N. C.

N. W. JOHNSON, JR., Student, Carleton College, Northfield, Minn.

W. H. JOHNSON, M.S. (N.Y.U.) Mathematician, International Business Machines Corporation, New York, N. Y.

J. E. KIMBER, JR., Student, Massachusetts Institute of Technology, Cambridge, Mass.

LEONARD KLAFTER, A.B. (George Washington) Mathematician, Army Map Service, Washington, D. C.

I. I. KOLODNER, Ph.D. (N.Y.U.) Research Associate, New York University, N. Y.

M. D. KRUSKAL, M.S. (N.Y.U.) Research Associate, Princeton University, N. J.

M. R. LAUDANTE, Student, Polytechnic Institute of Brooklyn, N. Y.

B. S. LEVINE, M.A. (Washington S. C.) Grad. Student, Columbia University, New York, N. Y.

DAVID LOEV, Student, University of Pennsylvania, Philadelphia, Pa.

A. P. LOOMER, M.A. (Acadia, Canada) Professor, Milton College, Wis.

G. R. MACLANE, Ph.D. (Rice) Asst. Professor, Rice Institute, Houston, Tex.

R. A. MAGDA, Student, University of Detroit, Mich.

D. M. MANDELBAUM, Student, Hillside High School, N. J.

G. J. MANN, M.A. (Loyola, Chicago) Asst. Professor, U. S. Naval Academy, Annapolis, Md.

J. F. MANOGUE, Student, Catholic University, Washington, D. C.

JON MATHEWS, Student, Pomona College, Claremont, Calif.

J. C. MCPHERSON, B.S. (Alabama Poly.) Assistant, Duke University, Durham, N. C.

H. J. MILLER, B.S. (Seton Hall U.) Teacher, St. Patricks High School, Elizabeth, N. J.

B. B. MISRA, M.S. (Michigan) Asst. Professor, Ravenshaw College, Cuttack, India

C. T. MOLLOY, Ph.D. (N.Y.U.) Head of Mathematics Section, Kellex Corporation, New York, N. Y.

P. M. MOSKOWITZ, B.S. (Columbia) Physicist, New York Naval Shipyard, Brooklyn, N. Y.

E. O. NELSON, M.A. (Minnesota) Instr., University of North Dakota, Grand Forks, N. D.

W. J. NEWMAN, B.S. (De Paul) Grad. Student, De Paul University, Chicago, Ill.

R. V. NIEDRAUER, Student, University of Buffalo, N. Y.

O. L. PERKINS, M.A. (Atlanta) Instr., Campbell Street High School, Daytona Beach, Fla.

W. J. PERVIN, Student, University of Michigan, Ann Arbor, Mich.

K. F. POWELL, B.S. (Pittsburgh) Customer Engineer, International Business Machines Corporation, New York, N. Y.

P. C. ROGERS, M.A. (Boston U.) Grad. Assistant, University of Maryland, College Park, Md.

BERTHOLD SCHWEIZER, B.S. (M.I.T.) Mathematician, Naval Ordnance Laboratory, White Oak, Md.

RAYMOND SEDNEY, D.Sc. (Carnegie Tech.) Aerodynamicist, Douglas Aircraft Company, Santa Monica, Calif.

H. F. SIEBER, Major, Signal Corps, Ft. Monmouth, N. J.

JEROME SIMONEL, B.A. (Paris) Justice, French Oceania High Court, Papeete, Tahiti Island, French Oceania

SISTER MARY MICHAIL, M.S. (Marquette) Chairman of Mathematics Department, Clarke College, Dubuque, Iowa

T. T. TANIMOTO, Ph.D. (Pittsburgh) Asst. Professor, Allegheny College, Meadville, Pa.

R. P. TAPSCOTT, Student, University of Kentucky, Lexington, Ky.

A. W. THOMAS, JR., Student, George Pepperdine College, Los Angeles, Calif.

R. H. THOMPSON, M.A. (Kansas) Professor, Sterling College, Kans.

R. F. TIDD, M.A. (Buffalo) Instr., Canisius College, Buffalo, N. Y.

J. E. WARD, M.S. (Northwestern) Instr., University of Wisconsin, Kenosha Extension Center, Wis.

G. P. WILLIAMS, A.B. (Columbia) Grad. Student, University of Cincinnati, Ohio

C. E. WOOD, M. A. (Columbia) Teacher, North Syracuse High School, N. Y.

E. W. WRIGHT, Student, University of Idaho, Moscow, Idaho

FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The twenty-eighth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Mississippi State College, State College, Mississippi, on February 16 and 17, 1951. Professor H. F. Schroeder, Chairman of the Section, presided at the Friday afternoon session and Professor S. B. Murray, Mississippi Vice-Chairman, presided at the Saturday morning session.

There were eighty persons presented including the following thirty-six members of the Association: T. A. Bickerstaff, G. J. Corley, Mamie M. Davis, W. L. Duren, Jr., Virginia I. Felder, L. M. Garrison, M. E. Gillis, F. Monica Goen, A. C. Grimes, J. A. Hardin, R. H. Hopkins, A. S. Householder, L. H. Kanter, H. T. Karnes, C. G. Killen, R. J. Koch, A. C. Maddox, J. W. Mc-

Climans, Betty McKnight, R. A. Miller, Benjamin Ernest Mitchell, S. B. Murray, Arthur Ollivier, R. L. O'Quinn, P. K. Rees, T. L. Reynolds, F. A. Rickey, H. F. Schroeder, S. W. Shelton, C. D. Smith, P. K. Smith, W. H. Spragens, M. M. Temple, V. B. Temple, B. B. Townsend, A. D. Wallace.

The following officers were elected for the coming year: Chairman, Arthur Ollivier, Mississippi State College; Louisiana Vice-Chairman, C. G. Killen, Northwestern Louisiana State College; Mississippi Vice-Chairman, Virginia I. Felder, Mississippi Southern College; Secretary-Treasurer, F. A. Rickey, Louisiana State University.

The time and place of the next annual meeting of the Section were set as February 15 and 16, 1952 at Northwestern Louisiana State College, Natchitoches, Louisiana.

The invited speaker for the meeting was Dr. A. S. Householder, Oak Ridge National Laboratory. His subject for an address at the Friday evening dinner held jointly with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics was *Mathematics at Oak Ridge*. At the Saturday morning session, Dr. Householder spoke on *Nervous Systems and Computing Machines*.

The following short papers were presented:

1. *A note on certain integral products*, by Professor R. A. Miller, University of Mississippi.

In this note certain relations are determined which are satisfied by two functions of a single variable such that the product of the indefinite integrals of the two functions should equal a non-zero constant multiplied by the indefinite integral of the product of the functions. The relations are such that if one of the functions is assumed, it is possible in certain cases to determine the second function. Several different cases are treated, and as a by-product, the primitive function of certain complicated integrands is determined with greater facility than by the usual methods.

2. *The induced homomorphism theorem and some of its applications to elementary group theory*, by Professor A. D. Wallace, Tulane University.

In this note it is shown how the Induced Homomorphism Theorem enables one to give simple and direct proofs of some of the isomorphism theorems commonly considered in elementary modern algebra. It is also noted that the Induced Homomorphism Theorem applies to rings. One can thus formulate analogs, for the ring structure, of the isomorphism theorems for groups.

3. *Cycloids and roses*, by Professor V. B. Temple, Louisiana College.

It is shown that, for every n-arched hypocycloid, there is an n-petaled inscribed rose; and for every n-arched epicycloid, there is an n-petaled circumscribed rose. The cycloids and their accompanying roses are defined as pedal curves. (A pedal curve is the locus of the vertex of a right angle as it moves according to given conditions). Equations for these curves are given in generalized form along with drawings showing the special cases for $n=3$ and $n=4$.

4(a). *Geometry of the circulant*, by Mr. R. A. Stokes, student, University of Mississippi, introduced by Professor B. E. Mitchell.

A circulant is a determinant in which any row is obtained from the preceding one by passing the last element over the others into the first position. When geometric meaning is given these elements, the circulant can be interpreted geometrically. The power of a point with respect to a circle with a real trace may be expressed as a circulant of the second order. There is a one-param-

eter family of conics whose discriminant is a circulant and which has the following properties: (1) The family contains one circle (imaginary); (2) The family contains two parabolas (one proper and one degenerate); (3) The family contains one equilateral hyperbola; (4) Each member of the family is tangent to every member at points (ω, ω^2) and (ω^2, ω); there are no other points of intersection; (5) The center of every member lies on the line $X = Y$.

(b) *Brachistocrone of ascent*, by Mr. Jerry McCall, student, University of Mississippi, introduced by Professor B. E. Mitchell.

Given a plane with constant rotation about a vertical axis in the plane, the impressed field of force composed of a horizontal force $\omega^2 r$ and vertical force g acts on every particle in the plane. The path of free motion of any particle is the square of an inverse hyperbolic function. Introducing a parabolic constraining curve, there is no motion of the particle. The orthogonal trajectory of this curve is logarithmic and is a brachistochrone.

5. *Approximation by Walsh-Rademacher functions*, by Professor W. H. Spragens, University of Mississippi.

The Walsh-Rademacher system of orthogonal functions is of interest because of the simple structure of the functions and the use made of the binary number notation. This paper is a brief exposition of the development of a given function in terms of this system and contains illustrations of the nature of the "step-function" approximations afforded by the partial sums.

6. *On free vectors*, by Professor W. L. Duren, Jr., Tulane University.

As is well known, a two-dimensional vector space X over the real numbers can be represented by taking as vectors the directed line segments from the origin to each point of the plane. Another representation of the same space can be constructed by taking as vectors the so-called "free vectors" which consist of all directed line segments in the plane with a convention of equivalence. This is often done incorrectly in text books, however. The purpose of this note is to present the free vector space concept from the viewpoint of abstract algebra. It is the quotient space of the cartesian product space $X \times X$ modulo the diagonal in $X \times X$, the diagonal being a scalar product ideal. This quotient space is isomorphic with X.

7. *On a separation theorem for the zeros of the ultraspherical polynomials*, by Professor L. H. Kanter, Mississippi State College.

Let $R_n(x, \lambda) = \partial/\partial x(P_n(x, \lambda))$, where $P_n(x, \lambda)$ is the ultraspherical polynomial of degree n. If the identity $P_n(x_\nu, \lambda) \equiv 0$, where x_ν is the νth zero of $P_n(x, \lambda)$, be differentiated with respect to λ, the resultant equation may be used to determine separation theorems involving the zeros of $R_n(x, \lambda)$ and those of the corresponding ultraspherical polynomial $P_n(x, \lambda)$. Generalizations to the polynomials $R_n(x, \lambda) + c(\lambda)P_n(x, \lambda)$, where $c(\lambda)$ is an arbitrary constant depending on λ, follow without difficulty. Also it can be shown that $R_n(x, \lambda)$ may be expressed as $2\sum_{\nu=0}^{n} P_\nu(x, \lambda)T_{n-\nu}(x, \lambda)/(n-\nu)$, where $T_0/0$ is defined to be equal to 0, and T_ν is the Tchebychef polynomial of degree ν.

8. *Shadow loci*, by Professor P. K. Smith, Louisiana Polytechnic Institute.

This paper treats of procedures in finding the equations' curves followed by a shadow. The top of a vertical pole of variable length is allowed to generate a space curve. From the top of a fixed vertical pole a light is allowed to cast the shadow of the top of the variable pole upon the horizontal plane. Several systems of algebraic and differential equations are used in studying the relationship between the equations of the space curve and the shadow curve.

9. *Some statistical applications of sequential analysis*, by C. D. Smith, University of Alabama, by title.

In this paper are considered: the probability density function; the nature of the p_1, α and p_2, β relations; the functions d_1, $\bar{d}$, and d_2 in the random sampling process. An illustrative case from auditing practice is included.

10. *The testing program and standing committee report*, by H. T. Karnes, Louisiana State University.

This is the third of a series of annual reports of the testing program conducted by the departments of mathematics of colleges of Louisiana and Mississippi to entering freshmen. In order to accumulate more data, it was decided to continue the program indefinitely. The results of the program and work of the committee have already been used to advantage in getting the certification requirements for teachers of high school mathematics increased. The first annual Mathematics Institute was favorably reported on and the second one announced.

F. A. RICKEY, *Secretary*

APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the Mathematical Association of America was held in Dallas, Texas, April 27–28, 1951. Southern Methodist University served as sponsor.

About eighty persons attended the sessions of the Association including the following forty-seven members of the Association: J. C. Bradford, Ina M. Bramblett, H. E. Bray, L. A. Colquitt, J. V. Cooke, Don Cude, Terrell Ellis, B. T. Goldbeck, L. D. Gregory, Blanche B. Grover, E. H. Hanson, E. A. Hazlewood, E. R. Heineman, J. J. Henrick, C. M. Howard, C. E. Kelley, Ruth Kissel, W. M. Koger, Ruth O. Lane, W. I. Layton, A. S. Lee, J. W. Lindsay, G. R. MacLane, Hazel L. Mason, Dorothy McCoy, W. K. McNabb, Harlan C. Miller, Leo Moser, E. D. Mouzon, C. A. Murray, T. D. Oxley, H. C. Parrish, C. J. Pipes, L. W. Ramsey, Dorothy L. Rees, C. A. Rogers, C. R. Sherer, D. P. Shore, D. W. Starr, Jennie L. Tate, F. E. Ulrich, R. S. Underwood, H. S. Vandiver, Margaret M. Welch, Mabel Williams, F. L. Wrenn, C. B. Wright.

At the business meeting the following officers were elected for the next academic year: Chairman, D. W. Starr, Southern Methodist University; Vice-Chairman, C. B. Wright, East Texas State Teachers College; Secretary-Treasurer, C. R. Sherer, Texas Christian University.

At a banquet given Friday night in the Student Union Dining Hall, H. M. Hosford, Vice-President of Southern Methodist University, was the principal speaker.

The first six papers were given Friday afternoon and the remaining papers were given Saturday morning.

1. *Surfaces*, by Professor G. R. MacLane, The Rice Institute.

2. *On the equation* $\phi(n)=\pi(n)$, by Dr. Leo Moser, Texas Technological College.

As usual, let $\phi(n)$ denote Euler's totient function, and $\pi(n)$ the number of primes not exceeding n. Using elementary methods only, the equation $\phi(n)=\pi(n)$ was shown to have exactly eight solutions, namely $n=2, 3, 4, 8, 10, 14, 20, 90$. The inequality $\phi(n)<\pi(n)$ was shown to have seven solutions. A number of related partially solved, and unsolved problems were discussed.

3. *Theorem concerning exact differential equations*, by Professor R. S. Underwood, Texas Technological College.

Most texts on differential equations solve the exact equation $Mdx+Ndy=0$ by a method which is cumbersome in itself and does not generalize easily to the case of n variables. But if these texts would borrow a little from advanced calculus and use the simple idea of integrating an exact differential expression along a broken line path, the solution would be simplified materially whether there be two or n variables.

4. *A representation of a certain class of Riemann surfaces*, by Mr. J. D. Rice, The Rice Institute, introduced by the Secretary.

5. *A theory concerning point sets*, by Professor C. J. Pipes, Southern Methodist University.

Define an abstract set property by taking as postulates certain characteristics common to the classes of sets of measure zero and sets of the first category. Then, certain results that appear as dual results for sets of measure zero and sets of the first category are found to hold true likewise for any class of sets satisfying the postulates for the abstract set property. Two particular results are discussed.

6. *The use of complex elements in metric geometry*, by Professor F. E. Ulrich, The Rice Institute.

7. *Preparation and certification of teachers in Texas*, by Mr. Lee Wilburn, Assistant Commissioner for Instructors, T. E. A., introduced by the Secretary.

8. *Relative values in the teaching of mathematics*, by Professor F. L. Wren, George Peabody College for Teachers (By invitation).

C. R. SHERER, *Secretary*

MAY MEETING OF THE WISCONSIN SECTION

The nineteenth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at Carroll College, Waukesha, Wisconsin, on Saturday, May 12, 1951, in conjunction with the Wisconsin Mathematics Council. Professor E. G. Harrell presided at the morning and afternoon sessions.

About fifty persons were present, including the following thirty-one members of the Association: K. J. Arnold, R. H. Bardell, L. J. Berner, Leonard Bristow, R. H. Bruck, H. H. Conwell, H. P. Evans, J. V. Finch, L. E. Fuller, E. G. Harrell, R. C. Huffer, J. F. Kenney, Morris Marden, A. E. May, P. E. Meadows, Genevieve T. Meyer, Marianne S. Otto, O. E. Overn, H. P. Pettit, A. C. Schaeffer, M. E. Shanks, Sister M. Mechtildis, Sister Mary Felice, Sister Mary Olive, Sister Mary Petronia, Abraham Spitzbart, J. V. Talacko, K. R. Thompson, R. D. Wagner, N. M. Watermolen, Louise A. Wolf.

At the business meeting held after the morning program, the following officers were elected for the coming year: Chairman, P. E. Meadows, Carroll College; Secretary, Louise A. Wolf, University of Wisconsin in Milwaukee; Program Committee Chairman, R. H. Bing, University of Wisconsin.

The program consisted of the following papers:

1. *Induction*, by Professor A. C. Schaeffer, University of Wisconsin.

Induction, not as mathematical induction, but with the meaning customary in other sciences, was the topic of this paper. Induction in this sense is a useful device for discovering what may be true in mathematics. An example was given.

2. *Epistemology and manifolds*, by Professor M. E. Shanks, Purdue University.

The author discussed various postulates for physical space which are plausible on heuristic grounds from our immediate sense experience. After some simplification, for reasons of mathematical expediency, the following conjecture was formulated. Is a separable metric space which is connected, locally connected, locally compact, n-dimensional and homogeneous a manifold? The literature pertaining to the problem was reviewed and a possible attack outlined. This attack examines the group of all homeomorphisms of the space onto itself, and a set of conjectures concerning this group was formulated which would facilitate solution of the manifold problem.

3. *The Mathematical Association of America*, by Professor R. H. Bardell, University of Wisconsin in Milwaukee.

4. *Finite hemi-groups*, by Professor L. V. Toralballa, Marquette University, introduced by Professor H. P. Pettit.

A left hemigroup Σ is a system with one composition operation, X, satisfying the following: (1) For every two elements a and b of Σ there exists uniquely aXb in Σ; (2) Σ is associative with respect to X; (3) For every element a of Σ there exists an element of a^{-1} of Σ such that for every element b of Σ, $(a^{-1}Xa)Xb=b$. Properties: The equation $aX=b$ is always uniquely solvable. Two elements a and b are homologous if there exists an element c such that $ac=a$ and $bc=b$. Homology is an equivalent relation. The equation $Xa=b$ is solvable only if a and b are homologous, in which case there is one solution in each homology class. Each homology class is a group. Two elements a and b are congruent if the inverse of a in the class of b is also the inverse of b in this class. Congruence is an equivalence relation.

5. *The Wisconsin Mathematics Council*, by Miss Margaret Striegl, Wauwatosa High School, introduced by the Secretary.

The Wisconsin Mathematics Council is affiliated with the National Council of Teachers of Mathematics and was represented by Miss Margaret A. Striegl at the annual delegate assembly at Pittsburgh in April. The delegates of fifty-two groups through their discussions and recommendations to the Board of Directors are striving to coordinate and strengthen the work of national, state and local organizations. The Wisconsin Council of about two hundred and eighty members is a group organized to encourage an active interest in mathematics and to improve the teaching of mathematics at all levels. The group meets the first Thursday in November during the annual convention of the Wisconsin Education Association. The executive committee, consisting of the six officers of the council, the past president, the presidents of the mathematics sections of the state divisional associations and area mathematics clubs, and the state representative of the NCTM, meets at various times to transact business. The research committee, representing all levels of instruction, has numerous meetings to discuss the results of investigations in mathematics areas and to prepare teaching materials and aids to publish in the *Wisconsin Teacher of Mathematics* which is issued three times a year.

6. *Progress of mathematical curriculum changes in the Milwaukee schools*, by Mr. Howard M. Aker, Assistant for Research and Educational Studies, Milwaukee, introduced by the Chairman.

A background of mathematics teaching practices in the Milwaukee public schools was presented. Progress toward a common understanding of the need for change was outlined leading to the present efforts to maintain a two-track system of mathematics in the secondary schools and a workable sequence of topics in elementary classes. Improvements in guidance procedures affecting placement of students and enlargement of testing programs were presented and the need for a supervisor in the field of mathematics to coordinate efforts toward "a continuous, sequential program of mathematics from kindergarten through grade 12 for all students" and "an adequate maintenance and remedial program" was pointed up.

7. *Mathematics education in England*, by Mr. Trevor Evans, The University of Manchester, England, and the University of Wisconsin, introduced by Professor H. P. Evans.

8. *Summary report of a questionnaire on mathematics programs for the training of secondary teachers of mathematics in the Wisconsin institutions of higher learning*, by Professor E. G. Harrell, Platteville State Teachers College.

LOUISE A. WOLF, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1-2, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.

ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.

INDIANA, Indiana University, Bloomington, Spring, 1952.

IOWA, Coe College, Cedar Rapids, April 18–19, 1952.

KANSAS

KENTUCKY, University of Kentucky, Lexington.

LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Spring, 1952.

MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.

MINNESOTA, College of St. Catherine, St. Paul, May 10, 1952.

MISSOURI, Lindenwood College, St. Charles, May 2, 1952.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1952.

NORTHERN CALIFORNIA, Stanford University, Stanford, January 26, 1952.

OHIO, April 19, 1952.

OKLAHOMA

PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.

PHILADELPHIA

ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May, 1952.

SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.

SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.

SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.

TEXAS, East Texas State Teachers College, Commerce, April, 1952.

UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May, 1952.

WISCONSIN

Prentice-Hall
COLLEGE TEXTS

MEANINGFUL MATHEMATICS

by H. S. KALTENBORN, Chairman, Department of Mathematics, Memphis State College

The purpose of this book is to provide the mathematical part of a liberal education. Material ranging from primitive number systems to calculus is included, while the large number of problems provide adequate drill on fundamental processes. Many fascinating topics, seldom encountered in traditional courses, are treated: for example, there is an unusual and excellent chapter on "Consumer Mathematics."

384 pages 5½" x 8½" Published 1951

Basic Mathematics for General Education

by H. C. TRIMBLE, FRANK C. BOLSER, and THOMAS L. WADE, all of Florida State University

The new arrangement and treatment of topics in this book (geared to the modern theory of general education) is original with the authors. Emphasis is on mathematics as a language rather than on theory or how-to-do-it. Mathematical elegance is sometimes sacrificed to restrict discussion to issues most interesting and helpful to the freshman student.

313 pages 5½" x 8½" Published 1950

COLLEGE ALGEBRA

by MOSES RICHARDSON, Brooklyn College

Reasoning is stressed as the best road to the remembrance of fundamental concepts. Rigorous proofs are used whenever possible. When this seems beyond the capacity of the student, it is clearly pointed out that the discussion presented does not constitute a proof. A major feature of the book is Chapter I, a 46-page study of the number system. Over 3,000 carefully graded problems round out this eminently teachable text.

472 pages 6" x 9" Published 1947

PRENTICE-HALL, Inc., 70 Fifth Avenue, New York 11, N. Y.

MATHEMATICAL EXPOSITIONS SERIES

The Foundations of Geometry

By Gilbert de B. Robinson. "Those with a good knowledge of school geometry should find this book very helpful."—*Nature.* 180 pp. Second edition. $3.75

Non-Euclidean Geometry

By H. S. M. Coxeter. "It should be the standard textbook on non-Euclidean geometry for a long time to come."—*The Mathematical Gazette.* 296 pp. Second edition. $4.75

The Theory of Potential and Spherical Harmonics

By W. J. Sternberg and T. L. Smith. "The book will form a good basis for further reading and research in this fascinating branch of mathematics."—*The Mathematical Gazette.* 324 pp. $5.00

The Variational Principles of Mechanics

By Cornelius Lanczos. "This is an excellent book covering one of the most brilliant achievements of scientific theory, and the text will certainly be welcomed by teachers and students of analytical mechanics."—*Quarterly of Applied Mechanics.* 332 pp. $5.75

Tensor Calculus

By J. L. Synge and A. Schild. A mathematician unacquainted with tensor calculus is at a serious disadvantage in several fields of pure and applied mathematics. This book is a general brief introduction to this subject. 346 pp. $6.00

The Theory of Functions of a Real Variable

By R. L. Jeffery. The first section leads the reader by easy stages through the essential parts of the theory of sets and the theory of measure to the properties of the Lebesgue integral. The later chapters contain a thorough treatment of the relation between derivatives and integrals. $6.00

General Topology

By W. Sierpinski. Translated and revised by C. C. Krieger. Professor Krieger's translation of Sierpinski's earlier work on point-set topology was speedily recognized as the outstanding reference work on the subject in English. Here is a translation of an entirely new book, differing vastly in arrangement, outlook and presentation. $6.00

UNIVERSITY OF TORONTO PRESS

available spring →

just published ↓

MATHEMATICS of INVESTMENT

Paul R. Rider, Washington University

Carl H. Fischer, University of Michigan, 359 pp., $5.00

- values of interest at the lower rates of interest
- tables of compound interest functions are arranged primarily by interest rate with five functions listed on each page under each rate
- the three chapters on life annuities and insurance are the most accurate and teachable which have yet appeared
- chapter on bonds has been brought into line with the practices and language of the bond market
- over 1000 problems
- answers to odd numbered problems at the end of the problem
- a new treatment on installment contracts is included in the chapter on simple interest
- the case of the more general annuity certain is handled with only one simple formula for the present value and a corresponding formula for the accumulated amount
- provides modern mortality tables with commutation symbols at a modern rate of interest

TEACHING MATHEMATICS IN THE SECONDARY SCHOOL

Lucien B. Kinney, Stanford University

C. Richard Purdy, San Jose State College, probably 520 pages, $4.50

A complete coverage of the curriculum and instruction in junior high school, senior high school, and junior college general mathematics. It provides a description of applications of present-day methods and materials, including visual aids as derived from actual classroom experience.

Contents: Mathematics in Modern Life. Historical Background of the Mathematics Curriculum. Present-Day Problems of Curriculum and Instruction. Directing Learning in Mathematics. The Teaching of Advanced High School Mathematics. College Preparatory Courses in the Junior College. The Purpose and Nature of General Mathematics. General Mathematics in the Junior High School. General Mathematics in the Senior High School. General Mathematics in the Junior College. Teaching Aids in the Effective Classroom. Planning the Long Unit Assignment. Constructing and Using Tests. Recreational Mathematics.

RINEHART & CO.

232 madison ave. new york 16

Noteworthy Books . . .

COLLEGE ALGEBRA

By EARLE B. MILLER, *Illinois College;* ROBERT M. THRALL, *University of Michigan.* A FIRST YEAR college textbook which avoids the complexity of being too advanced, and the sterility of oversimplified presentation. Covers the number system of algebra; treats functions of real variables, equations involving these functions, polynomials. Considers mathematical induction as a *modus operandi* in mathematics, introducing this concept with progressions, illustrating it with the binomial theorem. Discusses permutations, combinations; probability; matrices; determinants, systems of linear equations; complex numbers.

ANALYTIC GEOMETRY

By ALFRED L. NELSON, KARL W. FOLLEY, WILLIAM M. BORGMAN, *of Wayne University.* PLANNED AS PREPARATION for the calculus rather than a study of geometry, the book is of value to future students of calculus, the basic sciences, and engineering. Attention is given to two important problems of analytic geometry: 1) given the equation of a locus, to draw a curve, or describe it geometrically; 2) given the geometric description of a locus, to find its equation. Brief tables of trigonometric, exponential, and logarithmic functions are included in this volume.

INTRODUCTION TO ANALYTIC GEOMETRY AND THE CALCULUS

By H. M. DADOURIAN, *Trinity College (Connecticut).* THIS TEXTBOOK was prepared for use in a combined course of Analytic Geometry and the Calculus such as is offered for liberal arts and science students. The book presents the fundamental concepts of the Calculus so as to give the student an idea of the methods and uses of this branch of mathematics. It is the author's belief that the student will thus acquire a comprehension of the nature of mathematics as a mental discipline. Little if any knowledge of trigonometry is required.

INTERMEDIATE ALGEBRA FOR COLLEGES

By EARLE B. MILLER, *Illinois College.* THIS TEXTBOOK is for students who have had only one year of algebra in high school. Important features include: full explanations, emphasis on techniques, a generous number of illustrations and worked examples, numerous notes, early introduction of function concept and graphic methods, formal proofs, helpful treatment of logarithms, and many carefully graded exercises. The book is based on the author's twenty-five years of teaching algebra at college level.

Send for examination copy direct from:

THE RONALD PRESS COMPANY
15 East 26th Street, New York 10

two recent texts in the Appleton-Century Mathematics Series

THE MATHEMATICS OF FINANCE

By Franklin C. Smith, St. Thomas College

Emphasizing basic principles rather than special formulas, this text offers an introductory course in the mathematics of finance to the student of business administration. An appendix of mathematical tables, a review of algebraic topics, and exercises are included. *356 pp. $4.00*

TEACHING THE MEANINGS OF ARITHMETIC

By C. Newton Stokes, Temple University

"The book is in harmony with a modern and sound approach of teaching arithmetic. Both preparatory and in-service teachers can find much to help them."

Dr. Howard F. Fehr
Teachers College, Columbia
531 pp. $4.50

APPLETON-CENTURY-CROFTS 35 West 32nd Street New York 1, N.Y.

*"almost unlimited suggestions . . ." **

The Main Stream of Mathematics

By EDNA E. KRAMER

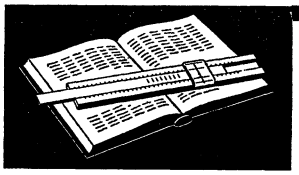
"A fascinating survey of mathematical history, lore, theory and practical application, beginning with the 'one, two, heap' of the savage through Euclid, Cardan, Descartes to Einstein and Georg Cantor's theory of the infinite. A delightful and diverting story."—*Los Angeles Times*

"A valuable addition to any teacher's personal library. . . . It would be hard to find any single volume so likely to prove interesting to a high school or junior college student. It offers almost unlimited suggestions for mathematics club programs and distinct possibilities as a text in a course in the appreciation of mathematics."—*School Science and Mathematics**

"The student of mathematics will read this book with pleasure. . . . The teacher of mathematics will find it a most valuable addition to his reference material."—*American Mathematical Monthly*

$5.00 at all bookstores or direct from the publisher

OXFORD UNIVERSITY PRESS 114 Fifth Ave., N.Y. 11



published by

D. C. Heath and Company

W. L. Hart's College Algebra, 3rd Edition

Offers all the usual topics of college algebra on a suitably mature level, preceded by an extensive presentation of intermediate algebra, for review or leisurely study. Intended for classes of varying degrees of preparation. Includes substantial supplementary content useful in engineering, the physical sciences, and statistics. $3.25

Mathematics of Investment, 3rd Edition

A standard course in annuities certain, with an introduction to life annuities and life insurance. Emphasizes problems under the **simple case** for annuities certain, with the general case in a final chapter. Very complete tables, particularly for small interest rates. Text $3.25. With Tables $4.25. Bound with Tables and Hart's **Essentials of College Algebra** $5.00. Tables separately $2.00

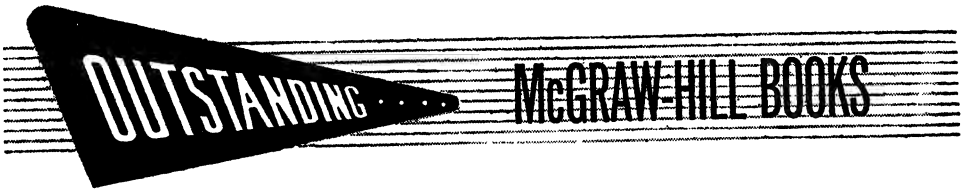
Curtiss and Moulton's Essentials of Analytic Geometry

Wilson and Tracey's Analytic Geometry

Tomlinson Fort's Calculus

D. C. HEATH AND COMPANY

Sales Offices: NEW YORK CHICAGO ATLANTA SAN FRANCISCO DALLAS Home Office: BOSTON



COLLEGE ALGEBRA

By Ross R. MIDDLEMISS, Washington University. Ready in March

The text covers all the topics usually taught in a standard course. The principal aim is to make the algebra course more valuable and more stimulating by emphasizing *reasoning and clear thinking*—and by thus combating the student's tendency toward mechanical operations unaccompanied by real thought.

TRIGONOMETRY, PLANE AND SPHERICAL

By LLOYD L. SMAIL, Lehigh University. 406 pages (with tables) $3.75

A basic text for a standard college course in trigonometry, with due attention to both the numerical and theoretical aspects of the subject. The arrangement has been planned to give utmost flexibility, so as to make the book adaptable to courses of varying lengths and needs.

ANALYTIC GEOMETRY

By Ross R. MIDDLEMISS. 316 pages, $3.50

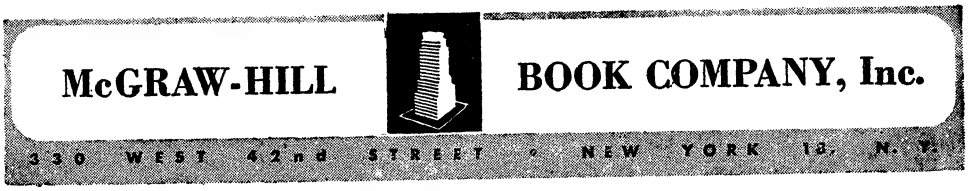
Presented in such a way as to make a maximum contribution to the general mathematical training of the student, and to give him a clear understanding of the fundamental methods of analytic geometry. More than the usual amount of attention is given to the exponential, logarithmic, trigonometric, and inverse trigonometric functions and their graphs.

DIFFERENTIAL AND INTEGRAL CALCULUS

By Ross R. MIDDLEMISS, Washington University. Second edition. 497 pages, $4.50

Offering a clear teachable presentation, the entire text has been revised to incorporate suggested improvements from the teaching of the first edition. A chapter on solid analytic geometry has been added; there are new illustrative examples; and the problems are largely new.

Send for copies on approval



from the reviews . . . .
proof of superiority

PRIMER OF COLLEGE MATHEMATICS — J. F. RANDOLPH

Julia Bower in THE AMERICAN MATHEMATICAL MONTHLY, May 1951: "This book is an interesting and well constructed text, readable, rigorous, discriminating in choice of material, and refreshing in manner of presentation. The format is good, the appearance of the pages uncrowded and attractive. . . . The author has made sure that important concepts recur and that problems and exercises give continual cumulative review. By use of this text the student should augment his knowledge by a very sound, interesting, and satisfying process. The instructor should enjoy teaching it, for it abounds with new and fruitful methods of dealing with traditional material." *1950—$4.75*

PLANE AND SPHERICAL TRIGONOMETRY — M. RICHARDSON

Cecil B. Read in SCHOOL SCIENCE & MATHEMATICS, May, 1950: "In general this text seems distinctly superior to many now on the market. It contains much more material than is the case with many books planned for a brief course, although some of this, as indicated in the text, could be omitted without destroying the continuity of the course. The reviewer was impressed by the care with which definitions were given and with which statements were so made that later work in mathematics would not contradict what was learned from the text. The explanations are often longer than customary, and should prove very helpful to the student . . . this text would seem to be one of the superior texts on the market." *1950, with tables, $3.75—without tables, $3.40*

A SHORT COURSE IN DIFFERENTIAL EQUATIONS — E. D. RAINVILLE

Cecil B. Read in SCHOOL SCIENCE & MATHEMATICS, January, 1951: "The text seems exceptionally well written, in particular one might comment favorably upon the discussion of the choice of the form in which a solution is written; likewise the care with which the author points out the need for rigorous proof of certain statements (and gives reference to which the reader may turn for such proof). There seems to be an ample supply of problems, with answers supplied for the majority of such problems. There is a brief chapter explaining hyperbolic functions, appearing before any use of these functions is made in the text." *1949—$3.00*

THE MACMILLAN COMPANY, 60 FIFTH AVE., N.Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 59 NUMBER 2

CONTENTS

Mathematics and Intellectual Honesty . . . Moses Richardson 73
The Origin of Polar Coordinates . . . . . J. L. Coolidge 78
On the Different Distances Determined by n Points . Leo Moser 85
Note on a Second Functional Equation Connected With the Function $\varphi(z)$ . . Hari das Bagchi and Phatik Chand Chatterjee 91
Mathematical Notes . . . . . D. G. Miller, Mark Lotkin, J. E. Maxfield, E. M. Wright 93
Classroom Notes . . . . . M. R. Spiegel, W. H. Witty, Richard Lyon and Morgan Ward, A. W. Walker 99
Elementary Problems and Solutions 104
Advanced Problems and Solutions 109
Recent Publications 117
Clubs and Allied Activities 122
News and Notices 127
Mathematical Association of America 133
New Members 133
March Meeting of the Southeastern Section 134
April Meeting of the Nebraska Section 139
April Meeting of the Rocky Mountain Section 142
April Meeting of the Iowa Section 144
Calendar of Future Meetings 148

FEBRUARY 1952

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL
PATRICK DU VAL
HOWARD EVES
F. A. FICKEN
J. W. GIVENS
L. M. KELLY
H. D. LARSEN
IVAN NIVEN
EDITH R. SCHNECKENBURGER
GEORGE SPRINGER
E. P. STARKE
G. B. THOMAS
E. P. VANCE

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, SAUNDERS MACLANE, University of Chicago
Honorary President, W. D. CAIRNS, Oberlin College
First Vice-President, F. L. GRIFFIN, Reed College
Second Vice-President, JEWELL H. BUSHEY, Hunter College
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo
Editor, C. B. ALLENDOERFER, University of Washington

Additional Members of the Board of Governors: E. B. ALLEN, C. F. BARR, T. A. BICKERSTAFF, J. C. BRIXEY, R. H. CAMERON, G. R. CLEMENTS, T. F. COPE, H. S. M. COXETER, W. M. DAVIS, J. M. EARL, G. M. EWING, L. R. FORD, TOMLINSON FORT, J. S. FRAME, A. E. GAULT, R. E. GILMAN, E. H. HANSON, M. R. HESTENES, AUGHTUM S. HOWARD, C. G. JAEGER, B. W. JONES, M. S. KNEBELMAN, R. E. LANGER, D. H. LEHMER, C. C. MACDUFFEE, W. E. MILNE, H. P. PETTIT, J. C. POLLEY, G. E. RAYNOR, E. B. ROESSLER, R. G. SANGER, F. H. STEEN, EARL WALDEN, MARIE J. WEISS, F. B. WILEY

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

MATHEMATICS AND INTELLECTUAL HONESTY*

MOSES RICHARDSON, Brooklyn College

An old legend tells of the explorer who lived for years among the lions and learned to speak their language. When he returned to human society, he reported that all the lions were certain that God is a lion. At one time or another, God has been nominated to honorary membership in the Mathematical Society, the Physical Society, and doubtless many others, as well as to the post of sponsor of many armies. I am aware of this universal tendency to deify the virtues of one's own kind. I am also aware of the claims of other subjects, and of the difficulty of defining Mathematics to the satisfaction of all mathematicians. Despite all this, I propose the following, if not as a definition, then at least as a partial description: *Mathematics is persistent intellectual honesty.* In the remainder of this talk, I wish to amplify this statement and point out some of its implications for teachers.

First, let us examine what Mathematics means to a modern mathematician. The most familiar example of mathematical thinking is doubtless Euclidean geometry. Let us imagine a conversation between an Unusually Patient Mathematician and an Uncommonly Intelligent Layman. The Layman begins the conversation by asking whether the Pythagorean theorem is true. The Mathematician responds by proving it in a usual way, namely by deducing it from theorems on similar triangles. The Layman perceives that his argument is of the airtight variety of strict deductive logic, but proves merely that if these theorems on similar triangles were true then so would the Pythagorean theorem be true. The Mathematician then deduces these from still simpler propositions, and so on. But since he must avoid both circularity and infinite regress, he must stop ultimately and base the entire discussion on a set of primitive propositions, called postulates, which are left without proof. The Layman then admits that if these postulates were true so would be the Pythagorean theorem, but contends that he is puzzled because he does not understand the meaning of certain terms which were used, such as "triangle." The Mathematician responds by defining "triangle" in the usual way in terms of "point," "line-segment," *etc.* The Layman says he would now understand the meaning of "triangle" if he were not puzzled about the meaning of the terms "line-segment," *etc.* The Mathematician then defines these in terms of still more primitive terms, and so on. But again, since he must avoid both circularity and infinite regress, he must stop ultimately and base all his definitions on a set of primitive terms which are left undefined.

At this point, he has converted Euclidean geometry into an *abstract mathematical science*, that is, a collection of statements of which some (the postulates) are unproved, but all others (the theorems) are logical consequences of these, and in which some terms are undefined but all others are defined in terms of

* An address delivered at the first meeting of the Association of Mathematics Teachers of New York State held at Syracuse University on May 12, 1951.

these.

The Layman then asks, "Are there objects in the universe for which these propositions are true?"

The Mathematician then answers, "We have proved that if there were objects such as points, etc., satisfying the postulates, then the theorems would be true of them too. This is all that Pure Mathematics is concerned with. Questions of truth belong to Applied Mathematics. Since the validity of a deductive argument is independent of the meanings of the terms involved, we regard our undefined terms as dummy symbols in Pure Mathematics. Thus we might write 'two x's determine a y' instead of 'two points determine a line' in the abstract mathematical science of Euclidean geometry. *Pure Mathematics* is the totality of abstract mathematical sciences. If concrete meanings are substituted for the undefined terms in an abstract mathematical science, then we have before us a *concrete interpretation* or *application* of it. *Applied Mathematics* is the totality of such concrete interpretations, and *Mathematics* consists of both Pure and Applied Mathematics. If we somehow knew that for a given interpretation the postulates were true, then we would be sure of the truth of the theorems also. But we are seldom certain concerning the truth of our postulates. We try to make them as simple and as plausible as possible, but we cannot guarantee their truth. In Applied Mathematics we use our postulates as good working hypotheses as long as they produce usable theorems, but we are unable to promise that their usefulness will be eternal. If a scientific hypothesis implies a single conclusion that is not in accord with observation, then we know decisively that it is not true as it stands. If all its implications so far drawn do appear to be in accord with observation, we cannot know decisively that it is true, but can only regard it *pro tempore* as a good working hypothesis."

The Layman then asks, "But is not this situation the same in other subjects as well as in Mathematics?"

The Mathematician responds, "I think you are right. If you attempt to organize any body of subject matter logically, you will ultimately cast it in the form of an abstract mathematical science, and if you apply it to reality you will use a concrete interpretation of this abstract mathematical science. It is in this sense that Mathematics is basic to all sciences even more than in the old sense of Mathematics as the science of space and quantity. The mural in the Hall of Science at the Chicago World's Fair, entitled the Tree of Knowledge, exhibits this notion by placing Mathematics at the base of the Tree, the fundamental sciences as the older, better developed branches of the Tree, and the practical arts and applied sciences as the younger, less developed branches which must draw their strength from below. With our definition of Pure and Applied Mathematics, it is true that all deductively organized sciences ultimately become branches of Mathematics. This has been considered unfortunate even by some mathematicians, but attempts to restrict the meaning of the word 'Mathematics' restrict it too much. Rather than throw out the baby with the bath water, we have grudgingly become reconciled to the embracing nature of our subject."

The Layman then says, "You have indeed opened inspiring vistas to me, and I begin to perceive why mathematicians are so enamored of their subject. But you have not yet told me whether the Pythagorean theorem is true."

The Mathematician says, "I have shown you that if there existed objects in reality which had the properties demanded of points, *etc.*, in the postulates of Euclidean geometry, then the Pythagorean theorem would be true of these objects. While in most situations Euclidean geometry seems to work well enough for practical purposes, I cannot assert that there are such objects."

The Layman then asks, "But did not the Greeks say that the axioms of geometry were true because they were self-evident?"

The Mathematician responds, "This was a weakness on their part which modern mathematicians have outgrown. The history of science is a road strewn with the decaying bones of assumptions which were once considered self-evident and later found to be false. In fact, much fundamental progress has been made by questioning assumptions which sometimes were considered so self-evident that they were made tacitly. For the practical purposes of geometry other postulates may be used as well as Euclid's, and these non-Euclidean geometries contradict Euclidean geometry in many places. I am afraid that if you mean the word 'truth' in the somewhat naive sense of absolute truth, then I cannot assert that Euclidean geometry is true. Pontius Pilate is said to have asked 'What is truth?' long ago, and, so far as I know, no thoroughly satisfactory answer has been given. Fortunately, science progresses without requiring absolute truth. Its successive theories provide better and better approximations to truth in the sense that a new theory must give correct predictions of a more inclusive set of observational facts than the theory it supplants. Every scientist knows how difficult it is to discover truth even in this tentative evolving sense, *a fortiori* in any absolute sense. It might be a better world if others besides scientists were more aware of this difficulty, for it is an excellent antidote for fanatical dogmatism. Throughout history, people who have persuaded themselves that they were already in possession of absolute truth have tended to suppress free inquiry and censor free expression, sometimes on the somewhat surprising grounds that, while the absolute truth was self-evident to them, others would be easily misled by arguments supporting contrary views, and have exhibited at times a distressing willingness to kill those who disagreed with them. A mature individual must learn to live with some unavoidable uncertainty. I recall an old comic strip character who said, 'It ain't ignorance that causes all the trouble—it's the things people know that ain't so'."

Our Layman, who is, you must remember, an Uncommonly Intelligent Layman, then says, "You have shown me that Mathematics is indeed the epitome of intellectual honesty, carried to rather unusual lengths. You do not assert more than you can prove. Even when you deduce your theorems from postulates which you could easily persuade me to believe to be self-evident truths, you warn me against believing this, and point out and investigate the possibility of alternative assumptions. Your insistence on explicit statement of assumptions

and undefined terms, on strict logical proof of theorems, and on clarity in definitions is inspiring. Mathematics is the antithesis of dogmatism and fanaticism and authoritarianism, and I suspect it is no coincidence that authoritarians have often fought and even liquidated mathematicians and other scientists. I intend to study mathematics further as soon as I can find the time."

At this happy outcome, let us take leave of our Mathematician and our Layman, because I am afraid of what might happen if we were to follow our Layman's subsequent career in various mathematics classrooms. I fear greatly that he might not find these ideals of intellectual honesty very much in evidence in every single classroom.

Bertrand Russell remarks somewhere that one of the hardest things in the world for a man to do is to stand before an audience and refrain from saying more than he knows. To be sure, we may all sometimes make a hasty statement or an honest mistake. I am not asking for perfection. But I fear that the teaching of mathematics and of natural science does not always live up to the high ideals that our Unusually Patient Mathematician has been expounding. The liberating influence of scientific rationalism seems nowadays to be lost too often, and instruction turns into authoritarian dicta on the part of the teacher, and regurgitated undigested rote responses on the part of the student. A college president once remarked within my hearing that after all, geometry was mostly a matter of memory. Is it possible that he could have been referring to the same study that I have been using as my prototype of mathematics today? I am not at all impugning the honesty of his reaction or his recollection of his own studies. I fear that he may well have been entirely accurate about the way in which he was taught geometry. But I deny that it need be so. I have met teachers of secondary mathematics who apparently taught geometry with due traditional regard for reasoning, but routinized their instruction in algebra and trigonometry until these subjects became nothing more than an elaborate game of tic-tac-toe with symbols. The same thing can be said of some college teachers who turn algebra and trigonometry and even calculus and differential equations into cook book courses. I deny that this need be so.

It is well known that men have a striking ability to compartmentalize their minds. People who are habitually honest critical thinkers in their mathematical work may fall far short of that ideal in other subjects, or in their personal dealings which may be governed by emotional reactions, ingrained prejudices, jealousies, *etc.* Even people who are sound critical thinkers in one branch of mathematics may fall short of that ideal in other branches of mathematics itself. I do not contend that the study of mathematics will automatically produce intellectually honest people. Nevertheless, I do contend that mathematics is the ideal subject in which to point out to the student the virtues of intellectual honesty. In this age of the advertising man and his more vicious cousin, the propaganda technician, is there any more important function for education to perform?

But this can be done only if mathematics is properly taught. Ideals of intel-

lectual honesty can be communicated to the student by instruction and by example only if the teacher is himself saturated with them and will seize the abundant opportunities to stress them in the classroom. However, if mathematics is presented as meaningless rote learning, as mere drill, pure unrationalized technique, routine mechanical manipulation of symbols justified only by the teacher's authority, will these ideals be served? If incorrect arguments are presented as valid proofs whose invalidity may or may not be disclosed in subsequent courses, will these ideals be served? I recall a remark of H. G. Forder, asking if it can be contended that "an unsound proof has an educational value not possessed by a sound one." I do not mean at all that all instruction must be rigorous. I have no objection whatever to omitting too difficult proofs and replacing them by properly labelled informal heuristic discussions. I mean merely that unless a proof is sound, it should not be presented as if it were. Otherwise the teacher is in a somewhat less defensible position than a passer of counterfeit currency whose innocent dupes suffer only in material wealth.

Mathematics should be taught so as to give the student:

1. An appreciation of the natural origin and evolutionary growth of the basic mathematical ideas;

2. A critical logical attitude, and a wholesome respect for correct reasoning, precise definitions, and clear grasp of underlying assumptions;

3. An understanding of the role of mathematics as one of the major branches of human endeavor and its relations with other branches of the accumulated wisdom of the human race;

4. A discussion of some of the important problems of pure mathematics and its applications;

5. An understanding of the nature and practical importance of postulational thinking.

An attempt should be made to emphasize the distinction between familiarity and understanding, between logical proof and routine manipulation, between a critical attitude of mind and habitual unquestioning belief, between scientific knowledge and both encyclopedic collections of facts and mere opinion and conjecture.

These remarks are intended to apply not merely to teachers of mathematics at secondary and collegiate levels, but also to those who are training such teachers. If a teacher is to carry out these purposes, he must have the background for it. If you will pardon me for perverting the modern educational scripture, he must not only know the student, he must also know his subject. By this I do not mean that a secondary teacher needs to know a great deal about differential geometry or topology or the calculus of variations. But he should be well grounded in the fundamental concepts of elementary mathematics. Unfortunately, he is frequently expected to absorb these fundamentals by osmosis. Too often, this process does not take place, and the prospective teacher emerges from his mathematics major with an equipment of heterogeneous techniques,

and little or no understanding of the historical development, philosophy, fundamental concepts, nature, spirit, or even *raison d'être* of his subject.

A student in a democracy should learn that the worth of a scientific theory is not to be determined by authoritarian dictates, nor even by popular acclaim, but only by dispassionate and free inquiry into the evidence for and against it and for and against possible alternative theories. This is a difficult task in controversial subjects. What better training can there be for this task than initiation into intellectual honesty through mathematics where controversy is comparatively absent and where standards of intellectual honesty are so high? And although there are certainly other objectives as well involved in the teaching of mathematics, is there a more important aim to be served in this perilous age when freedom of thought and expression is being assailed on all sides?

Bibliography

C. B. Allendoerfer, Mathematics for liberal arts students, this MONTHLY, vol. 54, 1947, pp. 573–578.

E. T. Bell, The Search for Truth, Reynal and Hitchcock, 1934.

C. J. Keyser, The Human Worth of Rigorous Thinking, Columbia University Press, 1925.

E. P. Northrop, Mathematics in a liberal education, this MONTHLY, vol. 52, 1945, pp. 132–135.

F. S. Nowlan, Objectives in the teaching of college mathematics, this MONTHLY, vol. 57, 1950, pp. 73–82.

O. Ore, Mathematics for students of the humanities, this MONTHLY, vol. 51, 1944, pp. 453–458.

M. Richardson, On the teaching of elementary mathematics, this MONTHLY, vol. 49, 1942, pp. 498–505.

M. Richardson, The place of mathematics in a liberal education, Nat. Math. Magazine, vol. 19, 1945, pp. 349–358.

M. Richardson, Fundamentals in the teaching of undergraduate mathematics, this MONTHLY, vol. 58, 1951, pp. 182–186.

B. Russell, Human Knowledge, Its Scope and Limits, Simon and Schuster, 1948.

B. Russell, The Functions of a Teacher, in Unpopular Essays, Simon and Schuster, 1950.

THE ORIGIN OF POLAR COORDINATES

J. L. COOLIDGE, Harvard University

1. The transformation of curves. The student of analytic geometry is likely to run across polar coordinates at some time during his first year's study of the subject, first in the plane, later, perhaps, in space. He may note a certain variety of definition, some writers allowing the radius vector to be negative, others insisting that only positive values are allowable. Except for this possible ambiguity, the interpretation and utility of such coordinates are well established.

The essential feature in the plane case is that the position of a point is perfectly established by the values of two parameters, a distance and an angle. The introduction of this angle is the essential feature wherein the parameters differ from those adopted by Fermat and Descartes. We have a new coordinate system,

applicable to every point in the plane. It is therefore a curious fact that considerable application was made of these coordinates to the transformation of particular curves without noticing that what was involved was a new technique, applicable to a variety of different questions. The first writer who should be mentioned in this connection is Bonaventura Cavalieri. In 1635 he published his highly original *Geometria indivisilibus continuorum.* In 1653 a second edition appeared in Bonn [1]. This is described in the words "*In hac postrema editione, ab erroribus expurgata*" which leads one to conclude that errors in the first edition were corrected but there is no suggestion that new material was introduced. I therefore accept the thesis that the edition of 1635 included the material of Book VI which deals with the spiral of Archimedes. He seeks especially the area within this curve. I will greatly shorten the labor by using differentials which, unfortunately, were first introduced nearly half a century later. Let the spiral have the equation which we should write

$$r = l\phi$$

We compare this with the parabola

$$y^2 = 2lx$$

The relation between the two curves may be written

$$(1) \qquad y = r, \qquad x = \frac{r\phi}{2}, \qquad dx = rd\phi$$

$$(2) \qquad \tfrac{1}{2}r^2d\phi = xdy.$$

This shows that the area inside the spiral is equal to that outside the parabola. If we integrate from 0 to 2π and remember that since the time of the Greeks it was known that the area outside a parabola, beginning at the vertex, is one third the product of the Cartesian coordinates of the end point, that is to say, one third the area of the including rectangle, we get Cavalieri's principal theorem, namely, that the area inside the first turn of the Archimedian spiral is one third of that of the circle whose radius is the radius vector to the end of the first turn. This is Archimedes' 24th proposition in his book on spirals.

Unfortunately there remains the slight possibility that Cavalieri was anticipated in this. In 1647 Saint Vincent brought out his *Opus Geometricum* with the claim that he had long been familiar with the method and had, in fact, explained it in 1625 to a certain Christopher Grienberger, who, unfortunately died in 1636 and so was not available in 1647. I will not go further into this unfortunate priority question, which is discussed at some length in Cantor [2].

There was a good deal of interest in the comparison of these curves, one of the curious being Pascal. He tells us in 1658 that Hobbes had supposed that the length of a parabolic arch was equal to a certain line segment but that Roberval had proved it equal to the arc of an Archimedian spiral. He goes on to say that Roberval's demonstration was not "Absoluement convainçaint" and some

geometers believed that he had made a mistake. Pascal himself did not share their doubts, but that in order to convince the doubters, he would give a demonstration of his own [3]. This is based on Cavalieri's transformation, though I find no mention of Cavalieri, nor have I found the appropriate passage in Roberval. We are told that Pascal's demonstration was much admired; it was based on purely classical methods. When we come to writing differentials there is nothing to it.

$$\frac{rd\phi}{dr} = \frac{r}{l} = \frac{y}{l}. \tag{3}$$

But it was known since the time of Apollonius that this represented the tangent of the angle between the ordinate of the parabola and the tangent. At corresponding points on the two curves there were equal angles between the tangent to the curve and the ordinate or radius vector. But the elemert of arc of the curve is $dr=dy$ times the secant of this angle. Hence corresponding arc lengths are equal; Pascal mentions the equality of angles as a corollary.

The same transformation was introduced in 1668 by James Gregory in his *Geometriae, pars universales* [4]. Here is a general transformation between two curves called involutio and evolutio. The general idea is to start with a curve $y=f(x)$ in rectangular coordinates, then bend the axis in such a way that its length is not altered but that ordinates perpendicular to this line become concurrent. Here we have

$$y = r, \qquad dx = rd\phi$$
$$\int ydx = \int r^2 d\phi \tag{4}$$

so that the area under the first curve is double the second. On the other hand

$$\frac{dr}{rd\phi} = \frac{dy}{dx},$$

so that corresponding angles between curve and ordinate or radius vector are equal. The equality of corresponding lengths comes also from

$$dx^2 + dy^2 = r^2 d\phi^2 + dr^2.$$

I should also mention at this place that polar coordinates were also known to Johann Bernoulli in discussing the spiral of Archimedes [5]. The logarithmic spiral was first found about this time by Descartes [6]. There is a very long discussion of various spirals by Varignon [7], based on the transformation

$$y = r, \qquad x = l\phi. \tag{5}$$

Here I should mention an error of Wieleitner who writes: "Bemerkenswerth ist dass er bei Behandlung von gewissen abgeleiteten Kurven die er, nach Jak

Gregory, Involuten und Evoluten nennt, ebenfalls in Anschlusse an Gregory diejenige Transformation von rechtwinkeligen zu Polarkoordinaten anwendet die wir heute mittels $a\phi = x$, $\rho = y$ ausdrucken und die bisher P. Varignon zuge schrieben wird" [8]. This is certainly Varignon's transformation [5] but not Gregory's [4].

2. Polar coordinates in the plane. The first writer to visualize polar coordinates as a means of locating any point in the plane appears to be Newton [9]. My attention to this was called by Boyer [10]. The work [9] appeared in 1736 but was undoubtedly composed in 1671 and the underlying ideas had been in the author's mind for some half dozen years already. A large section is devoted to the determination of tangents. To us this means finding the equation of the tangent, but to writers of the period from Fermat on, the essential thing was to construct the tangent by finding the subtangent, or the polar subtangent, a length cut on a perpendicular to the radius vector. Newton gives a good deal of attention to the case where he starts with the Cartesian equation, finding the answer in terms of the "fluxions" of x and y, and he remarks [9, p. 51]: "And from hence (I imagine) it will be sufficiently manifest by what methods the tangents to all curves are to be drawn. However, it may not be foreign to our purpose if I also show how the Problem may be performed when the curves are referred to right lines after any other method whatsoever, so that having the choice of several methods, the easiest and most simple may always be used."

We have here involved the very radical idea of using different sorts of parameters to locate a point in the plane, an idea which I shall return to in connection with Leibniz. Newton uses in connection with different curves the coordinates which we should write (r, y) the biopolar ones (r_1, r_2) $(r, f(\phi))$ where f is a simple trigonometric function, and, for spirals $(r, a\phi)$ where a is a constant. But Newton does not seem to have gone further in developing these coordinates, so I turn to the man who is usually put down as their discoverer. This is James Bernoulli, who sees in them a means of locating any point in the plane. In the first article he is interested in showing the application of the new calculus of differences of Leibniz to that curve which is like a parabola except that the ordinates spring from a circle, not a straight line [11]. He uses x and y for his new coordinates, so we should write

$$y = a - r, \qquad x = a\phi, \qquad y^2 = lx. \tag{6}$$

We might say that the radius vector does not spring from a fixed point but from a fixed circle, a correction which he makes later, but on pp. 436–437 we find something really significant:

"Differentiale arcus ad differentiale tangentis et secantis rationem habent cognitam; puta; ad differentiale tangentis quam quadratum radii ad quadratum secantis; et ad differentiale secantis quam quadratum radii ad rectangulum sub tangenti et secante."

We should express this by the equations

$$d \tan \phi = \sec^2 \phi d\phi; \qquad d \sec \phi = \sec \phi \tan \phi d\phi.$$

He speaks of these as known formulae. Who discovered them? The first is found in Barrow's Geometrical Lectures, Lecture X, example 5. Did Bernoulli know of Barrow? This is not impossible as he speaks of Barrow in an article immediately following where, incidentally, his radius vectors start from a fixed point, but it would be interesting to find out just who was the first to differentiate the elementary trigonometric functions. The credit for this is usually given to a considerably later writer, Roger Cotes [12]. Soon after this article Bernoulli returns to polar coordinates in an article dealing with an ostensibly very different subject, the curvature of an elastic lamina [13]. Here he calculates the radius of the osculating circle to the general curve in polar coordinates. He writes y where we should write r, and x where we should write $a\phi$. If x be taken as independent variable he writes for the radius of curvature

$$\frac{adr^3}{dxdr^2 + dxdy^2 - ydxd^2y}.$$

He gives no proof of this but one is easily supplied when one remembers that the radius of curvature is

$$\frac{ds}{d\phi + d\chi}; \qquad \tan \chi = \frac{rd\phi}{dr}.$$

Except for Newton's determination of the tangent this formula of 1694 seems to be the first use of polar coordinates in connection with the general curve.

It is time to make a short mention of Leibniz. He writes in 1692: [14]: "*Ordinatim applicatas* vocare solent geometrae rectas quotcunque inter se parallelas, quae a curve ad rectam quandam (directricem) usque ducuntur, quare cum ad directricem (tamquam axem) sunt normales, solent vocari ordinatae κατ'εοχην. Desarguesius rem prolatavit et sub ordinatim applicatis etiam comprehendit rectas *convergentes* and unum punctum commune aut ab eo *divergentes*. Et sane parallelae sub convergentibus aut divergentibus comprehendi possunt, fingendo punctum concursus infinite ab hic distare. Verum quia multis aliis modis fieri potest, ut infinitae dici intelligantur lineae secundum legem legem quandam communem, quae tamm non sint parallelae, vel convergentes ad punctum, aut a puncto omnibus commune divergentes, ideo nos tales lineas generaliter vocabimus *ordinatim ductas*, vel ordinatim (positione) datus." He means by this that lines constructed by any established rule may be called ordinates, and presently gives as an example solar rays that have been reflected or refracted according to some definite law. He has a much wider view of what may be called ordinates, though they are still straight lines. There is no reason to suppose that Desargues had any idea of polar coordinates, though he did clearly realize the concept of a point at infinity.

Before passing into three dimensions, I should just mention Jacob Hermann. [15]. His three coordinates are r, $r\cos\phi$, $r\sin\phi$ not far from the Cartesian system.

The meaning of polar coordinates was also well expressed by Euler somewhat later, that is to say in 1748, when he wrote "Natura harum curvarum commodissime explicatur per aequationem inter cujusque Corvae Puncti M a centro C distantium CM, et angulum ACM quaere haec recta cum recta positione data CA constituit" [16].

3. Polar coordinates in space. The development of polar coordinates in three space lags well behind that in the plane. This is not surprising. Plato in the *Republic* complains bitterly of the "ridiculous state" of three dimensional geometry compared to the high development of geometry "in plano" and a corresponding lag is found in the analytic geometry of Descartes. Take our present case of polar coordinates. There is some doubt as to even what the words mean. A. P. Rogers in a note to the fifth edition of Salmon's Geometry of three dimensions distinguishes between polar coordinates, and radio-angular ones, and I will maintain his distinction, though with some doubts as to its validity.

It is usual to give a credit for discussing either of these systems to Jean Baptiste Clairaut, though this apparently is for what he might have done, not what he actually accomplished which seems to have been nothing at all. We find him writing: [17] "A l'égard des courbes à double courbure, dont les coordonnées partent d'un point, ou dont les coordonnées représentent des lignes courbes, elles demandent une méthode particuliere, et peuvent fournier le sujet d'un autre Traité que je compte donner au public." The public never saw his proposed Traité, perhaps the eighteen year old author was frightened at the thought of undertaking it. He apparently had in mind "lineae ordinatim applicatae" that radiated from a fixed point, how he meant to deal with the angles does not appear.

We naturally turn to Euler. Strangely enough we find nothing in his *Mechanica* of 1736, but in the *Introductio* of 1748 we find the famous Euler angles ζ and θ [18]. He introduces these in various places, notably in [19] where we have the relations to Cartesian coordinates expressed by the equations.

$$(7) \qquad x = t - s\sin\theta\sin\zeta, \qquad y = u - s\sin\theta\cos\zeta, \qquad z = v - s\cos\theta.$$

Two years later Lagrange brings in the same variables, with slightly different notation with the comment "Une des transformations les plus utiles et les plus ordinaires" [20].

When it comes to radio angular coordinates we must turn to Euler again. We find in his "*Theoria motus corporum*" [21] the angles α, β, γ connected by the obvious equation

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

so that

$$x = r \cos \alpha, \qquad y = r \cos \beta, \qquad z = r \cos \gamma.$$

This notation became classic, especially with French writers. Half a century later we find Monge writing the coordinates of the foot of the perpendicular from the origin on the plane

$$Ax + By + Cz + D = 0$$

in the curious form

$$\cos = \frac{C}{\sqrt{A^2 + B^2 + C^2}}; \qquad \cos = \frac{B}{\sqrt{A^2 + B^2 + C^2}}; \qquad \cos = \frac{A}{\sqrt{A^2 + B^2 + C^2}}.$$

As between polar and radio-angular coordinates it is fair to notice that the former are more compact and involve only a minimum number of variables, the latter are more symmetric, and in fact are called by some writers "symmetric polar coordinates." In conclusion it is perhaps worth mentioning another compact form of coordinates due to Ossian Bonnet [23]. They are useful in studying the geometry of the oriented plane [24].

$$x = \frac{r(\alpha + \beta)}{\alpha\beta + 1}, \qquad y = \frac{ri(\beta - \alpha)}{\alpha\beta + 1}, \qquad z = \frac{r(\alpha\beta - 1)}{\alpha\beta + 1}.$$

The independent variables are, two of them, the parameters for the rectilinear generators of a sphere, which may seem a bit esoteric to some geometers. We may compare the squared distance element in the three systems

$$d^2 = ds^2 + s^2(d\theta^2 + \sin^2 \theta d\zeta^2)$$

$$d^2 = dr^2 + r^2(\sin^2 \alpha d\alpha^2 + \sin^2 \beta d\beta^2 \gamma d\gamma^2)$$

$$d^2 = dr^2 + \frac{4s^2 d\alpha d\beta}{(\alpha\beta + 1)^2}.$$

References

1. Geometria indivisilibus continuorum, P. Bonaventura Cavalieri Bonn, 1653, p. 436.
2. Cantor, Geschichte der Mathematik, Second Ed., vol. 2, Leipzig, 1900, p. 850.
3. Pascal, Oeuvres, Edition Brunschwicg, Boutroux aud Grazier, vol. VIII, Paris, 1914, pp. 250 ff.
4. James Gregory, Tercentenary Memorial Volume, London, 1939, pp. 493 ff.
5. Die Differentialrechnung von Johann Bernoulli im Jahre 1691–92, Leipzig, 1926, p. 27.
6. Descartes, Oeuvres, Edition of Adam and Tannery, vol. II Correspondence, Paris, 1898, p. 360.
7. Varignon, Nouvelle formation des spirales, Histoire de l'Académie Royale des Sciences Year 1704, Paris, 1745, pp. 70, 71.
8. Wieleitner, Geschichte der Mathematik, Part II, vol. 2, Berlin, 1921, p. 23.
9. Newton, The Method of Fluxions, London, 1736.

10. Boyer, Newton as an Originator of Polar Coördinates, this MONTHLY, vol. 56, 1949, pp. 73–78.

11. Bernoulli, Spécimen calculi differentialis in dimensione parabolae helicoidis, Acta Eruditorum, 1691, Opera Genva 1744, p. 431.

12. Cotes, Harmonia mensurarum, London, 1722. Cf. Cantor, Vol. I, Second Ed., vol. III, p. 413 and Von Braunmühl, Geschichte der Trigonometrie, Leipzig, 1903, vol. II p. 79.

13. Bernoulli, Curvatura laminale elasticae, Acta Eruditorum, 1694, Opera vol. II, p. 578.

14. Leibniz, De lineis in lineis in numero ordinatim ductis, Acta Eruditorum, Leipzig, 1692, Schriften, Edition Gerhardt, Part 2, vol. I, p. 266, Leipzig, 1858.

15. Jacob Hermann, Consideration Curvarum etc. Commentarii Academiae Petropolitanae IV, 1729, pp. 37 ff.

16. Euler, Introductio in Analysin infinitorum, vol. II, p. 301 Opera IX, p. 305.

17. Clairaut, Recherches sur les courbes à double courbure, Paris, 1731, last page of Introduction.

18. Euler, Introductio in Analysin infinitorum, vol. II, p. 366, Opera XI, p. 349.

19. Euler, De solidis quorum superficies in planum explicare licet, Novi Commentarii Petropolitanae, vol. 16, 1771, p. 11.

20. Lagrange, Sur l'attraction des sphèroides elliptiques, Memoires de l'Académie de Berlin, 1773, Oeuvres vol. 3, Paris, 1869, p. 305.

21. Euler, Theoris Motus corporum, Opera Series 2, vol. 3, p. 307.

22. Monge, Feuilles d'analyse appliquées à la géometrie, Paris, 1801, pp. 5, 6.

23. Bonnet, Memoires sur l'emploi d'un nouveau système de variables, Liouville, vol. 5, 1860. Much better in Darboux, Théorie générale des surface, vol. I, Paris, 1887, pp. 245 ff.

24. Coolidge, Equilong Transformation of Space, Transactions American Math. Soc., vol. IX, 1908.

ON THE DIFFERENT DISTANCES DETERMINED BY n POINTS*

LEO MOSER, University of Alberta

1. Introduction. n distinct points will always determine $\binom{n}{2}$ distances. However, many of these may be the same. We are concerned here with the number of different distances determined. Let $f(n)$ be the least number of different distances determined by n points in a plane. The vertices of an equilateral triangle show that $f(3)=1$ and from the square and the regular pentagon one easily sees that $f(4)=f(5)=2$. P. Erdös** has shown recently that

$$cn/\sqrt{\log n} > f(n) > \sqrt{n-1}-1, \tag{1}$$

where c is a fixed constant. The upper bound was obtained by considering the points of a square lattice. Erdös conjectured that $f(n)>n^{1-\epsilon}$ for every $\epsilon>0$ and n sufficiently large. The smallness of the lower bound is therefore rather striking.

* The author wishes to express his thanks to Professor A. Brauer for his valuable suggestions and kind assistance in preparing the manuscript.

** P. Erdös. On Sets of Distances of n Points. this MONTHLY, pp. 248–250, Vol. 53, 1946.

Nevertheless Erdös wrote that he had long sought to improve it without success. In Section 2 we will prove Theorem 1:

$$f(n) > \frac{n^{2/3}}{2\sqrt[3]{9}} - 1. \tag{2}$$

If the n points form the vertices of a convex polygon, then the least number $f^*(n)$ of different distances determined is much larger. In this case Erdös conjectured that

$$f^*(n) = [n/2], \tag{3}$$

which is strongly suggested by the vertices of the regular polygons, which show that $f^*(n) \leqq [n/2]$. In Section 3 we will prove Theorem II:

$$f^*(n) \geqq [(n+2)/3]. \tag{4}$$

2. Case of n arbitrary points. We first prove two lemmas.

LEMMA 1. *Let r be a positive integer, ϵ a real number, $0<\epsilon\leqq 1$. Let P be the point $(r+\epsilon, 0)$ and Q and R two points in the first quadrant, equidistant from P, and whose distances from the origin lie between r and $r+1$. Then $\overline{QR}<2$.* Note Figure 1.

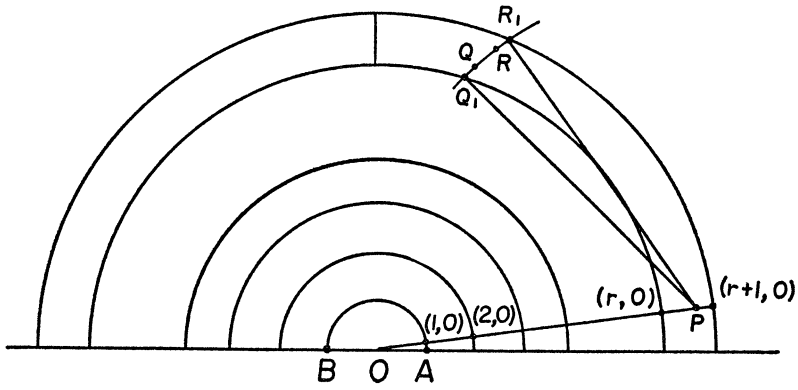


FIG. 1

Proof. Let the circle with center at P and passing through Q and R, cut the circles $x^2+y^2=r^2$ and $x^2+y^2=(r+1)^2$ in $Q_1(x, y)$ and $R_1(x+\Delta x, y+\Delta y)$, respectively. Clearly $\overline{Q_1R_1} \geqq \overline{AR}$ so it will suffice to show that $\overline{Q_1R_1}<2$ or $\Delta x^2+\Delta y^2<4$. We have

$$x^2 + y^2 = r^2 \tag{5}$$

$$(x + \Delta x)^2 + (y + \Delta y)^2 = (r + 1)^2 \tag{6}$$

while $\overline{PQ_1}=\overline{PR_1}$ yields

$$(x - r - \epsilon)^2 + y^2 = (x + \Delta x - r - \epsilon)^2 + (y + \Delta y)^2. \tag{7}$$

Simplification of (7), using (5) and (6), yields

$$\Delta x = 1 + \frac{1 - 2\epsilon}{2r + 2\epsilon}, \tag{8}$$

so that

$$1 - \frac{1}{2r} < \Delta x < 1 + \frac{1}{2r} \leqq 3/2. \tag{9}$$

It is interesting to note that Δx is independent of $\overline{PR}$. From (5) and (6) we have

$$2x \cdot \Delta x + 2y \cdot \Delta y + \Delta x^2 + \Delta y^2 = 2r + 1. \tag{10}$$

Further, we may assume $\Delta x^2 + \Delta x^2 > 3$, for otherwise our lemma is proved. Hence

$$x \cdot \Delta x + y \cdot \Delta y < r - 1. \tag{11}$$

Now, using the left hand side of (9) to estimate Δx and the fact that $x \leqq r$ and $x \geqq 0$, we obtain from (11)

$$\Delta y < \frac{r - 1 - x \cdot \Delta x}{y} < \frac{r - x}{y} = \frac{r - x}{\sqrt{r^2 - x^2}} = \sqrt{\frac{r - x}{r + x}} \leqq 1. \tag{12}$$

The lemma now follows from (12) and the right hand side of (9).

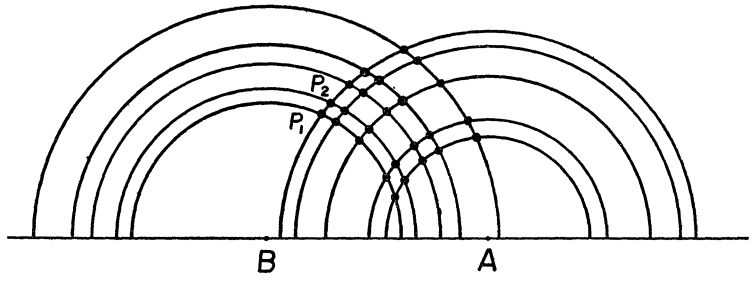


FIG. 2

LEMMA 2. *Given two points A and B and n other points $P_1, P_2, \cdots, P_n$, lying on, or to one side of the line AB. Of the distances $AP_1, AP_2, \cdots, AP_n$, $BP_1, BP_2, \cdots, BP_n$, at least $\sqrt{n}$ are distinct.*

Proof. Consider all semicircles to one side of AB having centers at A and B passing through the points $P_1, P_2, \cdots, P_n$ (Fig. 2). Let the number of distinct semicircles with centers at A and B be a and b, respectively. If $\max\{a, b\} \geqq \sqrt{n}$ the lemma is clearly true. If, however, $a < \sqrt{n}$ and $b < \sqrt{n}$ then the number of intersection points is at most $a \cdot b < \sqrt{n} \cdot \sqrt{n} = n$, since any two of the semicircles intersect at most once. But this yields a contradiction since each of the n distinct points $P_1, P_2, \cdots, P_n$ is an intersection point of two of these semicircles.

Hence the lemma is proved.

We note that this lemma is best possible in that, in it, it is not possible in general to replace $\sqrt{n}$ by a larger number. Further, we note that by choosing A and B to be consecutive points on the convex cover of the set of n points the lemma yields

$$f(n) \geqq \sqrt{n-2}, \tag{13}$$

which is already a little better than the right hand side of (1). Indeed, from (13) and the regular heptagon we may deduce that $f(7)=3$.

We proceed to the proof of Theorem I. Let A and B be two points determining the minimum distance in the set of n points. Let us denote this distance by 2. One of the half planes determined by AB, including the points on the line AB, contains at least $n/2$ points. Henceforth we will deal only with these points. Let O be the midpoint of AB. With center at O we construct semicircles of radii 1, 2, 3, $\cdots$ cutting the half plane into half-annular boxes which we make open on the inside and closed on the outside (Fig. 1). Let s be a number, not necessarily an integer which for the present we restrict only by $1 \leqq s \leqq n$. We consider 2 cases.

Case 1. Some box contains at least s points.

Case 2. No box contains as many as s points.

In Case 1, we take a box containing at least s points. This we cut into two equal parts by a line through O, and retain only one half which contain in the interior or on its boundary at least $s/2$ points. Let P be a point in this region (Fig. 1) making the angle AOP as small as possible. Suppose there are two other points Q and R in the region equidistant from P. By Lemma 1 we have $QR<2$, which contradicts the fact that 2 is the minimum distance determined. Hence, no two points in the region are equidistant from P. We obtain therefore at least $s/2-1$ different distances determined from P so that, in Case 1, we have

$$f(n) \geqq \frac{s}{2} - 1. \tag{14}$$

In Case 2, we divide the half-annular boxes into 3 classes, putting a box in class i if its outer radius $r \equiv i \pmod 3$, $i=1, 2, 3$. Now if A and B are excluded, at least one of the classes will contain at least the following number of points:

$$\frac{(n-2)}{2} \cdot \frac{1}{3} = \frac{n-2}{6}.$$

We retain now only A and B and the points of such a class of boxes. If d is any distance determined between A or B and any of the points of our class in a box outer radius r, then,

$$r - 2 < d \leqq r + 1,$$

while for a distance d' determined between A or B and a point of a neighboring box of the class, say the one of outer radius $r+3$,

$$r + 1 < d' \leqq r + 4.$$

Hence, there will be no overlap of distances determined between A and B and the points of the retained ith and jth boxes, respectively, for i. e. $i \neq j$. Suppose now that there remain A and B and t non-empty boxes containing $n_1, n_2, \cdots, n_t$ points. Then by our Lemma 2, and the above remarks, we have

$$f(n) \geqq \sqrt{n_1} + \sqrt{n_2} + \cdots + \sqrt{n_t}, \tag{15}$$

$$\frac{n-2}{6} \leqq n_1 + n_2 + \cdots + n_t, \tag{16}$$

moreover

$$s > n_i, \qquad i = 1, 2, 3, \cdots, t, \tag{17}$$

since no box contains as many as s points.

Now, by a well known inequality,

$$\sqrt{n_i} + \sqrt{n_j} \leqq 2\sqrt{\frac{n_i + n_j}{2}}, \tag{18}$$

so that the right hand side of (15) is not increased by replacing n_i and n_j by their arithmetic mean. Hence, in seeking a lower bound for $f(n)$, we may take

$$n_1 = n_2 = \cdots = n_t = \frac{n-2}{6t}. \tag{19}$$

By (15) and (19), we have

$$f(n) \geqq t\sqrt{\frac{n-2}{6t}} = \sqrt{\frac{(n+2)\cdot t}{6}}. \tag{20}$$

But now (17) and (19) give

$$t \geqq \frac{n-2}{6s}, \tag{21}$$

so that, by (20) and (21), we have

$$f(n) \geqq \frac{n-2}{6\sqrt{s}} > \frac{n}{6\sqrt{s}} - 1. \tag{22}$$

If we now combine (14) and (22), we obtain, in any case,

$$f(n) \geqq \min\left\{\frac{s}{2} - 1, \frac{n}{6\sqrt{s}} - 1\right\}, \qquad 1 \leqq s \leqq n. \tag{23}$$

To obtain the best value for s we let $s/2=n/6\sqrt{s}$ and find $s=n^{2/3}/\sqrt[3]{9}$. This value of s in (23) yields (2). Thus, we have completed the proof of Theorem I:

$$f(n) \geqq \frac{n^{2/3}}{2\sqrt[3]{9}} - 1.$$

This is better than the right hand side of (1) for $n>5184$.

3. Convex Case. We now turn to the case where the n points form the vertices of a convex polygon, our object being to prove (4). First we consider the following

LEMMA 3. *Let A and B be two points at the vertices of a sector of a circle less than or equal to a semicircle. Let $P_1, P_2, \cdots, P_m$ be m other points inside or on the sector. If all the points considered form the vertices of a convex polygon, then from A (and from B) exactly $m+1$ distances are determined.*

Proof. If $\overline{AP_i}=\overline{AP_j}$ then clearly, $AP_i \cdot BP_j$ cannot be part of a convex polygon. Note Figure 3.

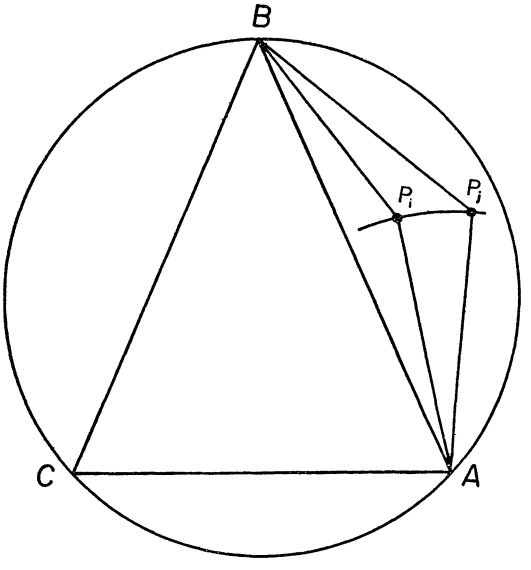


FIG. 3

Now consider the smallest circle containing our n points. If only two points lie on this circle, then they must be diametrically opposite. In this case, one of the semicircles will contain at least $(n+1)/2$ points in or on it, and Lemma 3 will yield

$$f^*(n) > \frac{n}{2} \geqq \left[\frac{n+2}{3}\right], \qquad (n \geqq 3).$$

Suppose then, that at least 3 points lie on the circle. We can then find three points A, B, C, on it which form a triangle having no angle greater than $\pi/2$, for otherwise the circle could be further contracted by shifting it in the direction of the angle greater than $\pi/2$ (since the center of the circumscribed circle is in the exterior of the triangle). Since our points are the vertices of a convex polygon, there are no points inside the triangle ABC, and hence at least one of the sectors determined by AB, AC, BC, will contain at least $[(n+5)/3]$ points in or on it, so that by Lemma 3 we obtain Theorem II:

$$f^*(n) \geqq \left[\frac{n+2}{3}\right].$$

NOTE ON A SECOND FUNCTIONAL EQUATION, CONNECTED WITH THE FUNCTION $\wp(z)$

HARI DAS BAGCHI, AND PHATIK CHAND CHATTERJEE, Calcutta University

It is known* that the Weierstrass function $\wp(z)$ satisfies the functional equation:

$$f(x+y) - f(x-y) = -\frac{f'(x)f'(y)}{\{f(x)-f(y)\}^2}. \tag{1}$$

We propose to determine all the analytic† solutions, having at most poles in the finite plane. If $f(x)$ satisfies (1), so does $f(x)+c$ where c is an arbitrary constant. To within this additive constant we shall show that the only solution is $\wp(x)$.

It is clear that $f(x)$ must have a pole at $x=0$, since otherwise as $x\to y$, the L.S. of (1) would remain finite for general y, whereas the R.S. would become infinite. Hence

$$f(x) = \frac{g(x)}{x^n}, \tag{2}$$

where n is a positive integer and $g(x)$ is regular and not zero at $x=0$.

Divide (1) by $2y$ and let $y\to 0$. We obtain

$$f'(x) = -\frac{1}{2}\cdot f'(x) \times \lim_{y\to 0}\left[\frac{f'(y)}{y\{f(x)-f(y)\}^2}\right];$$

and on using (2) (for the variable y) it is easily calculated that, in order that the limit on the right may exist and have the right value, we must have $n=2$

* See Whittaker & Watson, Modern Analysis, (1915), Ex 1, p. 449.

† On clearing (1) of fractions, the new functional equation has the solution $f(x)=$const.; but this trivial case is ruled out.

and $g(0)=1$.

Again, interchanging x and y in (1) we see that

$$f(x-y)=f(y-x);$$

so $f(x)$ is an *even* function. Consequently

$$f(x)=\frac{1}{x^2}+h(x), \tag{3}$$

where $h(x)$ is an *even* function, regular at $x=0$. Since we already know that to a solution can be added an arbitrary constant, it is no restriction to assume, as we now do, that $h(0)=0$.

Let x be in a region of regularity of $f(x)$; so that, for y small, the functions $f(x\pm y)$ are regular and can be expanded in power series in y:

$$f(x\pm y)=\sum_{r=0}^{\infty}\frac{f^{(r)}(x)}{r!}\cdot(\pm y)^r.$$

Using this and (3) (for the variable y) in (1), we obtain:

$$2\left[f'(x)+\frac{y^2}{3!}\cdot f'''(x)+\frac{y^4}{5!}\cdot f^{(5)}(x)+\cdots\right]\left[-1+y^2\{f(x)-h(y)\}\right]^2$$
$$=-f'(x)\cdot\{-2+y^3h'(y)\};$$

and on equating coefficients of y^2 on both sides and using $h(0)=0$, we get the differential equation:

$$f'''(x)=12f(x)f'(x).$$

A first integration yields:

$$f''(x)=6\{f(x)\}^2-\tfrac{1}{2}g_2,$$

where g_2 is an arbitrary constant; and on setting $u=f(x)$, we have:

$$\left(\frac{du}{dx}\right)^2=4u^3-g_2u-g_3,$$

where g_3 also is an arbitrary constant. Thus

$$x=\int_u^{\infty}\frac{dt}{\sqrt{4t^3-g_2t-g_3}},$$

since $u\to\infty$, when $x\to 0$; so $f(x)$ coincides with $\wp(x)$.

The desired result is thus obtained.

In conclusion we beg to express our heart-felt gratitude to the learned referee for his valuable suggestions and constructive criticisms, which have been of particular aid in the preparation of this paper.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tennessee.

POSTULATES FOR BOOLEAN ALGEBRA

D. G. MILLER, Urbana, Illinois

1. Introduction. This note will present a set of seven postulates for Boolean Algebra in terms of the ring operations. This set, like a recent short set of Byrne's [1], has only four transformation axioms, and they seem to be somewhat more familiar in appearance than those of Byrne's. The set is somewhat more condensed than the standard sets [2], [3], [4], and is just as workable. Its practicality will be exemplified in §3 where equivalence to a standard set will be demonstrated.

Consistency and independence examples will be exhibited. These examples, except the one for A, have only two elements. Hence when A is modified to read, "There are exactly two elements in K," this formulation will also be an independent set of postulates for the Two-valued Logic.

2. The postulates. The undefined ideas for this formulation are: A class of elements K; a binary operation $\cdot$, called logical multiplication; and a binary operation $+$, called ring addition. For brevity $a \cdot b$ will be written ab. Prefix to E, F, and G, "Whenever the elements and combinations indicated belong to K. . . . " The postulates follow:

A. There are at least two distinct elements in K.

B. ab is a unique element in K if a and b are in K.

C. $a+b$ is a unique element in K if a and b are in K.

D. There exists an element 1 in K such that for every a, $a1=a$.

E. $a+(b+b)=a$

F. $[a(bb)]c=(cb)a$

G. $a[(b+c)+d]=a(d+c)+ab$*

Postulates F and G indicate how the most familiar axioms can be combined, the associative, commutative, and idempotent laws for multiplication all being brought together in F, the distributive law and the associative and commutative laws for addition being combined in G. For this reason, the above set may have some pedagogical uses.

3. Equivalence. This set will be proved equivalent to one due to B. A. Bernstein [2].

The necessity part can be easily verified by the reader from Bernstein's set. The sufficiency will be shown in the following ten theorems.

* I am indebted to Prof. H. E. Vaughan for his helpful advice, especially in pointing out this simplified version of G.

3.1 $ab=ba$
For $ab=[a1]b=[a(11)]b=(b1)a=ba$
by D, D, F, D.

3.2 $1a=a$ by 3.1, D.

3.3 $aa=a$
For $aa=1(aa)=[1(aa)]1=(1a)1=1a=a$
by 3.2, D, F, D, 3.2.

3.4 $a(bc)=b(ca)$
For $a(bc)=(bc)a=[b(cc)]a=(ac)b=b(ac)=b(ca)$
by 3.1, 3.3, F, 3.1, 3.1.

3.5 $a+b=b+a$
For $a+b=1(a+b)=1([a+(1+1)]+b)=1[b+(1+1)]+1a$
$=[b+(1+1)]+a=b+a$
by 3.2, E, G, 3.2, E.

3.6 $a+(b+c)=b+(c+a)$
For $a+(b+c)=(b+c)+a=1[(b+c)+a]=1(a+c)+1b=(a+c)+b$
$=b+(a+c)=b+(c+a)$
by 3.5, 3.2, G, 3.2, 3.5, 3.5.

3.7 $a(b+c)=ab+ac$
For $a(b+c)=a([b+(1+1)]+c)=a[c+(1+1)]+ab=ac+ab=ab+ac$
by E, G, E, 3.5.

3.8 $(a+a)+b=b$ by E, 3.5.

DEFINITION I $a'=1+a$

3.9 a' is in K by Definition I, D, C.

3.10 $(a+a')b=b$
For $(a+a')b=[a+(1+a)]b=[1+(a+a)]b=1b=b$
by Definition I, 3.6, E, 3.2.

This completes the proof of sufficiency since C, 3.6, 3.8, B, 3.4. 3.3, 3.7, 3.9, 3.10, and A are Bernstein's P_1-P_{10} respectively.

4. Consistency and independence. The consistency and independence of A–G will be proved by the systems in the table below. S_0 is the consistency system. S_A is the independence system for postulate A, etc. Any blanks in the systems mean that there is no element of K for that particular operation.

S_0 $K = 0,1$

$\cdot$	0	1
0	0	0
1	0	1

$+$	0	1
0	0	1
1	1	0

S_A $K = 0$

$\cdot$	0
0	0

$+$	0
0	0

		·	0	1	+	0	1
S_B	$K = 0,1$	0	−	0	0	0	1
		1	0	1	1	1	0
		·	0	1	+	0	1
S_C	$K = 0,1$	0	0	0	0	0	1
		1	0	1	1	1	
		·	0	1	+	0	1
S_D	$K = 0,1$	0	0	0	0	0	1
		1	0	0	1	1	0
		·	0	1	+	0	1
S_E	$K = 0,1$	0	0	0	0	0	0
		1	0	1	1	0	0
		·	0	1	+	0	1
S_F	$K = 0,1$	0	0	0	0	0	0
		1	1	1	1	1	1
		·	0	1	+	0	1
S_G	$K = 0,1$	0	0	0	0	0	0
		1	0	1	1	1	1

5. A related briefer set. It is possible to transform A-G into an equivalent six postulate set by replacing D by D′, and by consolidating F and G into F′.

D′ written in full is

D.′ There exists a 1 in K such that for every a in K

a. $a1 = a$

b. $1a = a$*

The postulates briefly stated are:

A. K has at least two distinct elements.

B. ab is in K.

C. $a+b$ is in K.

D′. $a1 = 1a = a$.

E. $a+(b+b) = a$.

F′. $([a(bb)]c)[(d+e)+f] = [(cb)a][f+e]+[(cb)a]d$.

The set A–F′ can easily be derived from A–G. Postulates A–G can be deduced from A–F′ in the following four steps.

* This sort of postulate which has two subsections has been used, for example, by E. V. Huntington in "Sets of Independent Postulates for the Algebra of Logic," Trans. Am. Math. Soc., vol. 5, 1904, pp. 288–309; and by B. A. Bernstein in "A Simplification of the Whitehead-Huntington Set of Postulates for Boolean Algebras," Bull. Am. Math. Soc., vol. 22, 1916, pp. 458–459.

i. G by setting $b=c=1$ in F′.

ii. $a(1+1)=a+a$ by setting $b=c=d=f=1$, $e=1+1$ in F′.

iii. $a+b=b+a$ by setting $a=b=c=1$, $e=1+1$ in F′.

iv. F by setting $d=e=f=1$, and using *ii* and *iii*.

This proves the equivalence of the two sets.

The independence of A–F′ is given by the systems S_A, S_B, S_C, S_D, S_E, S_F, respectively.

References

1. Lee Byrne, Short Formulations of Boolean Algebra, Canadian Jour. Math., vol. 3, no. 1, 1951, pp. 31–33.

2. B. A. Bernstein, Postulates for Boolean Algebra Involving the Operation of Complete Disjunction, Annals of Math., vol. 37, 1936, pp. 317–325.

3. B. A. Bernstein, Postulate Sets for Boolean Rings, Trans. Am. Math. Soc., vol. 55, 1944, pp. 393–400.

4. E. R. Stabler, Sets of Postulates for Boolean Rings, this MONTHLY, vol. 48, 1941, pp. 20–28.

POLYNOMIALS HAVING A ROOT APPROACHING π

MARK LOTKIN, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland

In the course of examining a new quadrature formula we were led to the study of sequences of polynomials with the interesting property that one of their roots approaches π.

The object of this note is to show how these polynomials arise, and to demonstrate the property mentioned above.

The quadrature formula*

$$(1)\qquad I(h) \equiv \int_{x_0}^{x_1} f(t)\,dt = (h/2)(f_0 + f_1) + (h^2/10)(f_0' - f_1') + (h^3/120)(f_0'' + f_1'') + R,$$

with $x_1=x_0+h$; $f_{(j)}^{i}=(d^i f/dx^i)_{x=x_j}$; $i=0,\ 1,\ 2$; $j=0,\ 1$; $f_j^{(0)}=f_j$, is highly accurate; an estimate of the error R is given by

$$|R| \leqq (h^7/100{,}800)\ \max |f^{(6)}(x')|, \qquad x_0 \leqq x' \leqq x_1.$$

Let us apply our formula to $f(x)=\cos x$, $x_0=0$, $x_1=h_k=\pi/2^k$, for $k=1,\ 2,\ 3,\ \cdots$; obviously:

$$(2)\qquad I(k) \equiv \int_0^{h_k} \cos t\,dt = \sin h_k.$$

On the other hand, by (1), making use of

* See M. Lotkin, A New Integrating Procedure of High Accuracy, to appear soon in the Journal of Mathematics and Physics.

$$1 + \cos h_k = 2\cos^2 h_{k+1}, \qquad \sin h_k = 2 \sin h_{k+1} \cos h_{k+1},$$

$$I(k) = h_k(1 - h_k^2/60)\cos^2 h_{k+1} + 2(h_k^2/10)\sin h_{k+1}\cos h_{k+1} + R.$$

Consequently, because of (2),

$$E(k) \equiv \cos h_{k+1}[h_k(1 - h_k^2/60)\cos h_{k+1} - 2(1 - h_k^2/10)\sin h_{k+1}] + R = 0.$$

It may thus be concluded that the cubics

$$E(k, x) \equiv x_k(1 - x_k^2/60)\cos h_{k+1} - 2(1 - x_k^2/10)\sin h_{k+1} = 0, \tag{3}$$

with $x_k = x2^{-k}$, have at least one real root x which approaches the value π as k becomes unbounded.

This fact may also be recognized independently of the foregoing in the following manner: Realizing that

$$1 - x_k^2/10 = 6(1 - x_k^2/60) - 5,$$

equation (3) may be expressed as

$$(1 - x_k^2/60)(x_k \cos h_{k+1} - 12 \sin h_{k+1}) + 10 \sin h_{k+1} = 0. \tag{4}$$

Now, as k becomes large, $\sin h_{k+1}$, $\cos h_{k+1}$ approach 0, 1, respectively, and the values of x_k are seen to approach $\pm\sqrt{60}$ and zero.

Only the root ξ for which $\lim_{k\to\infty}(\xi 2^{-k}) = 0$, has a finite limit λ. For this root we get by (4)

$$\xi 2^{-k}\cos h_{k+1} - 2\sin h_{k+1} \doteq \xi 2^{-k} - \pi 2^{-k} \doteq 0, \tag{5}$$

whence

$$\lambda = \pi.$$

The rapidity of convergence is indicated in the table below, showing the values of ξ for $k = 1, 2, 3, 4, 5$.

k	1	2	3	4	5
ξ	3.142	3.1416	3.1415928	3.14159 2654	3.14159 26536

In computing these values use was made of the relationships

$$\cos h_{k+1} = (1/2)\sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}} \quad (\sqrt{2} \text{ repeated } k \text{ times})$$

$$\sin h_{k+1} = (1/2)\sqrt{2 - \sqrt{2 + \cdots + \sqrt{2}}}.$$

While formula (1) gives rise to the cubics (3), the weaker formula

$$I(h) \doteq (h/2)(f_0 + f_1) + (h^2/10)(f_0' - f_1')$$

clearly leads to the quadratic equations

$$x_k(x_k \sin h_{k+1} + 5 \cos h_{k+1}) - 10 \sin h_{k+1} = 0. \tag{6}$$

Here, too, we have a root η for which $\lim (\eta/2^k) = 0$; for this root η again clearly (5) holds, so that $\eta = \pi$.

SUMS AND PRODUCTS OF NORMAL NUMBERS

J. E. MAXFIELD, University of Oregon and Naval Ordnance Test Station

It is known that the product of a normal number and a rational number and the sum of a normal number and a rational number are normal [3]. Also the set of numbers normal to the base r is not closed under addition since it follows easily from the definition that 0 is not normal and that if x is normal then $-x$ is also normal. The purpose of this note is to prove that every number is the sum (product, if $\neq 0$) of two normal numbers in at least one way, and incidentally to show that the normal numbers are not closed under multiplication.

DEFINITION 1. A number σ is simply normal to the scale r if in the decimal expansion of σ to the scale r $\lim_{n\to\infty} n_c/n = 1/r$, for each c, where n_c is the number of occurrences of the digit c in the first n digits of σ.

DEFINITION 2. A number σ is normal to the scale r if $\sigma, r\sigma, r^2\sigma, \cdots$ are each simply normal to all the scales $r, r^2, \cdots$.

THEOREM. *A number a is the sum [product for $a \neq 0$] of two normal numbers in at least one way.*

Proof: The theorem is immediate for $a = 0$. Let $a \neq 0$. Let N be the set of normal numbers, N^* the set of abnormals or non-normals. The set N^* is of measure zero [2]. Define $f(y) = a - y$ [$f(y) = a/y$, for $a^{1/2}/2 \leq y \leq 2a^{1/2}$]. These mappings and their inverses are absolutely continuous. Thus there exists σ, a normal number, such that $a - \sigma$ [a/σ] is normal. For assuming the contrary, every $a - \sigma$ [a/σ] would be in N^*. By a theorem of Caratheodory [1, page 538, Theorem 2], the set N would have measure zero. This is a contradiction. Thus there exists a σ from N such that $f^{-1}(\sigma)$ is normal, and $f^{-1}(\sigma) + \sigma = a$ [$\sigma f^{-1}(\sigma) = a$].

COROLLARY. *The set of numbers normal to the scale r is not closed under multiplication.*

Proof: The proof is immediate.

References

1. C. Caratheodory, Über Reele Functionen, Second Edition, Chelsea Publishing Company, 231 W. 29th St., New York 1, New York, 1948, p. 538.

2. G. H. Hardy and N. M. Wright, An Introduction to the Theory of Numbers, Oxford at the Clarendon Press, 1938, pp. 123–127.

3. D. D. Wall, Normal Numbers, Ph.D. Thesis, 1949, University of California, Berkeley, California.

CORRECTION

E. M. WRIGHT, University of Aberdeen

The following corrections should be made in my note "A Prime-Representing Function" in the November 1951 MONTHLY:

1. Page 617, line 4: Replace "Ingham [2]) is fairly deep" by "Ingham [4]) is fairly deep."
2. Add to the list of references: 4. A. E. Ingham, On the difference between consecutive primes, Quart. Jour. of Math., vol. 8 (1937), pp. 255–266.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

STUDENT'S CHOICE IN MATHEMATICS—FUNDAMENTAL REASONING OR BLIND SUBSERVIENCE TO RULES

M. R. SPIEGEL, Rensselaer Polytechnic Institute

In a recent note (this MONTHLY, vol. 58, p. 188), F. Hawthorne proposed the following simple endpoint maxima and minima problem:

Find that point on the circle $x^2+y^2=1$ which is nearest to (2, 0). As he states "—the unsuspecting student sets up this problem using x as independent variable and obtains for the square of the distance from (x, y) to (2, 0)" the quantity $L^2=5-4x$. Differentiating and equating the result to zero, the student obtains $-4=0$. He is baffled. But why should he be? The reason, of course, is obvious. He has been taught in his calculus course a rule which he must memorize and apply blindly. To find the maximum or minimum—differentiate and set equal to zero. A machine can be designed to do the same.

If the student were taught some of the fundamental concepts of the calculus instead of blind subservience to rules and formulae, he might realize that his supposedly sacred rule does not apply in this case. He might be taught after discovering his predicament, to reason, for example, as follows. $L^2=5-4x$ is a positive quantity to be made as small as possible. This can be accomplished by making x as large as possible. Since, $x^2+y^2=1$, x will be largest when $y=0$, in which case $x=1$. Thus, the required point is (1, 0). He has not used calculus *per se* in the sense that he has not taken a derivative. He has, nevertheless, used calculus concepts.

Of course this fundamental reasoning cannot be acquired by the student in a day. He must receive constant practice in such approaches to problems. I believe that this blind attitude on the part of the student is but one of the sad consequences of some of our teaching methods discussed by M. Richardson in

his article (this MONTHLY, vol. 58, p. 182) in which he endeavors to encourage more teaching of fundamentals in our undergraduate courses.

It may be interesting to quote a more elementary example of our student's desire to obtain results from memorized formulae rather than by fundamental reasoning. I pose the following problem:

A motorist drives from New York to Troy at an average speed of 30 miles per hour and from Troy to New York (being in a hurry) at 60 miles per hour. What is his average speed for the entire trip assuming that the distance travelled from New York to Troy is the same as from Troy to New York?

I have given this problem to many students (to say nothing of some high-powered professors—not in mathematics, of course), and all of them have given the wrong answer. What?—did you say 45 miles per hour? Perhaps then, you had better try some fundamental reasoning too.

GENERALIZATION OF CLAIRAUT'S DIFFERENTIAL EQUATION

W. H. WITTY, Winona, Miss.

The purpose of this note is to show the existence of a differential equation of the nth order whose solution is obtained by the same method as Clairaut's equation. After differentiation it will have as a factor $y^{(n+1)}$. Clairaut's equation is the special case where $n=1$.

The equation

$$y = xy' - \frac{x^2y''}{2!} + \cdots + \frac{(-1)^{n-1}}{n!}x^n y^{(n)} + f[y^{(n)}] \tag{1}$$

upon differentiation yields

$$\left[f'[y^{(n)}] + \frac{(-1)^{n-1}}{n!}x^n\right]y^{(n+1)} = 0. \tag{2}$$

From the vanishing of the second factor we obtain

$$\begin{aligned}
y^{(n)} &= c_1 \\
y^{(n-1)} &= c_1x + c_2 \\
&\vdots \\
y'' &= \frac{c_1x^{n-2}}{(n-2)!} + \frac{c_2x^{n-3}}{(n-3)!} + \cdots + c_{n-2}x + c_{n-1} \\
y' &= \frac{c_1x^{n-1}}{(n-1)!} + \frac{c_2x^{n-2}}{(n-2)!} + \cdots + c_{n-1}x + c_n.
\end{aligned}$$

Substitution of these values into (1) leads to

$$y = \left[\frac{1}{(n-1)!} - \frac{1}{2!(n-2)!} + \frac{1}{3!(n-3)!} + \cdots + \frac{(-1)^{n-4}}{3!(n-3)!} + \frac{(-1)^{n-3}}{2!(n-2)!} + \frac{(-1)^{n-2}}{(n-1)!} + \frac{(-1)^{n-1}}{n!}\right] c_1 x^n + \left[\frac{1}{(n-2)!} - \frac{1}{2!(n-3)!} + \cdots + \frac{(-1)^{n-4}}{2!(n-3)!} + \frac{(-1)^{n-3}}{(n-2)!} + \frac{(-1)^{n-2}}{(n-1)!}\right] c_2 x^{n-1} + \cdots + c_{n-1}\frac{x^2}{2!} + c_n x + f(c_1).$$

This reduces to

$$y = \frac{c_1 x^n}{n!} + \frac{c_2 x^{n-1}}{(n-1)!} + \cdots + \frac{c_{n-1}}{2!}x^2 + c_n x + f(c_1), \tag{3}$$

which is the general solution.

Considering the other factor on the left in (2) we note that it is $\partial(1)/\partial y^{(n)}$. By eliminating $y^{(n)}$ from the two equations $\partial(1)/\partial y^{(n)} = 0$ and $(1) = 0$ we obtain the $y^{(n)}$ discriminant. Thus

$$f'[y^{(n)}] + \frac{(-1)^{n-1}}{n!}x^n = 0.$$

If this can be solved for $y^{(n)}$ we get

$$y^{(n)} = G(x).$$

Substituting this in (1), and rearranging, we have

$$\frac{(-1)^{n-1}}{(n-1)!}x^{n-1}y^{(n-1)} + \cdots + \frac{x^2y''}{2!} - xy' + y = H(x). \tag{4}$$

This is a Cauchy linear differential equation and can be easily solved. It has a singular solution of the form

$$y = a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-2}x^2 + a_{n-1}x + I(x) \tag{5}$$

where $I(x)$ is the particular integral.

As an example we take the differential equation

$$y = xy' - \frac{x^2y''}{2!} + \frac{x^3y'''}{3!} + (y''')^2 \tag{6}$$

whose general solution by (3) is

$$y = \frac{c_1x^3}{3!} + \frac{c_2x^2}{2!} + c_3x + c_1^2$$

and whose singular solution by (5) is

$$y = a_1x^2 + a_2x - \frac{x^6}{1440}$$

which satisfies both (1) and (4).

THE LIMIT FOR e

RICHARD LYON and MORGAN WARD, California Institute of Technology

The following way of showing that $(1 + 1/n)^n$ tends to e as n tends to infinity seems simpler than the current proofs in elementary texts.

Define e as $\lim_{n\to\infty} \sum_0^n 1/r!$ Then the result is evident from the inequality

$$\sum_0^n \frac{1}{r!} > \left(1 + \frac{1}{n}\right)^n > \sum_0^n \frac{1}{r!} - \frac{3}{2n}, \qquad n \geqq 3.$$

This inequality may be proved as follows. Assume $n \geqq 3$. Then by the binomial theorem,

$$\left(1 + \frac{1}{n}\right)^n = 2 + \sum_2^n \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{r-1}{n}\right)\frac{1}{r!}.$$

Clearly

$$1 > \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{r-1}{n}\right).$$

Hence

$$\sum_0^n \frac{1}{r!} = 2 + \sum_2^n \frac{1}{r!} > \left(1 + \frac{1}{n}\right)^n.$$

On the other hand,

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) = 1 - \frac{(1+2)}{n} + \frac{2}{n^2} > 1 - \frac{(1+2)}{n}.$$

Hence

$$\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3}{n}\right) > \left(1 - \frac{(1+2)}{n}\right)\left(1 - \frac{3}{n}\right) > 1 - \frac{(1+2+3)}{n},$$

and so on. Consequently if $r \geqq 2$,

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right) \geqq 1-\frac{(1+2+3+\cdots+r-1)}{n}$$

$$= 1-\frac{r(r-1)}{2n}$$

with equality only when $r=2$. Consequently since $n \geqq 3$

$$\left(1+\frac{1}{n}\right)^n > 2+\sum_2^n\left(1-\frac{r(r-1)}{2n}\right)\frac{1}{r!} = \sum_0^n \frac{1}{r!}-\frac{1}{2n}\sum_0^{n-2}\frac{1}{r!}$$

$$> \sum_0^n \frac{1}{r!}-\frac{3}{2n}, \quad \text{since} \quad \sum_0^{n-2}\frac{1}{r!} \text{ is less than 3.}$$

AN ELEMENTARY INTEGRAL

A. W. WALKER, University of Toronto

The logical approach to the integral $\int dx/\sqrt{x^2+c}$ is, of course, through hyperbolic functions, but it is often desirable to postpone the introduction of these until a later stage of the student's mathematical development. Most texts simply include the result in a preliminary table of basic forms, remarking that it can be verified by differentiation; the writer feels, however, that it is preferable to limit this table to an absolute minimum, concentrating instead on the standard techniques by which a wide variety of integrals can be reduced to these basic forms.

A suggested classroom procedure for discussion of this integral is as follows: —(a) Make the substitution $\sqrt{x^2+c}=u$ and show that this method breaks down when applied in the usual way. (b) Make a trigonometric substitution (reminding the class of the success of this method for integrals involving $\sqrt{a^2-x^2}$). Show that this method is not so successful here, and remark that this is not surprising, because the answer does not involve inverse trigonometric functions. However, our efforts have not been wasted, for we see how we can obtain the values of $\int \sec u\,du$, $\int \csc u\,du$ as soon as our present problem has been solved. (c) Show how the technique in (a) *can* be made to work in two ways, as follows:

(i) First, make the "reciprocal substitution" $x=1/z$; *then* let $\sqrt{1+cz^2}=u$. This is considerably longer than method (ii) below, and need not be carried through to the answer; it is only necessary to indicate how the problem can be solved by this quite straightforward approach.

(ii) Let $\sqrt{x^2+c}=u$; then $x\,dx=u\,du$, and therefore

$$\frac{dx}{u}=\frac{du}{x}=\frac{dx+du}{u+x}=\frac{d(x+u)}{x+u}=d\log(x+u)$$

and our problem is solved. The student is likely to be quite impressed by this

neat "dodge"; a word of warning as to its highly specialized character is not amiss.

Various associated integrals can now be found, using integration by parts, *etc.*, the work being minimized and a certain symmetry being preserved by making full use of the symbol u. Note, in particular, that

$$d \log (x + u) = \frac{udx}{u^2} = \frac{xdu}{x^2} = \frac{udx - xdu}{c}$$

and $d(ux) = udx + xdu$, from which we can obtain $\int udx$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1001. *Proposed by C. S. Ogilvy, Syracuse University*

Three cowboys who have to watch over cattle in a square pasture divide the square into three equal rectangles by parallel lines in the obvious way. Each then stations himself in the center of one of the rectangles and is responsible only for the cattle in his rectangle.

Show that the square can be divided into three equal areas in another way and a cowboy can be stationed at a point of each, so that (1) the three cowboys have the same "maximum ride," which is the distance from a cowboy's station to the most distant point of his area; (2) this maximum ride is less than it was before; and (3) every point in the square is entrusted to the man whose station is nearest it.

This problem is posed but not solved in Steinhaus, *Mathematical Snapshots* (1950), pp. 34–35.

E 1002. *Proposed by T. M. Apostol, California Institute of Technology*

Let A, B, C represent polynomials in a given number-field F, and let A' denote the derivative of A. Suppose B is an m-fold factor of A, *i.e.*, $A = B^m C$ and $B \nmid C$. It is well known that B is at least an $(m-1)$-fold factor of A'. The example $B = (x-1)^3$, $A = (x-1)^3(x^3-1)^2$ shows that it is possible for B to be an m-fold factor of A'. Is it possible for B to be an $(m+1)$-fold factor of A'?

E 1003. *Proposed by J. G. Wendel, Yale University*

Let $f(x, y)$ be continuous on the unit square, $0 \leqq x, y \leqq 1$. Show that $f(x, y)$ can be written in the form $f(x, y) = g(x) + h(y)$ if and only if

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \{f(x, y) + f(u, v)\}^2 dxdydudv = \int_0^1 \int_0^1 \int_0^1 \int_0^1 \{f(x, y) + f(u, v)\}\{f(x, v) + f(u, y)\} dxdydudv.$$

E 1004. *Proposed by L. R. White, Washington, D. C.*

(1) The approximation $\sqrt[n]{a} = 1 + (a-1)/n$ for n large and a close to 1 is well known and frequently used. Show that for large n and all a

$$\sqrt[n]{a} = 1 + (\ln a)/n$$

is a good approximation.

(2) Find

$$\lim_{n \to \infty} \{(k - 1 + \sqrt[n]{a})/k\}^n.$$

E 1005. *Proposed by D. W. Dubois, University of Oklahoma*

Let $x_0 \neq 0$ and $a > 0$ be two real numbers. Define

$$x_{n+1} = x_n/2 + a/2x_n, \qquad n = 0, 1, 2, \cdots.$$

Find all values of x_0 and a for which the sequence $\{x_n\}$ converges, and find the limits.

SOLUTIONS

A Cryptic Division

E 971 [1951, 417]. *Proposed by A. W. Willcox, Washington, D. C.*

Reconstruct the division problem:

```
* * ) * * * * 2 * ( * * * * *
    * *
    -----
    * * *
      * *
      -------
        * * *
        * * *
        -----
            * *
            * *
            ---
```

Solution by L. A. Ringenberg, Eastern Illinois State College. Complete the array of asterisks and 2 with zeros to form an 8 by 13 matrix. Let a_{ij} denote the digit in the ith row and the jth column. The conditions $a_{23} \neq 9$, $a_{44} = 9$, and $10a_{55} + a_{56} < d$, where d is the divisor, imply that there is an integer n, $2 \leqq n \leqq 9$, such that $95 < nd < 100$ and that $d \neq 11, 12, 14, 16, 24, 32$. If $d = 33$, the conditions $a_{75} = a_{76} = 0$ imply that $a_{1,11} = 3$, 6, or 9, which implies that $a_{65} = 0$ or that 33 is an integral divisor of a number between 40 and 60. If $d = 48$, then $a_{55} \geqq 4$ implies $a_{1,12} = 8$ or 9, which implies that $a_{76} \neq 0$ or $a_{77} = 0$. The only remaining possibility, $d = 49$, yields the unique dividend 591528.

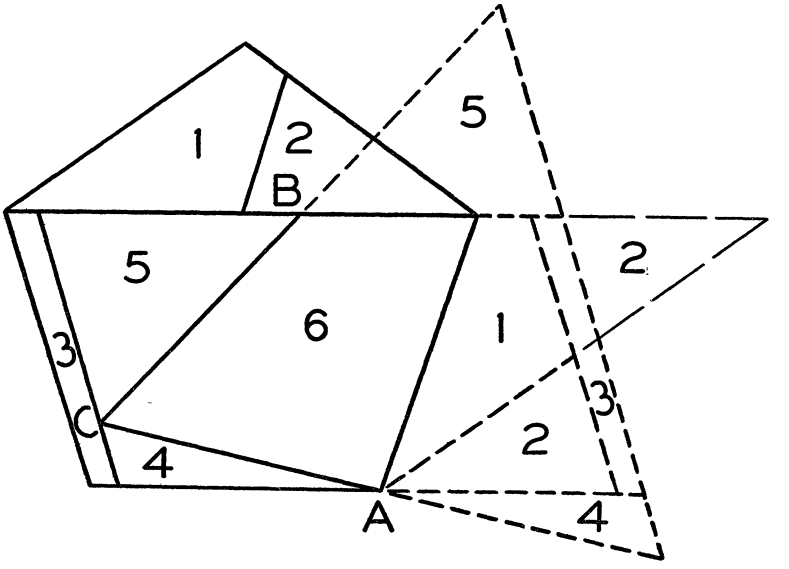
Also solved by R. V. Andree, Leon Bankoff, H. G. Bergmann, H. H. Berry, J. H. Braun, D. H. Browne, W. Buker, Monte Dernham, I. A. Dodes, C. V. Fronabarger, E. St. John Gough, Louisa Grinstein, Frank Herlihy, Sidney Kravitz, Elmer Latshaw, J. W. Layman, H. R. Leifer, John Lombardi, P. J. Malie, D. C. B. Marsh, Prasert Na Nagara, Margaret Olmsted, F. D. Parker, Martin Pearl, P. W. A. Raine, W. R. Ransom, J. A. Richman, J. W. Ridley, Jr., Azriel Rosenfeld, J. E. Sanders, B. B. Stoddard, C. W. Trigg, G. W. Walker, L. E. Ward, Jr., J. E. Weidlich, B. K. Gold, A. M. Wenner, and the proposer.

Trigg showed that if the dividend be prefaced by another asterisk, the new problem will have two solutions. He also showed that if in the original problem the 2 be replaced by an asterisk then there are fifty solutions. Only for a 2 or a 9 in this position is the solution unique.

Six Piece Dissection of a Pentagon into a Triangle

E 972 [1951, 417]. *Proposed by Michael Goldberg, Washington, D. C.*

Dissect a regular pentagon, by straight cuts, into six pieces which can be



put together to form an equilateral triangle. (See Ball-Coxeter, *Mathematical Recreations*, top of p. 93.)

Solution by the Proposer. Remove an isosceles triangle (pieces 1 and 2) by a cut along a diagonal. Place it along the side of the remainder to form a trapezoid. At the midpoint of the long side of the isosceles triangle, make a cut parallel to the opposite side of the trapezoid. Shift the small piece (piece 2) to form a parallelogram. Let e be the length of the edge of the equivalent equilateral triangle. The edge e equals approximately 1.9933 times the edge of the pentagon. Inscribe an equilateral triangle ABC of side $e/2$ so that the vertex A coincides with a vertex which is the common vertex of three pieces, and the vertex B lies on the opposite edge of the parallelogram. Make the cuts AC and BC. Make another cut at an angle of 60° to each of AC and BC. Rearrange the pieces as shown in the diagram.

W. B. Carver found a six piece dissection of a regular pentagon of unit side into an isosceles triangle of base 2. The legs of the isosceles triangle are then equal to $\sqrt{41+10\sqrt{5}}/4=1.98998$. This is a curiously close approximation to the desired dissection.

A Disguised Solution of the Quadratic

E 973 [1951, 418]. *Proposed by Leo Moser, Texas Technological College*

Show that if

$$x = -(b/a)\cos^2\{(1/4)\cos^{-1}(\{b^2 - 8ac\}/b^2)\}$$

exists, then it is a solution of $ax^2+bx+c=0$.

Solution by M. R. Spiegel, Rensselaer Polytechnic Institute. Denote the expression in the outer pair of braces by u. Then

$$x = -(b/a)\cos^2 u$$

and

$$ax + b = b\sin^2 u.$$

Consequently

$$x(ax + b) = -(b^2/4a)\sin^2 2u = -(b^2/8a)(1 - \cos 4u) = -c,$$

which proves that $x=-(b/a)\cos^2 u$, if it exists, satisfies $ax^2+bx+c=0$.

Also solved by A. N. Aheart, P. M. Anselone, Leon Bankoff, H. G. Bergmann, Louis Berkofsky, J. H. Braun, A. L. Epstein, E. A. Franz, C. V. Fronabarger, Emil Grosswald, Simon Hellerstein, Frank Herlihy, S. J. Jasper, R. S. Kingsbury, M. S. Klamkin, Sidney Kravitz, P. J. Leonard, Jr., David Mandelbaum, E. W. Marchand, D. C. B. Marsh, George Millman, Martin Moliver, Margaret Olmsted, F. D. Parker, Martin Pearl, W. R. Ransom, L. A. Ringenberg, J. H. Simester, O. E. Stanaitis, C. W. Trigg, B. K. Gold, R. E. Horton, and the proposer.

Cross-ratios and Intersecting Cevians

E 974 [1951, 418]. *Proposed by Alan Wayne, Flushing, N. Y.*

Let L, M, N divide the sides BC, CA, AB of triangle ABC in the ratios $BL/LC=r$, $CM/MA=s$, $AN/NB=t$. Let R, S, T be the intersections of the pairs of cevians BM, CN; CN, AL; AL, BM. Show that the cross-ratios (AL, ST), (BM, TR), (CN, RS) are all equal to rst.

Solution by Roscoe Woods, State University of Iowa. From a figure it is seen that $(AL, ST)=(AC, M'M)$, where M' is the intersection of BS and AC. Let $CM'/M'A=s'$. Then we have $(AC, M'M)=(AM'/M'C)(CM/MA)=s/s'$. But from Ceva's theorem it follows that $rs't=1$, whence $1/s'=rt$. It then follows that $(AL, ST)=rst$. By treating the other two cross-ratios in a similar manner, the theorem is established.

Also solved by C. V. Fronabarger, Robert Gunning, D. C. B. Marsh, George Millman, Martin Moliver, Margaret Olmsted, Martin Pearl, O. J. Ramler, C. W. Trigg, R. E. Horton and the proposer.

A Curious Function

E 975 [1951, 418]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute*

Prove that

$$\int_0^1 dx/x^x = \sum_{i=1}^{\infty} 1/i^i.$$

Solution by A. E. Livingston, University of Oregon. We have

$$\int_0^1 x^{-x}dx = \int_0^1 e^{-x\log x}dx = \int_0^1 \left[\sum_{n=0}^{\infty} (-x\log x)^n/n!\right]dx$$

$$= \sum_{n=0}^{\infty} (1/n!)\int_0^1 (-x\log x)^n dx = \sum_{n=0}^{\infty} 1/(n+1)^{n+1}.$$

The interchange in the order of integration and summation is permissible since the series converges uniformly to x^{-x}. That

$$\int_0^1 (-x\log x)^n dx = n!/(n+1)^{n+1}$$

follows by n integrations by parts, $(-x)^n$ being used as the dv at each stage.

Also solved by D. E. Amos, D. H. Browne, M. S. Klamkin, George Millman, L. A. Ringenberg, W. Seidel, W. D. Serbyn, J. H. Simester, O. E. Stanaitis, D. L. Thomsen, Jr., L. E. Ward, Jr., R. E. Wild, Chaslaw Stanoyewich, and the proposer.

The proposer remarked that it would be interesting to discover the general class of functions having the property that

$$\int_0^1 f(x)dx = \sum_{n=1}^{\infty} f(n).$$

Klamkin located the given problem, with a solution, in Polya-Szego, *Aufgaben und Lehrsatze*, vol. I, p. 29.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4473. *Proposed by J. B. Kelly, Institute for Advanced Study*

Show that the differences of order $(p+1)/2$ of the sequence

$$\left(\frac{1}{p}\right), \left(\frac{2}{p}\right), \cdots, \left(\frac{x}{p}\right), \cdots, \left(\frac{p-1}{p}\right)$$

are divisible by the prime p. Here (x/p) is the Legendre symbol. Show, conversely, that the only sequences, $\{f(x)\}$, of length $p-1$ consisting only of 1's and -1's, and having this property are $f(x) \equiv 1$, $f(x) \equiv -1$, $f(x) = (x/p)$, $f(x) = -(x/p)$, $(x = 1, 2, \cdots, p-1)$.

4474. *Proposed by Ky Fan, University of Notre Dame*

Let the zeros $z_i (1 \leqq i \leqq n)$ of the polynomial

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1} + z^n$$

be so arranged that $|z_1| \geqq |z_2| \geqq \cdots \geqq |z_n|$. It is a classical result due to Carmichael and Mason that

$$|z_1| \leqq \sqrt{1 + \sum_{i=0}^{n-1} |a_i|^2} = B. \tag{1}$$

Using Jensen's inequality, M. Fujiwara obtained the stronger inequality

$$|z_1 z_2 \cdots z_k| \leqq B, \qquad (1 \leqq k \leqq n). \tag{2}$$

Prove that this result (2) can still be slightly improved as follows:

$$|z_1 z_2 \cdots z_k|^2 \leqq \tfrac{1}{2}(B^2 + \sqrt{B^2 - 4|a_0|^2}), \qquad (1 \leqq k \leqq n). \tag{3}$$

4475. *Proposed by D. J. Newman, Harvard University*

Let $f(z)$ be continuous throughout the complex z-plane, and suppose that $f(z)/z \to 1$ as $z \to \infty$. Show that $f(z)$ must have a zero.

4476. *Proposed by T. M. Apostol, California Institute of Technology*

Let $p(n)$ be the number of unrestricted partitions of n, and let $\sigma(n)$ be the sum of the divisors of n. Prove

$$p(n) = \sum_{m=1}^{n} \sum_{n_1,n_2,\cdots,n_m} \frac{1}{m!} \frac{\sigma(n_1)\sigma(n_2)\cdots\sigma(n_m)}{n_1 \cdot n_2 \cdots n_m}.$$

For each fixed m, the inner sum is to be taken over all positive integers n_1, $n_2, \cdots, n_m$ whose sum is n, the order of the summands being taken into consideration.

4477. *Proposed by C. E. Springer, University of Oklahoma*

Given a triangle with angles α, β, γ. Show that the ratio of the area of the Morley triangle of the given triangle to the area of the given triangle is

$$\frac{\left(\prod \sin^2 \frac{\alpha}{3}\right)\left(1 - 4\sum \cos^2 \frac{\alpha}{3} + 16 \prod \cos \frac{\alpha}{3}\right)}{(\prod \sin \alpha)\left(\prod \sin \frac{\pi - \alpha}{3}\right)}.$$

(If the angles of a triangle be trisected, the intersections of the pair of trisectors adjacent to each side determine an equilateral triangle called the Morley triangle of the given triangle. See also [1943, 552].)

SOLUTIONS

Roots of the Equation tan $x = x$

4400 [1950, 420]. *Proposed by C. D. Olds, San Jose State College*

Every positive root of the equation $\tan x = x$ can be expressed as follows: $x = (p + \frac{1}{2})\pi - \theta$ where p is an arbitrary integer $\geqq 0$ and

$$\theta = C_0\xi + C_1\xi^3 + C_2\xi^5 + \cdots, \quad \xi = 1/(p + \tfrac{1}{2})\pi.$$

The coefficients C_0, C_1, $C_2 \cdots$ are positive rational numbers. Show that, for n very large,

$$C_n = 12^{-1/6}\Gamma(1/3)\pi^{-2/3} \cdot (\pi/2)^{2n+1}(2n + 1)^{-4/3}(1 + \omega_n),$$

where $\omega_n \to 0$ when $n \to \infty$, so that the above series remains convergent even for $p = 0$.*

Solution by G. Szekeres, University of Adelaide, Australia. We may write the

* Originally proposed as no. 3290 [1927, 491] by J. V. Uspensky.

equation $\tan x = x$ in the form

$$\cot\theta = 1/\xi - \theta, \qquad x = (p + \tfrac{1}{2})\pi - \theta = 1/\xi - \theta,$$
$$\xi = (\theta + \cot\theta)^{-1} = \theta(1 - \tfrac{2}{3}\theta^2 + \tfrac{7}{15}\theta^4 + \cdots),$$

hence $\theta = f(\xi) = \sum_{n=0}^{\infty} c_n \xi^n$ where $c_{2m} = 0$ and $c_{2m+1} = C_m$. Obviously the C_m are rational since $f(\xi)$ is the inverse of a power series with rational coefficients.

Now the power series $f(\xi)$ defines an analytic function which has algebraic singularities at the points $\xi = \pm 2/\pi$ and no other singularities on the circle of convergence. In fact,

$$\frac{d\xi}{d\theta} = \left(\frac{\cot\theta}{\theta + \cot\theta}\right)^2 = 0$$

at $\theta = \pm\pi/2$ where $\xi = \pm 2/\pi$, and the circle $|\xi| = 2/\pi$ is mapped by the function $\theta = f(\xi)$ into a curve J which goes through the points $\theta = \pm\pi/2$ and is situated in the hexagonal region bounded by the lines

$$(1) \qquad \begin{aligned} &-\tfrac{1}{2}\pi + (1 \pm i)t, \quad \tfrac{1}{2}\pi - (1 \pm i)t, \quad 0 \leqq t \leqq 1, \\ &i + x, \quad -i + x, \quad -\tfrac{1}{2}\pi + 1 \leqq x \leqq \tfrac{1}{2}\pi - 1. \end{aligned}$$

This can be shown either by plotting the curve J in the θ-plane or by checking that $|\theta + \cot\theta| \leqq \tfrac{1}{2}\pi$, $|\xi| \geqq (2/\pi)$ on the lines (1). Clearly $\theta + \cot\theta$ does not vanish in the hexagon bounded by the lines (1) (on the imaginary axis, $\theta + \cot\theta = 0$ at $\theta = \pm\beta i$ where $\beta = \coth\beta$, i.e., when β is very nearly 1.2), hence $(\theta + \cot\theta)^{-1}$ is regular on and inside J.

Now

$$\xi = (\theta + \cot\theta)^{-1} = \frac{2}{\pi} - \frac{4}{3\pi^2}(\tfrac{1}{2}\pi - \theta)^3 + \cdots,$$
$$= -\frac{2}{\pi} + \frac{4}{3\pi^2}(\tfrac{1}{2}\pi + \theta)^3 + \cdots$$

around the points $\theta = \tfrac{1}{2}\pi$ and $\theta = -\tfrac{1}{2}\pi$ respectively, hence the following expansions are valid around the branch points $\pm 2/\pi$:

$$\theta - \tfrac{1}{2}\pi = -\left(\frac{3\pi}{2}\right)^{1/3}(1 - \tfrac{1}{2}\pi\xi)^{1/3} + \cdots,$$
$$\theta + \tfrac{1}{2}\pi = \left(\frac{3\pi}{2}\right)^{1/3}(1 + \tfrac{1}{2}\pi\xi)^{1/3} + \cdots.$$

Since real values of θ correspond to real values of ξ the expressions $(1 - \tfrac{1}{2}\pi\xi)^{1/3}$ and $(1 + \tfrac{1}{2}\pi\xi)^{1/3}$ represent the principal branches of the functions which take the value 1 at $\xi = 0$. It follows from a well known theorem of Darboux (see G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloquium Publications, v. 23, p. 201) that the asymptotic behavior of the coefficients c_n is determined by

the leading terms in the above expansions. Thus

$$c_n = \left(\frac{3\pi}{2}\right)^{1/3}\{1 + (-1)^{n+1}\}\binom{1/3}{n}\left(\frac{\pi}{2}\right)^n\{1 + O(n^{-1/3})\}.$$

This and the well known relation

$$\binom{1/3}{n} \simeq \tfrac{1}{3}\{\Gamma(\tfrac{2}{3})\}^{-1}n^{-4/3} = \frac{1}{3\pi}\sin\frac{\pi}{3}\,\Gamma(\tfrac{1}{3})n^{-4/3}$$

give the required result.

Editorial Note. With but slightly more elaborate analysis, the Proposer has obtained analogous expressions for C_n when the above problem is generalized to give the roots of $\tan x = bx$, where b is any non-zero real number. For example, if $b>1$, we have

$$C_n \simeq \left(\frac{\tan\alpha}{\pi}\right)^{1/2}(2n+1)^{-3/2}(\alpha + \tfrac{1}{2}\sin 2\alpha)^{2n+3/2}.$$

Slight changes are required for other values of b. He has also found the corresponding expansion when the roots of $x = \cot x$ are considered.

Brocard Angle

4401 [1950, 420]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In the triangle ABC for which the cotangent of the Brocard angle is equal to 2, show that the sides satisfy the relation

$$5(a^4 + b^4 + c^4) = 6(b^2c^2 + c^2a^2 + a^2b^2).$$

Show also that it is impossible for the lengths of the sides to be all integers.

Solution by C. S. Venkataraman, Trichur, South India. We shall denote the Brocard angle by ω, and apply the well known results,

$$\cot\omega = \sum\cot A, \qquad \sum\cot A\cot B = 1, \qquad \cot A = \frac{b^2 + c^2 - a^2}{4K},$$

and analogous formulas for cot B and cot C. K is the area of the triangle ABC. Then we have

$$\sum\cot^2 A = (\sum\cot A)^2 - 2\sum\cot A\cot B = \cot^2\omega - 2$$
$$= 4 - 2 = 2\sum\cot A\cot B.$$

Therefore

$$\frac{\sum(b^2 + c^2 - a^2)^2}{16K^2} = 2\,\frac{\sum(b^2 + c^2 - a^2)(c^2 + a^2 - b^2)}{16K^2}$$

which is easily reduced to

$$5\sum a^4 = 6\sum a^2b^2$$

as required. By the addition of $3\sum a^4$ to both sides this may be put in the form

$$8\sum a^4 = 3(\sum a^2)^2. \tag{1}$$

Now suppose, if possible, that a, b, c are all integers. Put $a=ra_1$, $b=rb_1$, $c=rc_1$, where r is their greatest common divisor and a_1, b_1, c_1 have no common divisor >1. Then (1) becomes

$$8\sum a_1^4 = 3(\sum a_1^2)^2.$$

Thus $\sum a_1^2$ is a multiple of 4, but this is clearly impossible unless a_1, b_1, c_1 are all even, contrary to their definition. Hence the sides cannot all be integers.

Also solved by Daniel Finkel, N. T. Seely, Jr., and L. A. Ringenberg.

The Euler Functions $\mathrm{dl}^n(x)$

4402 [1950, 421]. *Proposed by Carl Cohen, Cambridge Junior College, Cambridge, Mass.*

Let the series of functions $\mathrm{dl}^n(x)$ be defined as

$$\mathrm{dl}^{(n)}(x) = \int_0^x \mathrm{dl}^{(n-1)}(t)\,dt/t, \tag{1}$$

$$\mathrm{dl}^{(1)}(x) = -\int_0^x \log(1-t)\,dt/t. \tag{2}$$

Show that for $|x| < \pi/2$,

$$\text{(i)} \qquad \tan x = \sum_{n=1}^{\infty} (2/\pi)^{2n}\left[\mathrm{dl}^{(2n-1)}(1) - \mathrm{dl}^{(2n-1)}(-1)\right]x^{2n-1},$$

$$\text{(ii)} \qquad \sec x = 1 + i\sum_{n=1}^{\infty} (2/\pi)^{2n+1}\left[\mathrm{dl}^{(2n)}(-i) - \mathrm{dl}^{(2n)}(i)\right]x^{2n}.$$

Solution by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y. Consider the well known infinite product

$$\cos\frac{\pi}{2}v = \prod_{k=1}^{\infty}\left[1 - \frac{v^2}{(2k-1)^2}\right].$$

Taking logarithmic derivatives, we find easily

$$\tan\frac{\pi}{2}v = \frac{4v}{\pi}\sum_{k=1}^{\infty}\frac{1}{(2k-1)^2}\left[1 - \frac{v^2}{(2k-1)^2}\right]^{-1}$$

$$= \frac{4v}{\pi}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{v^{2n-2}}{(2k-1)^{2n}}$$

valid for $|v| < 1$. Since the double series is absolutely convergent we may interchange the order of summation to obtain

$$(A) \qquad \tan \frac{\pi}{2} v = \frac{4}{\pi} \sum_{n=1}^{\infty} \left[\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}} \right] v^{2n-1}.$$

Now from (2)

$$\mathrm{dl}^{(1)}(x) = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \cdots$$

valid for $|x| \leqq 1$, and by (1) we find

$$\mathrm{dl}^{(n)}(x) = \frac{x}{1^{n+1}} + \frac{x^2}{2^{n+1}} + \frac{x^3}{3^{n+1}} + \cdots$$

so that

$$\mathrm{dl}^{(2n-1)}(1) - \mathrm{dl}^{(2n-1)}(-1) = 2 \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{2n}}.$$

Upon substituting this in (A) and replacing $\pi v/2$ by x, we have (i).

To prove (ii) start with formula 4, Whittaker and Watson, *A Course of Modern Analysis*, New York (1945), p. 136, which may be put

$$\begin{aligned} \sec \frac{\pi}{2} v &= \frac{4}{\pi} \left[\frac{1}{1 - v^2} - \frac{3}{9 - v^2} + \frac{5}{25 - v^2} - \cdots \right] \\ &= \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2k-1} \left[1 - \frac{v^2}{(2k-1)^2} \right]^{-1} \\ &= \frac{4}{\pi} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^{k-1} v^{2n}}{(2k-1)^{2n+1}} \\ &= \frac{4}{\pi} \sum_{n=0}^{\infty} \left[\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^{2n+1}} \right] v^{2n}. \end{aligned}$$

But

$$i[\mathrm{dl}^{(2n)}(-i) - \mathrm{dl}^{(2n)}(i)] = 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2k-1)^{2n+1}}.$$

Making use of this relation and replacing $\pi v/2$ by x we have (ii).

Also solved by A. E. Livingston, G. Lumer, O. E. Stanaitis, and the proposer.

Altitude Quadric

4403 [1950, 421]. *Proposed by J. H. Butchart, Arizona State College*

A necessary and sufficient condition that a hyperboloid of one sheet be an altitude quadric (*i.e.* any four rulings of one regulus may be altitudes of a tetrahedron) is that any plane section perpendicular to a ruling is an equilateral hyperbola. (Compare the Proposer's note, *The altitude quadric of a tetrahedron* [1940, 383].)

Solution by G. Lumer, Instituto de Matematica Y Estadistica, Montevideo, Uruguay.

The condition is necessary. Denote by (R) and (R'), the two reguli which correspond to the given hyperboloid. Consider four rulings, a, b, c, d, of (R), and the tetrahedron $ABCD$ (where A is on a, B on b, . . .)whose altitudes are a, b, c, d (this tetrahedron exists by hypothesis.) The orthogonal projections of a, b, c on the plane ABC give the altitudes of triangle ABC which pass through the orthocenter H of ABC. Then the line through H and perpendicular to the plane ABC necessarily meets a, b, and c. This shows that this line belongs to (R') and so H is a point of the section of the hyperboloid made by the plane ABC. Now, since A, B, C belong to this section, it is a conic which contains together with A, B, C, the orthocenter of triangle ABC. It is therefore an equilateral hyperbola. Further, plane ABC is orthogonal to d, so that every plane orthogonal to d will cut the hyperboloid in an equilateral hyperbola. Since d is arbitrary, our assertion is proved.

The condition is sufficient. Consider four arbitrary rulings, a, b, c, d, of (R); and consider a plane moving so as to remain orthogonal to d. There exists one (and only one) position of this plane for which the line that lies on it and meets a and b is also orthogonal to c (because as our plane moves the considered lines take all directions which are contained in the plane-orientation orthogonal to d). Consider now this position of the plane and denote by A, B, C the points which it intercepts on a, b, c, respectively. Let h be the ruling of (R') parallel to d; as h meets a, b, c, they are projected onto the plane ABC in lines a', b', c' which pass through the point H where h cuts ABC. As AB is orthogonal to c, c' is an altitude of the triangle ABC; but the conic in which the hyperboloid is cut by the plane ABC is by hypothesis an equilateral hyperbola which contains A, B, C, and H; and as H is on an altitude of ABC, it is its orthocenter. From this it follows that also BC and CA are orthogonal to a and b, respectively, so that there exist planes passing through AB, BC, CA and orthogonal to c, a, b; they determine a point D. To complete the proof we have now only to show that D is on d. But the same method just used for the plane ABC, when applied, say, to ABD, shows that the orthogonal projection of d on this plane is an altitude of the triangle ABD, and so passes through D. As the same is true for the planes BCD and CAD, D lies on d. We have then obtained the tetrahedron $ABCD$ which has a, b, c, d as its altitudes.

Simultaneous Equations with Greatest Integer Function

4404 [1950, 421]. *Proposed by Ky Fan, University of Notre Dame*

Determine all positive solutions x of the infinite system of equations

$$[[n/x]x - x] = n - 1, \qquad n = 1, 2, 3, \cdots . \tag{1}$$

The brackets denote, as usual, the greatest integer function.

Solution by Emil Grosswald, Institute for Advanced Study. We show that the only solutions are

$$0 < x \leqq \tfrac{1}{2} \quad \text{and} \quad x = p/(2p - 1), \qquad p = 1, 2, 3, \cdots .$$

(1) is equivalent to the two inequalities

$$n - 1 \leqq [n/x]x - x < n,$$

of which the second is always satisfied. Dividing by x, we see that (1) is equivalent to

$$n/x - [n/x] \leqq 1/x - 1. \tag{2}$$

If $0<x\leqq\frac{1}{2}$, we have $n-2x\geqq n-1$, whence

$$[n/x] - 1 > n/x - 2 \geqq n/x - 1/x$$

so that (2) is satisfied for all n.

If $x>\frac{1}{2}$ and rational, then $x=p/q$, $(p, q)=1$ and $p\geqq(q+1)/2$. Consider first $p=(q+1)/2$ whence $x=p/q=p/(2p-1)$. Then

$$n/x - [n/x] \leqq 1 - 1/p = 1/x - 1,$$

so that (2) holds for all n. If, on the other hand, $p>(q+1)/2$, since $(p, q)=1$, we may choose integers N and h such that $Nq-hp=1$, or $qN/p=h+1/p$. Let $n=(p-1)N$; then $n/x=(p-1)N\cdot q/p=(p-1)h+(p-1)/p$. Therefore $n/x - [n/x]=(p-1)/p$, but $1/x-1=q/p-1<(p-1)/p$, so that, for at least one value of n, (2) is not true.

Finally, if $x>\frac{1}{2}$ and irrational, then for arbitrary $\epsilon>0$, we can find two integers n and N such that $N-\epsilon<n/x<N$. Choose ϵ such that $\epsilon<1$ and $\epsilon<2-1/x$. Then $[n/x]=N-1$ and

$$n/x - [n/x] > 1 - \epsilon > 1/x - 1$$

so that, for n chosen as above, (2) does not hold.

Also solved by Michael Stimac, and the proposer.

Representations of an Arbitrary Number in Terms of a Given Sequence

4405 [1950, 489]. *Proposed by Paul Erdös, University of Aberdeen, Scotland*

Let $x_1, x_2, \cdots$ be a sequence of real numbers, $x_i>c>0$. We assume that they approach all their limit points from below. Then, for any A, there is only a finite number of solutions of the equation

$$A = \sum_{k=1}^{\infty} a_k x_k, \tag{1}$$

where the a_k are non-negative integers.

A special case occurs if $0 < x_1 < x_2 < \cdots$.

The case $x_k = 1 - 2^{-k}$ was communicated to the proposer by R. A. Rosenbaum.

Solution by William Scott, University of Kansas. If (1) is satisfied, then $\sum a_k < A/c$, and only a finite number of the a_k are different from 0, say $\alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_r$. Moreover there are only a finite number of sets $(\alpha_1, \cdots, \alpha_r)$ of integers such that $0 < \alpha_1 \leqq \alpha_2 \leqq \cdots \leqq \alpha_r$, $\sum \alpha_i < A/c$.

Suppose that there are an infinite number of distinct solutions of (1). Then infinitely many of them have the same set $(\alpha_1, \cdots, \alpha_r)$. Let these solutions be $\{a_k(n)\}$, $n = 1, 2, \cdots$; $a_{k(i,n)}(n) = \alpha_i$, $a_k(n) = 0$ otherwise. Now $c < x_{k(i,n)} \leqq A$. Hence by the Bolzano-Weierstrass theorem, there exists a subsequence with respect to n of $\{a_k(n)\}$ such that (using the same notation for the subsequence as for the original sequence) $\lim_n x_{k(i,n)}$ exists and equals, say y_i. Then

$$\alpha_1 x_{k(1,n)} + \cdots + \alpha_r x_{k(r,n)} = A,$$

$$\alpha_1 y_1 + \cdots + \alpha_r y_r = A.$$

If, for some i, $x_{k(i,n)}$ approaches y_i from below, then for a $j \neq i$, $x_{k(i,n)}$ approaches y_j from above, which is impossible. Hence, for each i, $x_{k(i,n)} = y_i$ for all but a finite number of n. Therefore, since all limit points of $\{x_j\}$ are approached from below, for each i, $k(i, n)$ takes on only a finite number of different values. But this would give an infinite number of identical solutions $\{a_k(n_1)\} = \{a_k(n_2)\} = \cdots$, which contradicts the distinctness of the solutions. Thus in any case we are led to a contradiction, and the theorem is true.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any other editors or officers of the Association.

The Mathematics of Finance. By F. C. Smith. Appleton-Century-Crofts Inc., New York, 1951. 12+356 pages. $4.00.

The preface of this textbook states that "the explanations have been prepared with the student of average mathematical ability in mind." Professor

Smith has successfully kept the student in mind for his explanatory material reads easily and clearly. The definitions, set out in italics, are carefully stated. Throughout the book, as new terms are introduced, they are clearly defined and new ideas are presented simply and logically. In each chapter there are many illustrative examples, with those on interpolation exceptionally good. Line diagrams are introduced in the chapter on compound interest and are used frequently throughout the book. A number of well chosen problems are given after each topic, while sets of review problems are found at the end of each chapter.

The author attempts to stress the use of fundamental principles rather than formulas. It seems that fewer special formulas could have been worked out in some places, particularly in the chapter on bonds. In the chapter on general annuities, the author discusses two cases, p payments per interest period and q interest periods per payment interval. It has always seemed, to this reviewer, that it is simpler and easier for the student to use the unified method which is mentioned in the last paragraph of the chapter.

The usual material is covered in the book. Chapter one consists of a review of logarithms and progressions, two helpful sections on significant digits and rounding off numbers. There are three chapters on actuarial work for the instructor who wishes to include some work on this material. This new text on the mathematics of finance should be a very teachable book, one that will be a pleasure both to the student and to the instructor.

D. E. SOUTH
University of Kentucky

Mathematics of Finance. By A. E. May. American Book Co., New York, 1951. 8+264 pages. $3.00.

A first glance at this book will probably give the reader an impression that it is too abbreviated. However, a little further examination discloses the fact that although the material is very compactly presented, none of the usual topics are omitted. Professor May states that "this text was designed to build the fundamentals of investment theory on just a few simple, important formulas." He has accomplished this aim better than most authors of texts on the mathematics of finance. However, by omitting the formula $S=P(1+rt)$ he seems to lose the important idea of the equivalent relation between present value and amount under a simple interest rate. This omission also results in the illustrative examples on simple interest appearing as exercises in arithmetic.

Each chapter contains a number of carefully prepared examples and an adequate number of problems. Simple annuities are introduced by three excellent illustrations which show how to handle these problems by geometric progressions before the annuity symbols are introduced. In the chapter on general annuities, however, it seems that the complicated processes explained in the examples would be simplified by using the general annuity symbols instead of avoiding them. The author's definition of the term of an annuity as "extending from one payment interval before the first payment through the day of the last

payment" appears to cause some confusion as to what is meant by the present value and amount of an annuity, particularly of an annuity due.

The preface and introduction, discussing the types of problems to be considered, are very well written and should be read both at the beginning and at the end of the course. The introduction closes with this excellent advice to the student, "above all, think as you work."

D. E. SOUTH
University of Kentucky

Lectures on Classical Differential Geometry. By D. J. Struik. Addison-Wesley Press, Inc., Cambridge, Massachusetts, 1950. 8+221 pages.

Here is a welcome addition to the books in English on metric differential geometry of curves and surfaces in ordinary space. Among the features which distinguish it as outstanding are the many interesting historical and biographical notes, the wealth of useful references to the original papers, the clear and concise presentation, and the numerous well drawn figures which will no doubt be effective in carrying out the author's purpose of "stimulating the student's visual understanding of geometry."

The comprehensiveness of the book is amazing for so small a volume. It is designed as a first course in the subject although it contains much more than most classes can assimilate in one semester. There are many well graded exercises ranging from those which require routine manipulation illustrating the text to those which require penetrating thought. Answers or hints for solution for all exercises are found at the back of the volume.

For the most part the notation used is that of the Gibbs form of vector analysis, which, according to the author, "after years of competition with other notations seems to have won the day." This is probably a matter of personal taste and experience, and a matter to be decided according to the future needs of the students. For those who will take only one course in the subject the vector analysis is quite adequate and probably advisable. However, for those who wish to continue to more advanced stages of Riemannian geometry of higher dimensions or to study relativity theory the tensor notation would be a valuable tool in the first course.

After treating the fundamental theory of space curves in the fifty-four pages of Chapter I, the author introduces in Chapter II the two fundamental quadratic forms to portray the elementary theory of surfaces. He is careful to point out the distinction between singular points of a surface and singular points of its parametric representation. For a model of clarity, one should note the article on the indicatrix of Dupin on page 83. The equations of Gauss and Weingarten are discussed in Chapter III. The distinction between intrinsic and extrinsic geometry of a surface is forcefully brought out. A treatment of geodesics, geodesic curvature, geodesic coordinates, and the Gauss-Bonnet theorem is found in Chapter IV. Transformation of coordinates receives little attention. Differential parameters appear in an exercise but their invariance is not mentioned.

In the last chapter the author begins with a discussion of envelopes of one parameter families of surfaces, continues with a study of conformal, isometric, and geodesic mapping, and concludes with minimal surfaces, ruled surfaces, and imaginaries in surface theory. On page 169 it is stated that equiareal mapping will be discussed but the concept appears only in three exercises.

Remarkably few typographical errors were noticed. A few minor lapses in the characteristic care with which the book was written might be mentioned. On page one "the equation is" is followed by *three* equations. On page five the axes are denoted by x, y, z in the figure but by X, Y, Z in the text. There are a few indications that the author has attempted to conserve space by avoiding connecting phrases. This detracts from ease of understanding on the part of readers who are not already familiar with the notation and subject matter. An instance of this is at the middle of page 10 where the statement appears "let us consider a plane $\mathbf{X}\cdot\mathbf{a}=p$; $\mathbf{X}$ generic point of the plane, $\mathbf{a}\perp$ plane, p a constant." A little more preparation for the equation of the plane in vector form might be supplied. The statement on page 11 that the osculating plane passes through "at least" three consecutive points of the curve calls for further explanation if it is to be of meaning to the uninitiated. Also on page 11 it is stated that three vectors are *coplanar* if they "can be moved into one plane." The student might argue that any three vectors can be "moved into one plane." Another instance of the author's tendency toward elliptical language is at the middle of page 134 where a differential equation and an integral of it are displayed on the same line. This practice renders the antecedent of "this" in the following line doubtful for the reader. The reviewer does not favor the introduction of the expression "natural equations" in lieu of the well established "intrinsic equations" of a curve. In exercise 15 on page 195, the meaning of T is not clear.

This is a useful volume which should be at hand for every student and teacher of differential geomtery, and it will surely become a popular text book for students who can devote only one semester to differential geometry.

C. E. SPRINGER
University of Oklahoma

Dictionary of Mathematical Sciences, Volume I (German-English). By Leo Herland. New York, Frederick Ungar Publishing Company, 1951. 235 pages. $3.25.

There has for many years been a real need for a good mathematics German-English dictionary. As a result, this dictionary should be well received, for the author and his collaborators have done an excellent job.

This first modern bilingual mathematics dictionary covers not only the major subjects of mathematics, such as geometry and algebra, but also many fields of application, such as mathematical logic, statistics, commercial arithmetic, and physics and astronomy in those concepts which lend themselves to mathematical treatment. Numerous cross references and model usages are included. The emphasis is on clarity and distinction. E. P. V.

NEW BOOKS RECEIVED

Advanced Dynamics (2 vols.). By H. E. Smart. London, Macmillan and Company, 1951. Vol. I, xi+419 pp. Vol. II, xi+419 pp.

Plane Trigonometry with Tables. Third Edition. By L. M. Kells, W. F. Kern, and J. R. Bland. New York, McGraw-Hill Book Co., 1951. xi+220 pp. +118 pp. of tables. $3.50.

The Algebra of Vectors and Matrices. By T. L. Wade. Cambridge, Mass., Addison-Wesley Press, 1951. 189 pp. $4.50.

Introduction to Number Theory. By Trygve Nagell. New York, John Wiley and Sons Co., 1951. 309 pp. $5.00.

Calculus and Analytic Geometry. By G. B. Thomas. Cambridge, Addison-Wesley Press, Inc., 1951. 685 pp. $6.00.

Differential Geometry. By D. J. Struik. Cambridge, Mass., Addison-Wesley Press, Inc., 1950, viii+221 pp.

Calculus. By Tomlinson Fort. Boston, D. C. Heath and Co., 1951. xii+560 pp. $4.75.

How to Study, How to Solve—Arithmetic—Calculus. By H. M. Dadourian. Cambridge, Mass., Addison-Wesley Press, 1951. vi+121 pp. $.60.

The Classical Theory of Fields. By L. Landau and E. Lifschitz. Cambridge, Mass., Addison-Wesley Press, 1951. ix+354 pp. $7.50.

Descriptive Geometry, A Pictorial Approach. By H. B. Howe. New York, The Ronald Press, 1951. x+332 pp. $4.00.

Dictionary of Mathematical Sciences. Vol. I. By Leo Herland. New York, Stephen Daye Press, 1951. 235 pp. $3.25.

British Scientists. By E. J. Holmyard. New York, The Philosophical Library, 1951. viii+88 pp. $2.75.

Tables to Facilitate Sequential t-Tests. Washington, Department of Commerce, National Bureau of Standards. Government Printing Office, 1951. 19+82 pp. $.45.

Tables of n! and $\Gamma(n+\frac{1}{2})$ *for the First Thousand Values of n.* Bureau of Standards, Government Printing Office, Washington, 1951. 4+10 pp. $.15.

Everyday General Mathematics (*Book* 2). By Wm. Betz, A. B. Miller, F. B. Mitchell, Elizabeth and H. C. Taylor, Boston, Ginn and Company, 1951. 10+438 pp.

The Arithmetic of Better Business. By Frank McMackin, J. A. March and C. E. Baten. Boston, Ginn and Company, 1951. 8+389 pp.

Teaching the Meanings of Arithmetic. By C. N. Stokes. New York, Appleton-Century-Crofts, Inc., 1951. 14+531 pp. $4.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to H. D. Larsen, Albion College, Albion, Michigan.

Pi Mu Epsilon, New York University

The New York University chapter of *Pi Mu Epsilon* held five meetings during 1950–51 at which the following papers were presented:

The principle of the computing machine, by Dr. Gerald Goertzel

Mathematics between the Scylla of physics and the Charybdis of logic, by Dr. John van Heijenoort

Probability theory and games of chance, by Dr. Gottfried E. Noether

The drunkard's path or the gambler's ruin, by Dr. Bernard Friedman

The foundations of physics, by Dr. Henry Margenau.

The officers elected for 1951–52 are: Director, Willard L. Miranker; Vice-Director, Walter Koppelman; Secretary, William C. Zoellner.

Mathematics Club, Montana State University

The following papers were presented during 1950–51 at the monthly meetings of the *Mathematics Club* of Montana State University:

Al-jabra u'al muqabahah, the origin of notation used in algebra, by Dr. T. G. Ostrom

The history of non-euclidean geometry, by George Craft

Math in card playing, by William Jameson

The number system, by Elsie Taylor

Careers in mathematics, by Thomas Bray.

The annual party was held February 28 at Dr. Chatland's home. The spring picnic with the Chemistry Club was held May 25 at the Montana Power Park.

Rachel Kinney and John Marvin were awarded prizes for outstanding work in beginning mathematics. The awards were *Handbooks* donated by the Chemical Rubber Company.

Officers elected for 1951–52 are: President, Tom Bray; Vice-President, Hollis McCrea; Secretary-Treasurer, Pat Lovely; Advisor, Dr. Marsaglia.

Kappa Mu Epsilon, Mount Mary College

The following papers were presented at the monthly meetings of the *Wisconsin Alpha* chapter of *Kappa Mu Epsilon* during 1950–51:

How I became interested in actuarial work, by Mary Hunt

Explanation of actuarial work, by Mr. T. V. Henningston, Mutual Insurance Company

Introduction to topology, by Ann Sanfelippo

Special theorems and ideas in topology, by Janet Haig

Teaching arithmetic in the primary grades, by Marilyn Briggeman

The Chinese abacus, by Adeline Madritsch

Mathematics and design, by Audrey Reiff
Calculating devices, by Betty Haendel
Probability, by Carole Sands
Number bases, by Dolores Wall
Arithmetic and geometric progressions, by Ruth Renwick
Numerology, by Pat Zimmer
Book review: "The psychology of invention in the mathematical world," by Mary Ann Pichotta.

Officers for the year were: President, Adeline Madritsch; Vice-President, Janet Haig; Secretary, Joan Weller; Treasurer, Carol Zaffrann.

Mathematics Club, Rutgers University

During 1950–51, the following papers were presented to the *Mathematics Club* of Rutgers University, the first three by undergraduate students:
Tic-tac-toe in three and four dimensions, by Peter Bender
Can machines think? by Robert Garfunkel
Some ideas from projective geometry, by Oscar W. Greenberg
Several geometric problems, by John C. Slonczewski
Theory of electrical networks, by Dr. Richard M. Cohn
How physical ideas can be used in solving problems in mathematics, by Dr. Louis M. Court
Three contradictions in classical mathematics, by Dr. Clifford D. Firestone
Philosophy and mathematics, by Dr. Charles H. Kaiser
Russell's paradox, by Dr. Robert E. Luce
Unicursal curves, by Dr. Hyman J. Zimmerberg.

Oscar Greenberg was appointed temporary chairman for the first meeting of next year, at which time officers will be elected. The faculty advisor for next year is Dr. Harold S. Grant.

Kappa Mu Epsilon, Upsala College

The *New Jersey Alpha* chapter of *Kappa Mu Epsilon* held six meetings in 1950–51 at which the following papers were presented:
The method of least squares, by Martin Netzler
Kepler's three laws, by William Stachel
Vector analysis, by Ethel Larson
Graphical analysis of complex numbers, by Dr. Howard Fehr
Cryptography: ciphers and codes, by Lloyd Johnson
Dynamic geometry, by Professor David Skolnik.

Dr. Howard Fehr of Columbia University was guest speaker at a joint meeting of the chapter and the Mathematics Club. Members of the New Jersey Beta chapter of Kappa Mu Epsilon were guests.

Officers for 1951–52 are: President, Juanita Sachs; Vice-President, William Stachel; Secretary, Ethel Larson; Treasurer, Martin Netzler; Historian, Donald Lindtvedt; Faculty Sponsor, M. A. Nordgaard.

Mathematics Club, Upsala College

The following papers were presented to the *Mathematics Club* of Upsala College during 1950–51:

Special topics in mathematics, by James Grimshaw and Lloyd Johnson

Map-making, by Juanita Sachs and Professor R. Reed

Partial fractions, by Martin Netzler

Possibility for sailing faster than the wind, by William Stachel

Rocket to the moon, by Paul Kolk

The log function is unique, by Ethel Larson

The cuneiform system in mathematics, by Vera Versfelt.

A Christmas party was held at the home of Professor Reed, faculty sponsor of the club.

Mathematics Club, University of Detroit

The University of Detroit *Mathematics Club* held six meetings in 1950–51. The following papers were presented, the first two by students:

The binary number system, by Gerald Simonds

Quality control, by W. McClintock

Topics in mathematics, by Dr. Lyle E. Mehlenbacher

Mathematics and philosophy, by Dr. Maxwell.

Movies constituted the last program of the year at a joint meeting of the Physics and Mathematics Clubs.

The officers for the year were: President, Gerald Simonds; Vice-President Patrick McNamara; Secretary-Treasurer, Doris Haines; Faculty Sponsors, Dr Emily Pixley, Miss Florence Tetreault.

Pi Mu Epsilon, University of Missouri

The following papers were presented in 1950–51 at meetings of the University of Missouri chapter of *Pi Mu Epsilon*:

Some applications of mathematics in electrical engineering, by Professor Bert Gastineau

Some mathematical aspects of human behavior, by Dr. Herman Betz

A demonstration of soap film surfaces, by Ben Jaeger

Some famous mathematicians and what they did, by Miss Mary Cummings

Importance of mathematics to the Navy, by Lt. Com. Cobb, U.S.N.

Infinite series and summability, by Dr. Paul Burcham.

The chapter conducted a competitive examination in calculus, awarding prizes as follows: first prize, Donald Garnett; second prize, Frederick M. Cash; third prize, Clifford H. Brown. The year's activities ended with a banquet at which Dr. L. M. Blumenthal spoke on *Mathematics as a career*.

Officers for 1951–52 are: President, Harley Newsom; Vice-President, Paul Sims; Secretary, Carl Spohr; Treasurer, Don Putnam; Faculty Sponsor, Miss Mary Cummings.

Mathematics Club, University of Miami

The following topics were presented at meetings of the University of Miami *Mathematics Club* during 1950–51:

Mathematical biophysics, by D. J. R. Foulis

Rings and fields of classes, by Ira Rosenbaum

Geometrical toys, by Carolyn Palmer

Geometries, by Dr. Mayme Logsdon

Abstract Galois theory, by Dr. Marc Krasner

Numbers, numbers, numbers, by Dr. Louis Mordell

Lattices, by Mrs. Georgia Del Franco

Vector spaces, by Edward Stratton

Astronomical triangle, by Capt. Robert

Mathematics and Naval officers, by Rear Admiral Richard Tuggle

Time, by John Maecher

Inductive proof for Descartes' rule of signs, by Mary Magner.

In addition to its regular meetings, the club had a Christmas party and a hayride.

A year's membership in the Mathematical Association of America was awarded to David Foulis for the best paper presented by an undergraduate.

Officers for 1951–52 are: President, David Foulis; Vice-President, Mary Magner; Secretary, John Maecher; Treasurer, Mary Magner.

Pi Mu Epsilon, University of Miami

The *Florida Alpha* chapter of *Pi Mu Epsilon* was installed at the University of Miami on March 21, 1951, with Professor Tomlinson Fort representing the national fraternity at the installation banquet. The following paper was presented at a subsequent meeting:

An axiomatic approach to trigonometry, by Dr. Meyer.

The officers are: Director, Mrs. Georgia Del Franco; Vice-Director, David Foulis; Secretary, Mary Magner; Treasurer, Tadeus Putla.

Undergraduate Mathematics Club, Cornell University

The *Cornell Undergraduate Mathematics Club* was organized on May 9, 1951. Two meetings were held in 1951 at which the following papers were presented:

The history of probability, by Professor Mark Kac

Jumps, oscillations, and Gibb's phenomena, by Professor Ralph Agnew.

The officers are: President, Jerome Sacks; Vice-President, Marcia Goldberg; Secretary-Treasurer, Roberta Torrance.

Delta X, University of Toledo

The following papers were presented at the monthly meetings of *Delta X*, mathematics club of the University of Toledo, during 1950–51:

Number concepts, by Phillip Miller

Mathematical melées, by Frank Sherbourne
The calendar, by Russell Bergquist
Non-euclidean geometry, by Lois Crew
Professional opportunities in mathematics, by Donald Ewing
Omar Khayyam, the mathematician, by Larry Reger
What is mathematics? by Professor Clair Blackall.

A banquet preceded the May meeting and a picnic was held in June. Awards were given to Phillip O'Neill and Melvin Batch as outstanding students in freshman mathematics.

Officers for 1951–52 are: President, Lois Crew and Phillip Miller; Vice-President, Larry Reger; Secretary-Treasurer, Carol Garn.

Mathematics Club, Harvard University

The 1950–51 season of the *Harvard Mathematics Club* featured the following talks by students and faculty members:

Maximal problems in the calculus, by Professor Joseph Walsh
The spectral theorem, by Professor George W. Mackey
Three pearls of number theory, by Dr. Raymond Ayoub
An algebra of power series, by Ernest Schlesinger
Random walk, gambling techniques, and absorption processes, by Paul Lochak
The magic of Diophantine equations, by Harry Gonshoe
A geometrical approach to Dirichlet's problem, by Jeremy Bernstein.

A picnic concluded the club's activities for the year.

The officers were: President, William Turanski; Vice-President, Ariel Zemach; Secretary, Harry Gonshoe; Treasurer, Hal Roydon.

Zeta Mu Tau, University of Washington

Zeta Mu Tau is an undergraduate mathematics honorary fraternity for students in engineering, mathematics, and the physical sciences. Three meetings were held during the 1950–51 academic year. The following papers were presented:

Some odd algebras and their application to logic, by Prof. Hewitt
Solution of one equation in two unknowns, by Prof. Jerbert.

The year's activities were climaxed by a banquet honoring the new members initiated during the year.

Officers elected for the year 1951–52 are: President, Kenneth Hammer; Vice-President, John Benoit; Treasurer, Robert Winders; Secretary, Paul Ellinger; Faculty Advisor, Prof. Haller.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

STANFORD UNIVERSITY COMPETITIVE EXAMINATION IN MATHEMATICS

The sixth Stanford University Competitive Examination in Mathematics (see AMERICAN MATHEMATICAL MONTHLY, vol. 53 (1946), pp. 406–409) was held on April 7, 1951, in 35 high schools in California; 155 senior high school students participated. The following problems were proposed:

1. The length of the perimeter of a right triangle is 60 inches and the length of the altitude perpendicular to the hypotenuse is 12 inches. Find the sides of the triangle.

2. A quadrilateral is cut into four triangles by its diagonals. We call two of these triangles "opposite" if they have a common vertex but no common side. Prove the following statements: (a) The product of the areas of two opposite triangles is equal to the product of the areas of the other two opposite triangles. (b) The quadrilateral is a trapezoid if, and only if, there are two opposite triangles equal in area. (c) The quadrilateral is a parallelogram if, and only if, all four triangles are equal in area.

3. We consider the frustum of a right circular cone. The plane that is parallel to the lower and upper bases of the frustum and at equal distance from both intersects the frustum in the "median circle." The frustum and a cylinder have the same altitude, and the median circle of the frustum is the base of the cylinder. Which one of these two solids has the greater volume, the frustum or the cylinder? Prove your answer! (A possible proof is by algebra: Express both volumes in terms of suitable data and transform their difference so that its sign becomes obvious.)

The writer of the best paper, Mr. Edward Barlow, student at Castlemont High School, Oakland, California, received a $660 Scholarship at Stanford University. Mr. J. F. Wolfe of Pasadena, California, received "Honorable Mention."

EMPLOYMENT OPPORTUNITIES BULLETIN OF THE BUREAU OF ORDNANCE

The Navy's Bureau of Ordnance is faced with an unprecedented expansion program. There is a need for Mathematicians in a salary range of $3,410 to $8,360 per annum.

The Bureau of Ordnance is responsible for the research, design, development, procurement, manufacture, distribution, maintenance, repair, alteration, and effectiveness of Naval Ordnance.

The Bureau publishes bi-monthly a 60-page booklet which summarizes current vacancies and job opportunities. It may be obtained without cost from

the Bureau of Ordnance, Department of the Navy, Washington 25, D. C.

PERSONAL ITEMS

Mrs. Jewell H. Bushey, chairman of the Department of Mathematics of Hunter College, was the official representative of the Association at the Sixteenth Educational Conference which was held in New York City on November 1–2, 1951.

Professor J. A. Ward of the University of Kentucky represented the Association at the inauguration of President F. A. Rose of Transylvania College on November 30, 1951.

St. Louis University announces: Assistant Professor W. A. Vezeau has been promoted to an associate professorship; Dr. J. J. Andrews has been promoted to an assistant professorship.

Syracuse University announces the following: Professor D. E. Kibbey, formerly acting chairman of the Department of Mathematics, has been appointed Chairman of the Department; Instructors R. M. Exner and Kathryn A. Morgan have been promoted to assistant professorships; Dr. D. G. Austin, previously a graduate student at the University of Chicago, has been appointed to an instructorship.

The United States Naval Academy makes the following announcements: Senior Professor L. T. Wilson has retired with the title of Professor Emeritus and is teaching now at Jacksonville State Teachers College, Alabama; Associate Professor H. C. Stotz has been promoted to a professorship; Assistant Professors H. L. Kinsolving and K. L. Palmquist have been promoted to associate professorships; Assistant Professor A. R. Craw has been recalled to active duty in the United States Navy; Assistant Professor A. H. Steinbrenner is on leave of absence during 1951–52.

At the University of Colorado: Associate Professor Albert Edrei of the University of Saskatchewan has been appointed to an assistant professorship; Assistant Professor A. B. Farnell is on leave of absence and is teaching at the United States Military Academy.

The University of Michigan announces the following: Assistant Professor Gail Young has been promoted to an associate professorship; Dr. E. E. Moise has been promoted to an assistant professorship; Dr. Raoul Bott, who has been studying at the Institute for Advanced Study, has been appointed to an instructorship; Dr. A. B. Clarke, formerly a graduate student at Brown University, has been appointed to an instructorship; Assistant Professor W. J. LeVeque is on leave of absence and is studying at the University of Manchester, England.

The University of North Carolina at Chapel Hill reports the following: Professor E. A. Cameron has been granted a leave of absence for 1951–52 and is visiting a number of universities under the provisions of an Educational Foundation Fund grant; Assistant Professor I. R. Hershner has been granted a leave of absence due to his recall to active service as an officer in the United

States Army.

The University of North Carolina, Woman's College, announces: Miss Frances E. Wolfe, previously an instructor at the University of Kansas, has been appointed to an instructorship; Assistant Professor Lila P. Walker has been granted a leave of absence for 1951–52 and is engaged in graduate work at the University of North Carolina at Chapel Hill.

The University of South Dakota makes the following announcements: Instructor Don Alkire of the University of Kansas and Instructor Norman Hoover of Washburn University have been appointed to instructorships; Professor W. W. Gutzman is on leave of absence and is located at Dahlgren, Virginia.

At the University of Toronto: Professor G. G. Lorentz, formerly special lecturer and holder of Lady Davis Foundation Fellowship, has been promoted to an assistant professorship; Dr. J. A. Jacobs, formerly lecturer in Applied Mathematics at Royal Halloway College, has been appointed Associate Professor of Applied Mathematics; Dr. Abraham Robinson, previously deputy head of the Department of Aerodynamics of the College of Aeronautics at Cranfield, has been appointed Associate Professor of Applied Mathematics.

Washington University announces the appointments of Mr. J. W. Diesel and Mr. D. L. Guy to graduate assistantships.

Wisconsin State College at LaCrosse announces the following: Professor L. K. Adkins, who has served as Head of the Department of Mathematics for thirty-three years, has retired with the title of Professor Emeritus; also, Professor Adkins was retired recently from the Reserve Component of the United States Army and given a commission as Colonel, Honorary Reserve; Mr. Arnold Temte has become a member of the mathematics staff.

Assistant Professor L. V. Albero of Western Reserve University has resigned from this position and is a member of the staff of the Laboratory of the National Advisory Committee for Aeronautics, Cleveland.

Dr. R. W. Allen, formerly a graduate student at St. Louis University, has been appointed to an instructorship at Xavier University, Cincinnati.

Mr. F. H. Applebaum, formerly a student at the University of Pennsylvania, is Assistant Life Actuary in the Insurance Department, Commonwealth of Pennsylvania, Philadelphia.

Associate Professor Iola M. Baker of Western Reserve University has been promoted to a professorship.

Instructor A. F. Bartholomay of Rutgers University has resigned from this position and is working at the Laboratory for Electronics, Boston.

Mr. S. K. Berberian, previously a graduate student at the University of Chicago, has been appointed to an instructorship at Southern Illinois University.

Professor L. M. Blumenthal is on leave of absence from the University of Missouri in order to join the staff of the Institute for Numerical Analysis, National Bureau of Standards, Los Angeles.

Assistant Professor H. D. Brunk of Rice Institute is employed now as

Mathematician in the Systems Evaluation Department, Sandia Corporation, Albuquerque.

Associate Professor A. B. Carson has been promoted to the position of Professor and Head of the Mathematics Department of the United States Air Forces Institute of Technology.

Dr. E. D. Cashwell, formerly of Ohio State University, is a member of the staff of Los Alamos Scientific Laboratory.

Mr. H. E. Chrestenson, who has been a teaching assistant at the State College of Washington, is now a teaching fellow at the University of Oregon.

Assistant Professor Eckford Cohen of Syracuse University is now at the Institute for Advanced Study.

Dr. B. H. Colvin, formerly of the University of Wisconsin, is with the Mathematical Services Group, Boeing Airplane Company, Seattle.

Professor N. A. Court of the University of Oklahoma has retired with the title of Professor Emeritus.

Mr. M. R. Demers, University of Nevada, has been promoted to an assistant professorship.

Mr. R. E. Dowds, previously a student at Kent State University, has accepted a position as Mathematician at the Naval Ordnance Plant, Indianapolis.

Assistant Professor W. H. Durfee of Dartmouth College is now at the National Bureau of Standards, Washington, D. C.

Mr. R. E. Ekstrom, formerly an assistant instructor at Westminster College, Missouri, has a position as Mathematician at the United States Naval Ordnance Plant, Indianapolis.

Mr. Norbert Ellman, previously of the Milwaukee School of Engineering, has been appointed to an instructorship at Marquette University.

Instructor J. C. Gibson of Rensselaer Polytechnic Institute is associated now with the Department of Applied Science, International Business Machines Corporation, Albany.

Associate Professor J. R. Hanna of the University of Wichita is on leave of absence during 1951–52 and is serving as a part-time instructor at the University of Colorado.

Instructor J. C. Harden, Jr., of Clemson Agricultural College has been promoted to an assistant professorship.

Mr. J. J. Hart has accepted a position as Mathematician at Redstone Arsenal, Huntsville, Albama.

Mr. F. C. Hatfield of Bemidji State Teachers College, Minnesota, has been appointed to an instructorship in the Department of Physics and Physical Science, Blackburn College.

Mr. Julius Honig, previously a mathematician at the National Bureau of Standards, Washington, D. C., has been appointed Research Analyst in the Department of Defense, Washington, D. C.

Mr. D. M. Houser, formerly a graduate assistant at Kansas State College,

has a position as Mathematician with the Electrical Staff Department, Boeing Airplane Company, Wichita.

Associate Professor L. W. Johnson of Purdue University has been appointed Professor and Head of the Mathematics Department of Oklahoma Agricultural and Mechanical College.

Research Assistant Professor Joseph Keller of Washington Square College, New York University, has been promoted to a research associate professorship.

Instructor Ruth E. Kennedy of Louisiana Polytechnic Institute is teaching now in Neville High School, Monroe, Louisiana.

Instructor A. E. Kinney of New York State Maritime Academy, Fort Schuyler, has been promoted to an assistant professorship.

Dr. P. E. Klopsteg has been granted a leave of absence from his position as Professor of Applied Science and Director of Research of Northwestern Institute of Technology so that he may serve as Assistant Director of the National Science Foundation for the Division of Physical, Mathematical and Engineering Sciences.

Assistant Professor C. E. Langenhop of Iowa State College has accepted a position as Acting Director of Analytical Research Group, Princeton University.

Mr. A. S. Lee, Jr., formerly a graduate student at Southern Methodist University, has accepted a position as Mathematician in the Research Laboratory, Magnolia Petroleum Company, Dallas.

Professor Charles Loewner of Syracuse University has been appointed to a professorship at Stanford University.

Mr. Roy Long, previously a graduate student at the University of Georgia, is now Job Analyst in the Civilian Personnel Office, Camp Gordon, Georgia.

Instructor Martin Maltenfort of Manhattan College has been promoted to an assistant professorship; at present he is on leave of absence and is serving with the United States Army in Washington, D. C.

Mr. F. H. McGar, Jr., has been appointed to an instructorship at Sweet Briar College.

Mr. A. E. Miller, formerly a student at the University of Pennsylvania, is now Assistant Research Engineer for the Burroughs Adding Machine Company, Philadelphia.

Dr. B. E. Mitchell, who was a graduate assistant at the University of Wisconsin, has been appointed to an assistant professorship at Alabama Polytechnic Institute.

Mr. J. S. Morison has accepted a position as Mathematician at Douglas Aircraft Company, Santa Monica.

Mr. D. E. Morrill, previously a graduate fellow at the University of Mississippi, is serving in the United States Army.

Lecturer Leo Moser of the University of Manitoba has been appointed Lecturer at the University of Alberta.

Associate Professor David Moskovitz of Carnegie Institute of Technology has been promoted to a professorship.

Graduate Assistant J. A. Nickel of Oregon State College is now a research assistant at Indiana University.

Mr. D. G. O'Connor, who has been Junior Engineer, Sylvania Electric Company, Bayside, Long Island, is employed as Design Engineer by International Business Machines Corporation, Endicott, New York.

Emeritus Dean J. R. Overman of the College of Liberal Arts, Bowling Green State University, has been named Dean of Faculties.

Mr. Frank Pomilla, formerly a member of the Mathematics Department of St. John's College, St. John's University, is now on the staff of the Department of Physics.

Dr. R. F. Reeves of Iowa State College has been appointed to an instructorship at Ohio State University.

Instructor A. A. Ritchie of the University of Tennessee has accepted a position as a teacher at Ponca City Senior High School, Oklahoma.

Mr. G. R. Schriro, who has been teaching at North Plainfield High School, New Jersey, is teaching now at Farmingdale High School, Long Island.

Instructor C. E. Shotwell of the University of the South has been promoted to an assistant professorship.

Mr. J. A. Silva, formerly an instructor at Duke University, has a position as Research Associate at the Institute for Cooperative Research, Johns Hopkins University.

Instructor R. L. Snider of the University of Missouri is teaching now at Community High School, Blue Island, Illinois.

Mr. R. H. Spohn has accepted a position as Mathematician at the Kellex Corporation, New York City.

Graduate Assistant R. A. Spong of Northwestern University is now Mathematician in the United States Navy Underwater Sound Laboratory, Fort Trumbull, New London, Connecticut.

Dr. C. G. Stipe, professor of mathematics and Dr. J. H. Service, professor of physics, Michigan College of Mining and Technology, have completed the monumenting of the water boundaries between the states of Michigan, Minnesota and Wisconsin; this work was done during the past two summers for the Joint Survey Commission of the three states.

Mr. E. J. Stulken has been promoted to the position of Chief Seismologist of Geophysical Service, Dallas.

Assistant Professor D. B. Sumner of McMaster University has been promoted to an associate professorship.

Miss Anna L. Taylor has been appointed Mathematician at the United States Naval Ordnance Test Station, Pasadena.

Mr. H. R. Uhl, formerly a graduate teaching fellow at Tulane University, is serving in the United States Navy.

Assistant Professor M. C. Waddell of Western Reserve University is now at the Applied Physics Laboratory, Silver Spring, Maryland.

Graduate Assistant A. M. Wedel of Iowa State College has been appointed to an assistant professorship at Bethel College, Kansas.

Instructor Anne M. Whitney of the University of Pennsylvania has been appointed to an assistant professorship at Goucher College.

Assistant Professor Louise A. Wolf of the University of Wisconsin in Milwaukee has been promoted to an associate professorship.

Instructor D. M. Young, Jr., of Harvard University has a position as Mathematician at Aberdeen Proving Ground, Maryland.

Graduate Assistant R. L. Young, University of Denver, has a position as Mathematician at Wright-Patterson Air Force Base, Dayton.

Assistant Professor H. J. Zimmerberg of Rutgers University has been promoted to an associate professorship.

Miss Elsie O. Bull, retired head of the Department of Mathematics, State Teachers College, Chester, Pennsylvania, died on November 2, 1951.

Professor L. C. Mathewson of Dartmouth College died on October 27, 1951. He was a charter member of the Association.

Emeritus Professor J. F. Messick of Emory University died on October 6, 1951. He was a charter member of the Association.

Mr. Justin Nicolet, a structural engineer for the Department of Subways and Highways, Chicago, died on October 22, 1951. He had been a member of the Association for thirty-three years.

Professor J. B. Rosenbach, head of the Department of Mathematics of Carnegie Institute of Technology, died on November 6, 1951. He had been a member of the Association for thirty-five years. He was a member of the Board of Governors of the Association and had served recently as Chairman of the Allegheny Mountain Section.

Dr. J. I. Vass, who was retired from his position as instructor at the University of Wisconsin in Milwaukee, died on March 31, 1951. He had been a member of the Association for twenty-seven years.

Emeritus Professor H. C. Wolff, Drexel Institute of Technology, died on October 30, 1951.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following thirty-one persons have been elected to membership by the Board of Governors on applications duly certified.

J. D. BAUM, M.A. (Yale) Grad. Student, Yale University, New Haven, Conn.

MRS. LOUISE G. BELAI, Ph.D. (Vienna, Austria) Head of Mathematics Department, Our Lady of Cincinnati College, Cincinnati, Ohio.

MRS. SHIRLEY A. BLACKETT, M.Ed. (Penna. State) Test Constructor, Educational Testing Service, Princeton, N. J.

K. H. CARLSON, M.S. (Iowa) Instr., Michigan State College, East Lansing, Mich.

R. J. CORMIER, B.S. (Chattanooga) Grad. Student, University of Alabama, University, Ala.

MELCHOIR DI CARLO-COTTONE, B.S. (Wagner) Mathematician, Department of the Air Force, Washington, D. C.

N. L. ELLMANN, M.S. (Marquette) Instr., Marquette University, Milwaukee, Wis.

G. F. FEEMAN, B.S. (Muhlenberg) Grad. Assistant, Lehigh University, Bethlehem, Pa.

J. F. FOSTER, JR., M.A. (Harvard) Instr., University of Arizona, Tucson, Ariz.

JOHN HILZMAN, Student, University of Rhode Island, Kingston, R. I.

L. B. HOUK, Student, Marquette University, Milwaukee, Wis.

R. B. KELLOGG, Student, Massachusetts Institute of Technology, Cambridge, Mass.

ROBERT KRANS, B.A. (Hope) Asst. Instructor, University of Arizona, Tucson, Ariz.

H. T. LABORDE, M.A. (South Carolina) Part-time Instructor, University of North Carolina, Chapel Hill, N. C.

R. M. LAKNESS, Ph.D. (California) Instr., University of California, Berkeley, Calif.

S. L. LIDA, B.A. (Kansas) Mathematician, Los Alamos Scientific Laboratory, N. Mex.

R. B. LYON, M.S. (Idaho) Asso. Professor, Arizona State College, Tempe, Ariz.

G. C. MCCOYD, Student, St. John's University, Brooklyn, N. Y.

D. E. RICHMOND, Ph.D. (Cornell) Professor, Williams College, Williamstown, Mass. (On leave and a Lecturer at M.I.T. for the year)

W. E. RUEL, Student, St. John's College, Brooklyn, N. Y.

D. H. SHAFFER, M.S. (Carnegie) Asst. Research Mathematician, Carnegie Institute of Technology, Pittsburgh, Pa.

LELIA V. SHORE, A.B. (Randolph-Macon Woman's C.) Teacher, Hickory High School, N. C.

ANNETTE SINCLAIR, Ph.D. (Illinois) Instr., University of Tennessee, Knoxville, Tenn.

C. R. SWENSON, M.A. (Emory) Asst. Professor, Georgia Institute of Technology, Atlanta, Ga.

R. E. THOMAS, M.A. (Ohio State) Instr., State University of Iowa, Iowa City, Iowa.

EUGENE USDIN, B.S.M.E. (Purdue) Research Engineer, Stanolind Oil and Gas Company, Tulsa, Okla.

F. B. VAN WYK, Student, Massachusetts Institute of Technology, Cambridge, Mass.

L. F. WAHLSTROM, Ph.D. (Wisconsin) Teacher, Eau Claire State College, Wis.

SUE R. WALDMAN, Student, Hofstra College, Hempstead, N. Y.

MARIA A. WEBER, Ph.D. (Cornell) Lecturer, University of California at Los Angeles, Calif.

J. H. WOLF, JR., M.A. (Buffalo) Dynamic Analysis, Bell Aircraft Corporation, Niagara Falls, N. Y.

MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held at Vanderbilt University and George Peabody College for Teachers, Nashville, Tennessee, on Friday and Saturday, March 16–17, 1951. Professor C. G. Phipps, Chairman of the Section, and Professor B. G. Clark, Vice-Chairman, presided over the general sessions, Professors H. S. Kaltenborn, I. E. Perlin, Augustus Sisk and F. L. Wren presided over subsections, and Vice-Chancellor C. M. Sarratt, Vanderbilt University, was the master of ceremonies Friday evening at the dinner in honor of Professor Saunders MacLane, guest of the Section.

There were about one hundred seventy present including the following

eighty members of the Association: E. S. Ashcraft, J. H. Banks, W. R. Baum, W. S. Beckwith, A. H. Black, R. V. Blair, Floyd Bowling, M. G. Boyse, C. L. Bradshaw, S. K. Bright, N. R. Bryan, B. F. Bryant, B. G. Clark, G. M. Conwell, R. W. Cowan, J. C. Currie, H. J. Dark, L. A. Dye, E. L. Eagle, R. B. Folsom, J. R. Garrett, D. B. Goodner, W. W. Graham, B. F. Hadnot, J. J. Hart, P. R. Hill, Sr., A. T. Hind, C. W. Hook, L. P. Hutchison, J. A. Hyden, S. L. Jamison, T. L. Jordan, Jr., H. S. Kaltenborn, J. W. LaGrone, J. N. P. Lawrence, R. J. Levit, F. A. Lewis, Lee Lorch, G. H. Lundberg, Saunders MacLane, Elna B. McBride, W. G. McGavock, S. W. McInnis, W. N. Mebane, Nellie P. Miser, W. L. Miser, R. H. Moorman, W. V. Neisius, T. A. Newton, Lillian G. Perkins, I. E. Perlin, C. G. Phipps, V. Elise Qualls, Ellen F. Rasor, L. T. Ratner, G. E. Reves, H. A. Robinson, E. B. Shanks, D. C. Sheldon, Augustus Sisk, C. B. Smith, E. L. Stanley, L. W. Stark, R. B. Stiles, H. E. Taylor, W. C. Taylor, Jr., J. M. Thomas, J. C. Thurman, H. S. Thurston, G. L. Tiller, E. A. Voorhees, Jr., D. T. Walker, J. G. Wall, J. R. Wesson, M. C. Wicht, W. L. Williams, R. L. Wilson, G. N. Wollan, R. S. Wollan, F. L. Wren.

At the business session the following officers were elected: Chairman, B. G. Clark, Vanderbilt University; Vice-Chairman, H. K. Fulmer, Georgia Institute of Technology; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The Section voted to hold its next meeting on March 21–22, 1952, at Georgia Institute of Technology and Agnes Scott College.

The program consisted of the following papers:

1. *A system of envelopes associated with the cuspidal cubic*, by Professor A. H. Black, Southern Illinois University.

From a point on the cuspidal cubic $ky^2=x^3$ there is one tangent to the cubic having point of contact T and x and y intercepts R and S. For all positions of Q the ratios of the distances between Q, R, S, and T are constant. At a point P, dividing any of the above segments into the ratio r_1/r_2, the line perpendicular to the tangent envelops a curve. The equations and the properties of this system of curves were discussed.

2. *On defining some elementary functions*, by Professor J. C. Currie, Georgia Institute of Technology.

Some classroom notes on the definitions of $\log x$, $\sin x$, and e^{ix} were discussed.

3. *Magic circles*, by Professor S. W. McInnis, University of Florida.

In this paper the author discussed the various types of magic circles, their essential properties, and their formation from certain magic squares.

4. *A simple method for estimating retirement allowances*, by Professor R. J. Levit, University of Georgia.

Upper and lower bounds were obtained for the amount of an annuity with periodic payments varying in accord with certain hypotheses. The result finds application in estimating the allowance to be expected under retirement systems of the usual contributory type.

5. *Maxima and minima under restraint*, by Professor C. G. Phipps, University of Florida. To appear in an early issue of this MONTHLY.

6. *Some notes on the Sylvester eliminant*, by Professor F. L. Wren, George Peabody College for Teachers.

The pivotal element method for evaluating a determinant, when applied to the Sylvester eliminant of two equations, is very effective in determining the highest common factor of the two functions and the roots common to the two equations.

7. *On the binomial series*, by Mr. G. N. Wollan, University of Georgia.

Facts regarding convergence and summability of the binomial series were summarized.

8. *Notes on graphical computation*, by Mr. M. D. Prince, Engineering Experiment Station, Georgia Institute of Technology, introduced by the Secretary.

This speaker presented a coordinate system on which sinusoidal functions plot as straight lines. Properties of this coordinate paper were discussed, including its application to graphical solutions, and its use in solving certain one and two parameter equations involving sine and cosine functions.

9. *Partition of space*, by Professor H. A. Robinson, Agnes Scott College.

This speaker considered as an application of combinatory analysis certain problems of the partition of space.

10. *Summation of polynomials*, by Mr. A. T. Hind, Jr., University of Georgia.

The formulas for the sums of squares and higher powers were proved by elementary methods of the calculus of finite differences. The summation procedure was also used in determining the sum of an infinite series and in solving "shot-piling" problems.

11. *The Clairaut partial differential equation*, by Professor I. E. Perlin, Georgia Institute of Technology.

A complete integral for the Clairaut partial differential equation can be obtained from the Lagrange-Charpit method by reducing the given equation to an ordinary Clairaut differential equation.

12. *On orthogonality properties of solutions of a certain linear differential equation*, by Professor R. W. Cowan, University of Florida.

A general solution of the second order differential equation containing two parameters is obtained by Frobenius method. Particular solutions are shown to satisfy several recurrence relations. Under certain conditions these particular solutions are proved orthogonal.

13. *Remarks concerning the limit of error for Simpson's rule*, by Mr. J. G. Wall, University of Georgia.

A simple and direct derivation of the limit of error for Simpson's rule is given. This derivation is not generally given in calculus text books.

14. *A method for summing convergent alternating series*, by Professor B. F. Hadnot, North Georgia College.

In this paper a method for summation of convergent alternating series by the use of summation by parts and the Euler-Maclaurin summation formula was established.

15. *Moisture stresses in certain types of wood laminates*, by Professor C. B. Smith, University of Florida.

This investigation is concerned with the elastic behavior of a two-layer wood plate subjected to a uniform change in moisture content. The plate considered is general since the direction of the grain in one layer is assumed to be inclined at any angle with the direction of the grain in the other layer. By means of a mathematical analysis the distribution of stress in infinite two-layer plates is discussed. It is also shown that all plates of this type will warp unless the direction of the grain is the same in both layers.

16. *A determinant for statistical moments*, by Professor W. V. Neisius, Georgia Institute of Technology.

An nth order determinant was developed for converting moments about the origin to moments about the mean. Also the determinant for μ_i' in terms of $\mu_1, \mu_2 \cdots, \mu_i$ was developed. In each case, the method of pivotal condensation was used. By means of one of the many convenient methods which have been developed for evaluating a determinant, these expressions prove useful for calculating purposes.

17. *An application of the theory of normal families to a problem in Riemann surfaces*, by Dr. H. E. Taylor, The Florida State University.

The classical theorem, "Every family of functions holomorphic in a domain D and which fail to take two values is a normal family" is used to prove that a certain Riemann surface is the image of the z-sphere less the point $z = \infty$. The proof is by contradiction.

18. *Solution and generalization of a problem in the* MONTHLY, by Mr. T. A. Newton, University of Georgia.

A solution of problem number 4374 in this MONTHLY, December, 1949, concerning the Abel-Dini theorem for divergent series, as well as a solution for a more general form of the problem, was obtained by applying a theorem of de La Vallee Poussin concerning divergent series.

19. *What makes students think?* by Professor Saunders MacLane, University of Chicago.

The question is a hard one to which in discouragement a teacher might answer: nothing. Nevertheless it is an important one, and subsumes the most important objective of our teaching. Making students think is more basic than making students use mathematics in connection with other disciplines. Current educational theory has failed in not sufficiently emphasizing this point. It is thus something with which the Mathematical Association is seriously concerned.

A still further question is: What makes students do original thinking? Various methods of training original thinkers at different mathematical centers may be compared. The results show that there are and can be effectual ways of making students think and think originally, but that there is no standardized way of doing this all in the same fashion, for all schools, all teachers, and all students. Thinking must fit the circumstances, so must teaching to think.

20. *Cardinal lattices*, by Professor L. T. Ratner, Vanderbilt University.

A cardinal lattice is defined to be an isomorph of the lattice of all T-space topologies inducible on a given point set. One topology is said to be "weaker" than another topology if every set open in the first topology is also open in the second; this partial ordering makes a complete lattice out of the topology totality. A cardinal lattice is modular if and only if it (or some isomorph) is generated by a set containing fewer than three points. Additional observations deal with sublattices generated when the base set is required to carry an algebraic structure, with all operations assumed continuous.

21. *A simplified Tschirnhaus transformation*, by Professor H. S. Thurston, University of Alabama.

An elementary method for determining the transformation $\rho=\phi(\eta)$, where η is a primitive number of the field $Ra(\rho)$, was presented. For the general irreducible equation satisfied by ρ, a function can be found whose vanishing is the necessary and sufficient condition that η be imprimitive.

22. *The definition of conics in analytic geometry*, by Professor L. A. Dye, The Citadel.

This paper develops the thesis that the study of the conics will proceed more smoothly and logically, if an analytic definition for these curves is used.

23. *On conic sections in the elliptic plane*, by Professor D. B. Goodner, The Florida State University.

Standard forms, developed by elementary methods, of the equations of the conic sections in the elliptic plane were shown to be special cases of the standard form of the equation of the ellipse. A method of classifying these standard forms was introduced.

24. *On identities and the use of the identity sign*, by Professor C. G. Phipps, University of Florida.

A conditional equation, written with a two-barred sign, *always* contains at least one dependent variable. An identity, written with a three-barred sign, *never* contains a dependent variable. Consequently, the sign in the equation defining the value of a measurable quality should have two bars, not three. This convention is frequently violated. On the other hand, equations relating to homogeneous functions which are true identities are seldom written as such. Furthermore, identities, since they impose no conditions upon the variables appearing in them, should never be counted among a set of equations to be solved simultaneously. Confusion in these matters can be avoided only by the proper designation of each type of equation, either by the use of the appropriate equality sign or by some other specific means.

25. *Connectivity and homotopy groups*, by Professor Saunders MacLane, University of Chicago.

The study of homotopy groups is currently one of the most active aspects of algebraic topology. The first homotopy group (fundamental group) of a topological space X has as elements equivalence classes of mappings of a unit interval into X, with both ends of the interval mapped into the chosen base point. Similarly, elements of the second homotopy group are equivalence classes of continuous mappings of the unit square into the space, with the whole boundary mapped into the base point. The multiplication of elements in the second and higher homotopy groups is abelian. The explicit computation of the higher homotopy groups of simple spaces is notoriously difficult, even for the homotopy groups of higher dimensional spheres. Recent spectacular results of J. P. Serre have given much new information; in particular, it is now known exactly which homotopy groups of which spheres are infinite. Recent results of Eilenberg-MacLane give an effective algebraic determination of all the singular homology groups of a space when only one homotopy group is not trivial.

26. *Poles and polars with respect to a triangle with an application*, by Professor Augustus Sisk, Maryville College.

The speaker derived the equations for the transformation carrying a point into its polar line. This transformation was used to solve a certain problem in which trilinear coordinates were employed.

27. *Integral right triangles with an integral acute angle bisector*, by Mr. D. T. Walker, University of Georgia.

Any integral right triangle has sides $2mt$, $t(m^2-n^2)$, $t(m^2+n^2)$, where m and n are integers such that m and n are relatively prime, of opposite parity, and unequal, and t is any integer. Some such triangles have an integral bisector of one acute angle; i.e., in triangle (28, 96, 100), 35 bisects the angle opposite 96. This paper discussed the form of such triangles having an integral acute angle bisector.

28. *The coefficients of a regular normal operator*, by Professor S. L. Jamison, Florida State University.

The coefficients of a normal operator $N(z)$ which is expressible as a convergent power series in the complex parameter z were shown to form an abelian system of normal operators. However, if the parameter involved is a real parameter, these coefficients need not even be normal.

29. *A synthetic device for G. C. D. process*, by Professor E. B. Shanks, Vanderbilt University.

Let f and g be polynomials of degrees m and n $(m \leqq n)$ and with leading coefficients f_0 and g_0 respectively. By means of the operation $f \cdot g = x(f_0 g - g_0 f x^{n-m})$ for polynomials and its analogue for n-tuples, a synthetic procedure was demonstrated for dividing any two polynomials, for obtaining their greatest common divisor, for determining the vanishing or non-vanishing of their resultant, for solving systems of linear equations and for evaluating determinants. This "synthetic elimination" procedure effects a great saving in time and is accomplished without use of the process of division.

30. *The historical development of the function concept*, by L. W. Stark, George Peabody College for Teachers.

Using a number of references from classical mathematical textbooks and articles, the author traced the broadening concept of function from the definition given by Jean Bernoulli to the definitions given by Dedekind and Hobson.

31. *A survey course in mathematics for college students*, by Professor H. S. Kaltenborn, Memphis State College.

The "cultural" type of course, for students interested in acquiring a general education, was discussed with regard to its objectives, selection of topics and method of presentation.

32. *An analysis and evaluation of college undergraduate mathematics*, by Professor N. R. Bryan, Clemson College.

Much greater coordination should be attained in undergraduate mathematics if each subject is developed according to the demands of the subject. The order of topics in general should be influenced by the demands of other courses which follow.

33. *Abridged multiplicaton*, by Professor P. R. Hill, Sr., University of Georgia.

The importance of abridged multiplication in college mathematics was shown. The method outlined operates on simple rules, easy to master and to apply. No unnecessary figures or marks were used in this thoroughly stream-lined multiplication.

H. A. ROBINSON, *Secretary*

APRIL MEETING OF THE NEBRASKA SECTION

The twenty-seventh annual meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska in Lin-

coln on Saturday, April 21, 1951. Professor Morris Dansky presided at the two sessions.

Thirty-three persons attended the morning session including the following twenty-one members of the Association: M. A. Basoco, H. W. Becker, A. K. Bettinger, Jessie W. Boyce, C. C. Buck, C. C. Camp, F. Marion Clarke, H. M. Cox, Morris Dansky, J. M. Earl, C. B. Gass, Edwin Halfar, R. E. Heath, E. H. C. Hildebrandt, L. M. Larsen, W. G. Leavitt, W. T. Lenser, E. J. Lowry, H. B. Ribeiro, H. L. Rice, Lulu L. Runge.

Officers elected at the meeting were: Chairman, W. G. Leavitt, University of Nebraska; Vice Chairman, Morris Dansky, Creighton University; Secretary, Edwin Halfar, University of Nebraska.

Professor H. T. Muhly of the University of Iowa was the guest speaker of the Association at both sessions. The afternoon session was held in conjunction with the Nebraska Section of the Council of Mathematics Teachers. At this session, Professor E. H. C. Hildebrandt of Northwestern University was also a guest speaker.

The following papers were presented:

1. *On the nature of algebra*, by F. Marion Clarke, University of Nebraska.

This speaker sketched the changes in the meaning of algebra from the first clumsy attempts to find a quantitative unknown to the modern structure theory in terms of a highly specialized set of symbols. It can be shown that the underlying spirit of algebra to reduce parts to a whole persisted through its development, and that the particular direction of this development was inspired by the need to formalize the physical problems of the age.

2. *Probleme des Timbres Poste*, by H. W. Becker, Electronic Radio-Television Institute, Omaha.

Perhaps the foremost unsolved problem of combinatory analysis is the determination of the number of ways of folding a strip of n duplicate areas into one area. This is isomorphically the number of planar permutations of n letters, that is, the number of Sainte-Lague permutation diagrams without crossovers. These are important in aesthetics and physics, being topologically arabesques, spirals, sine waves, hysteresis curves, etc. The speaker discussed recent work of Errera, Sade, Touchard (*Canadian Jour. of Math.*, II, 1950, pp. 385–398) and Sainte-Lague, *Avec Des Nombres Et Des Lignes* (Paris, 1937, 147–162). He exhibited tables of his own classifications of foldings through $n=10$ with respect to beginnings, progeny, inversions, etc. The latter bear a deceptive resemblance to the enumeration of series-parallel circuits.

3. *The renaissance in algebraic geometry*, by H. T. Muhly, University of Iowa, introduced by the Secretary.

During the decade of the 1930's there occurred a new burst of activity in the field of algebraic geometry. The new vistas which these efforts have revealed are today only partially explored, and yet the subject has already undergone profound changes both in method and scope. This activity, which is now in full swing, started when it was realized that modern arithmetical theories afford the proper tools for dealing with algebraic varieties in a rigorous manner. There resulted a union of geometry and arithmetic which has greatly enriched both branches. Any partial list of names of those responsible for bringing about this union would have to include O. Zariski, A. Weil, C. Chevalley, H. Hasse, F. K. Schmidt, and B. L. van der Waerden.

Surprisingly enough the effectiveness of arithmetical methods in the theory of algebraic functions was recognized as early as 1882 by R. Dedekind and H. Weber. Their arithmetic formulation

of the notion of a place of the Riemann surface of a field of algebraic functions of one variable can be regarded as pointing the way not only to Zariski's systematic application of the theory of valuations to birational transformations, but also to his definition of a simple point in terms of its local ring and his arithmetic formulation and solution of the problem of local uniformization.

4. *Study of an elementary proof of Schur's lemma*, by H. B. Ribeiro, University of Nebraska.

Schur's lemma is obtained as an immediate consequence of an almost obvious theorem on mappings of sets and very few basic facts of linear algebra. The used type of analysis is not new. The proof is intended to improve the usual elementary presentation of the subject.

5. *The stability of the cumulative mean mark*, by H. M. Cox, University of Nebraska.

The mean marks of 382 senior students in the University of Nebraska have been computed each semester, and coefficients of correlation have been calculated semester by semester and college by college. Although there are significant differences between college and semester means (analysis of variance), the correlations are generally satisfactory whether computed within groups or combined. The mean marks for the second semester of residence appear to be more valid and to have a higher predictive value than do the mean marks for the first semester of residence.

6. *The Chinese remainder theorem*, by W. G. Leavitt, University of Nebraska.

This paper introduced the Chinese remainder theorem and gave a few of the high points of its long history. It indicated how the theorem appears in the light of modern algebra, and showed, as illustration, some of its applications to domains of analytic functions.

7. *Some attitude scores*, by R. J. Agan, Nebraska State Teachers College, Wayne, Nebraska, introduced by the Secretary.

A comparison of attitude scores made on the Conover Attitude Inventory before and after a course in driving.

8. *Various pressures and trends affecting Nebraska's secondary mathematics program*, by M. W. Beckmann, University of Nebraska, introduced by the Secretary.

There was a brief introduction giving the reasons for a re-examination of the aims and content of mathematics instruction. Data were given showing the trends in enrollment in certain mathematics courses, the status of mathematics with several organizations in Nebraska, and the effect of the recommendations of the national committee, called The Commission on Post-War Plans, on Nebraska's secondary mathematics program.

9. *The role of the secondary schools in mathematical training*, by Edwin Halfar, University of Nebraska.

Taking into consideration the fact that most secondary schools in Nebraska must set up a mathematics program that must at the same time be adequate for those who are intending to do college work and those who are not, the author discussed the possible nature of such a program.

10. *What shall we do with mathematics?* by E. H. C. Hildebrandt, Northwestern University.

11. *A proposal to improve the undergraduate curriculum*, by H. T. Muhly, University of Iowa.

Since it is desirable for students in the natural science area to complete the calculus by the end of their sophomore year at the latest, and since it is not possible to assume that a beginning freshman has any knowledge of trigonometry, it is proposed that trigonometry be made an incidental part of a course in analytic geometry. The material of trigonometry can be presented as an application of the analytic method and can replace other illustrations. In particular, the theorem of Pythagoras in its analytic form (the "distance formula") leads immediately to the law of cosines and to the addition formulas. It was the opinion of the speaker that the point of view that this approach would entail would serve the student's needs far more than the excessive emphasis on the solution of triangles which is so often found in standard courses in trigonometry.

12. *A note on the effect of high school preparation in mathematics as measured by the Nebraska mathematics classification examination*, by H. M. Cox, University of Nebraska.

Questions on general mathematics (Part I) and questions on elementary high school algebra (Part II) differentiate sharply between students with two (or less) and three (or more) semesters of high school algebra. However, and for the effective use of the examination, there occur gradations in ascending order of mean score in accordance with the amount and variety of high school courses in mathematics. The Nebraska examination correlates satisfactorily with Section VI (Mathematics) of the Cooperative General Culture Test.

LULU L. RUNGE, *Secretary*

THE APRIL MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-fourth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the Colorado State College of Education, Greeley, Colorado, on Friday and Saturday, April 20 and 21, 1951. Professor Dale O. Patterson, Chairman of the Section, presided at all the sessions.

Of the approximately one hundred thirty persons who registered, the following fifty were members of the Association: C. F. Barr, W. E. Briggs, J. R. Britton, R. G. Buschman, F. M. Carpenter, A. G. Clark, C. H. Cook, G. S. Cook, David Devol, Mary C. Doremus, A. B. Farnell, F. N. Fisch, R. R. Gutzman, Leota C. Hayward, I. L. Hebel, LeRoy Holubar, Burrowes Hunt, C. A. Hutchinson, B. W. Jones, M. W. Jones, A. J. Kempner, Claribel Kendall, J. S. Leech, Garner McCrossen, H. C. McKenzie, M. L. Madison, D. C. B. Marsh, Jr., W. K. Nelson, Greta Neubauer, K. L. Noble, D. O. Patterson, H. C. Peterson, Lily B. Powell, G. B. Rice, O. H. Rechard, A. W. Recht, L. W. Rutland, Jr.. Nathan Schwid, W. N. Smith, L. C. Snively, M. E. Sperline, K. H. Stahl, P. O. Steen, J. F. Stockman, E. P. Tovani, E. L. Vanderburgh, V. J. Varineau, W. W, Varner, J. F. Wagner, Lillie Walters.

At the business meeting, it was voted to hold the next annual meeting at Western State College, Gunnison, Colorado, in May, 1952. The following officers were elected for the ensuing year: Chairman, Professor C. H. Cook, Western State College; Vice-Chairman, Professor B. W. Jones, University of Colorado; Secretary-Treasurer, Professor J. R. Britton, University of Colorado.

The program of papers for the Friday afternoon and Saturday morning sessions was as follows:

1. *Sidelights on certain topics in elementary statistics*, by Professor A. G.

Clark, Colorado A. & M. College.

Various topics of the elementary course in statistics were discussed. These included bounds for the coefficient of correlation, the median as the value of M that minimizes $\sum_i |M-x_i|$, and the probit diagram method for fitting distribution curves to sample data.

2. *Necessary and sufficient conditions on p and r that the equation $x^4+px^2+r=0$ be normal over the rational field*, by Mr. W. E. Briggs, University of Colorado.

Let t, $-t$, t', $-t'$ be the roots of the irreducible equation $x^4+px^2+r=0$, where t is an arbitrary root, and p and r are rational. The equation will be normal if $R(t)=R(-t)=R(t')=R(-t')$, where $R(t)$ is the field of all numbers of the form $a_0+a_1t+a_2t^2+a_3t^3$ with the a_i rational. The necessary and sufficient condition is that t' be an element of $R(t)$, or that $a_0=a_2=0$, with $a_1=p/\sqrt{r}$, $a_3=1/\sqrt{r}$, in which case the Galois group is the four group, or with $a_1=(p^2-2r)/\sqrt{p^2r-4r^2}$, $a_3=p/\sqrt{p^2r-4r^2}$, which gives the cyclic group of order four. This implies that either r or p^2r-4r^2 is a rational square.

3. *Remarks on complex numbers and their functions*, by Professor (Emeritus) A. J. Kempner, University of Colorado.

4. *Generalized functional dependence*, by Professor H. M. Jurney, Colorado School of Mines, introduced by Professor I. L. Hebel.

The functional dependence of n functions $u_i(x_1, \cdots, x_m)$, $i=1, 2, \cdots, n$, of m variables was discussed. The results may be expressed in the form of a theorem:

A relationship of the form $\phi(u_1, \cdots, u_n)=0$, exists for all values of $x_1, \cdots, x_m$ in some given domain of these variables if and only if the rank of the "Jacobian matrix" J_{mn} is less than n, where

$$J_{mn}=\begin{pmatrix} \partial u_1/\partial x_1, \cdots, \partial u_1/\partial x_m \\ \cdots\cdots\cdots\cdots \\ \partial u_n/\partial x_1, \cdots, \partial u_n/\partial x_m \end{pmatrix}.$$

5. *Periodic solutions of nonlinear differential equations*, by Professor A. B. Farnell, University of Colorado.

A discussion was given of the use of fixed point theorems in proving the existence of periodic solutions of nonlinear differential equations, and, by way of illustration, the proof of the existence of such a solution for a particular equation was given.

6. *On automorphs of conic sections*, by Professor B. W. Jones, University of Colorado.

The linear transformations $x=\alpha x'+\beta y'$, $y=\gamma x'+\delta y'$, with $\alpha\delta-\beta\gamma=1$ which leave invariant the quadratic form x^2+sy^2, $s\neq 0$, were shown to satisfy the conditions $\alpha^2+s\gamma^2=1$, $\alpha=\delta$, and, if $\alpha\neq 0$, $\beta=-s\gamma$. Hence β may be defined as the "conic cosine" of an angle θ and γ the "conic sine." If $s=1$ we have the circular functions, if $s=-1$, the hyperbolic functions. It was shown that such transformations may be used to eliminate the xy term in the equation of any conic section.

7. *The IBM card-programmed electronic calculator*, by Mr. W. W. Varner, University of Colorado.

This illustrated presentation described the physical appearance and operation of the recently released semi-portable IBM card-programmed electronic calculator. A specific problem was presented and the details of programming introduced to illustrate the versatility of the machine as well as the technique of programming. A brief discussion of the arrangement of the calculation to mini-

mize storage requirements was included to call attention to the critical problem of storage limitation.

8. *Do you enjoy the problem sections in the Monthly?* by Mr. Hans Stetter and Mr. Donald Tucker, Colorado A. & M. College, introduced by Professor M. L. Madison.

Representative problems selected from the advanced problems section of late issues of this MONTHLY were solved. The problems proposed in the MONTHLY can serve as a challenge to the undergraduate major in mathematics, and many of these problems can be solved by ingenious elementary devices.

9. *A note on income tax calculations*, by Professor W. K. Nelson, University of Colorado.

10. *Occupational outlets in industrial and business fields for majors in mathematics*, by Professor S. R. Smith, University of Wyoming.

In the unavoidable absence of Professor Smith, this paper was read by Professor Greta Neubauer.

11. *Recent efforts and achievements in the revision of the high school mathematics program, and their significance in college*, by Professor C. F. Barr, University of Wyoming.

Professor Barr presented a review of the content and grade placement of high school algebra and geometry. He then developed historically the opinions of well-known mathematicians and the various responses of mathematics teachers to these opinions. Two large resulting movements were described: one, the "two-track" movement in which algebra and geometry were taught to the superior pupils while a course with a utilitarian flavor was presented to those not capable of following the algebra-geometry track; the other movement being not the "two-track" one, but the socializing and popularizing of algebra and geometry, which were urged upon a majority, if not all, of the pupils. The objections to each of these programs were reviewed. The author proposed that a third program, consistent with the accepted purposes of mathematics, be considered, namely, the development of a course compiled from the everyday experiences of all normal citizens. This course he urged should be required of all pupils at some time in their high school program, regardless of their intellectual abilities, and that it be supplemented by algebra and geometry of the classical type if the student intended to pursue mathematics or if he expected to train in any technological field. The age level at which this course should be required was discussed briefly, with the observation that perhaps systematic experimentation alone would furnish any dependable answer.

The after-dinner address Friday evening was given by the guest speaker, Professor G. B. Price, University of Kansas. Professor Price gave an illustrated lecture on the topic, *Experiences of a Mathematician as an Operations Analyst with the Eighth Air Force in England.*

J. R. BRITTON, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The Iowa Section of the Mathematical Association of America held its thirty-eighth annual meeting at Wartburg College, Waverly, Iowa, on Friday and Saturday, April 20–21, 1951. The Chairman, Professor D. L. Holl of the Iowa State College, presided at both sessions.

The meeting was attended by approximately fifty persons including the following thirty-three members of the Association: W. S. Bicknell, H. D. Block, I. H. Brune, E. L. Canfield, E. W. Chittenden, Marian E. Daniells, W. M. Davis, R. M. Deming, L. E. Ernsdorff, R. W. Gardner, B. E. Gillam, Cornelius Gouwens, F. S. Harper, J. J. L. Hinrichsen, D. L. Holl, G. E. Kaldenberg, G. L. Keppers, O. C. Kreider, R. J. Lambert, C. E. Langenhop, F. W. Lott, Jr., R. B. McClenon, Fred Robertson, Ralph M. Robinson, Hazel M. Rothlisberger, Augusta L. Schurrer, M. F. Smiley, F. M. Stein, H. P. Thielman, C. E. Van Horn, Bernard Vinograde, A. M. Wedel, E. A. Zubay.

The following members were elected to serve as officers for next year: Chairman, Professor L. E. Ernsdorff, Loras College; Vice-Chairman, Professor L. A. Knowler, State University of Iowa; and Secretary-Treasurer, Professor Fred Robertson, Iowa State College.

The following papers were presented at the meeting:

1. *The Simpson quartic of a triangle*, by Professor C. E. Van Horn, Wartburg College.

Perhaps we catch the spirit of Euclid best when we so teach geometry that our students are encouraged to seek for new results (at least new to them). The illustrations show a few of the discoveries made in elementary geometry in modern times such as the so-called Simpson line of a point on the circumcircle of a triangle, the Euler line of a triangle, the nine-point circle, the mean or equilateral derivative of a triangle, and the Simpson quartic of a triangle with a few theorems connecting the above concept.

2. *Normal matrices and the characteristic root problems*, by Mr. R. J. Lambert, Iowa State College.

It is well known that if N is normal, the real parts of the characteristic roots of N are those of $(N+N^*)/2$ where N^* is the conjugate transpose of N. The imaginary parts of the roots are the characteristic roots of $(N-N^*)/2i$. The purpose of this paper is to use these facts as a possible means of obtaining the characteristic roots (real or complex) of a given square matrix A. It can be shown that there exists a positive definite Hermitian matrix H such that $HAH^{-1}=N$ is normal. To find the transforming matrix H we use the conditions of normality, i.e. $HAH^{-1}H^{-1}A^*H = H^{-1}A^*HHAH^{-1}$ or $JAJ^{-1}A^*=A^*JAJ^{-1}$ where $J=H^2$. If A is non-derogatory, then any matrix which commutes with A (or A^*) is a polynomial in A. Therefore $JAJ^{-1}=P(A^*)$ where $P(X)$ denotes a scalar polynomial in X. It can be shown that if A is real, $P(X)$ has real coefficients and $P(A)=J^{-1}A^*J$. For this case $P[P(A)]=A$. The coefficients of the polynomial have been worked out by the author for the second and third order cases. If only the characteristic roots are desired, the matrix J need not be found since the real parts of the roots of A are those $[A+P(A)]/2$ and the imaginary parts are the roots of $[A-P(A)]/2i$.

3. *Linear transformations on or onto a Banach space*, by Dr. H. D. Block, Iowa State College.

A simple application of the Baire category principle gives some information about any linear transformation whose domain (or range) is a Banach Space but which is not necessarily bounded, or closed, or one to one. Let T be a linear transformation from all of a Banach Space $\mathfrak{X}$ to a normed vector space Y. Then there is a number $m>0$ such that for any $x\in\mathfrak{X}$ there exists a sequence $x_n\to x$ such that $\|Tx_n\|\leq m\|x\|$, and $\{Tx_n\}$ converges in the sense of Cauchy. (The closed graph theorem is an immediate consequence of this.)

Similarly it is easily shown that if T is a linear transformation from a normed vector space X

onto all of a Banach Space Υ then there is a sequence $y_n \to y$ with $y_n = Tx_n$, $\|x_n\| \leqq m\|y\|$ and $\{x_n\}$ convergent in the sense of Cauchy.

4. *The structure of mathematics*, by Professor Fred Robertson, Iowa State College.

The author develops the operational formula $T \to S_E = S'_{E'}$ where T is an operator, S_E is the set of elements upon which T acts, and $S'_{E'}$ is the result. By a proper choice of the operator and the elements together with the class concept and the laws of nature the preceding formula is found to be basic in the study of mathematics. The operator T and its inverse T^{-1} under certain general conditions are shown to satisfy a certain partial differential equation. This equation provides a method of finding the operator inverse to the operator T.

5. *Application of Volterra transforms to some hypergeometric series*, by Mr. A. M. Wedel, Iowa State College.

Let $f(x, y)$ and $g(x, y)$ be two functions which are continuous in the closed region $a \leqq x \leqq y \leqq b$. The integral $\int_x^y f(x, t)g(t, y)dt$ is defined to be the product by composition of f and g. There is an isomorphism between those algebraic formulas which involve only addition and multiplication and those obtained from the algebraic formulas by replacing powers of the variable by powers by composition of f. It can easily be shown that if

$$t(n, a, u; z) = e^{-az}z^{u+1}\,{}_2F_1(n, -n; \tfrac{1}{2}; z)$$

where ${}_2F_1(n, -n; \frac{1}{2}; (1-z)/2)$ is the Tchebycheff polynomial of degree n, then

$$2t(n, a, u; z)t(m, b, v; z) = t(n + m, a + b, u + v + 1; z) + t(n - m, a + b, u + v + 1; z).$$

By replacing powers of z by powers by composition of $f(x, y) = 1$, we get an integral addition theorem which expresses the integral of the product of the sums of two Bessel functions with parameters a and b in terms of sums of Bessel functions with parameters $a+b$.

6. *The proofs of rank theorems*, by Professor Bernard Vinograde, Iowa State College.

In particular, the basic rank theorems for symmetric matrices are proved in simple vector fashion. The method is formalized into an abstract ordering process which defines rings of linear maps (of matric sets) whose factorization properties are analyzed.

7. *The transcendence of e*, by Professor Marian E. Daniells, Iowa State College.

This paper reviews the proof of the transcendence of e given by Hilbert in 1893. His method employs an integral of the form

$$\int_0^\infty \frac{z^r[(z-1)(z-2)\cdots(z-n)]^{r+1}}{r!}e^{-z}dz$$

devised by Hermite and two fundamental theorems in the theory of numbers.

8. *Matric rules*, by Professor M. F. Smiley, State University of Iowa. (Invited address.)

It has been shown recently that $2n$ is the minimum degree of a polynomial identity for the algebra F_n of all square matrices of order n with elements in a field F. (A. S. Amitsur and J. Levitski, *Minimal identities for algebras*, Proc. Amer. Math. Soc., 1, 1950, pp. 449–463). We give an exposition of the proof which shows that this result holds when F is replaced by a commutative

ring with unit element (N. H. McCoy, *Rings and Ideals*, Carus Mathematical Monograph, No. 8). The problem of determining necessary and sufficient conditions on a ring A with unit in order that A_n satisfy a standard identity of degree $2k$ with k greater than n is proposed and solved for the simplest case: $n=2$, $k=3$.

9. *The slide rule made meaningful*, by Professor G. L. Keppers, Iowa State Teachers College.

The author favored the plan of each student actually constructing a slide rule and using that rule to do some simple computations. He gave units of length for the actual construction which simplified the computation.

10. *On the application of stochastic methods in mathematical biology*, by Mr. A. T. Reid, Chicago. (Read by title.)

11. *Note on the relation between the properties of separation and connection*, by Mrs. Barbara B. Blair, University of Iowa, introduced by the Secretary.

A topology obtained by the axiomatization of the notion of separation of sets is compared with a topology, defined in terms of neighborhoods, which requires only that every point shall have a neighborhood and every neighborhood of a point contains the point. It is found that the separation topology is stronger than the neighborhood topology, and a necessary and sufficient condition is given that the two topologies coincide.

12. *Analogies in elasticity*, by Dr. D. L. Holl, Iowa State College.

Physical systems expressible in identical mathematical form are said to be analogous. The advantages and limitations of them as equivalent systems were pointed out. A complete correspondence was established between the torsion problem of a rod with a multiply connected section and the corresponding soap film with properly elevated disks. A similar analog was made for the flexure problem, with the necessary criteria for multiply connected sections. The analogs of plane strain, plane stress for a two dimensional plane problem with the corresponding plate deflection problem under edge deflections, edge moments and normal surface loads, were detailed.

13. *Remarks on method in teaching the definite integral in a first course in calculus*, by Professor E. W. Chittenden, State University of Iowa.

Professor Chittenden outlined a program for teaching the theory of definite integrals in a first course in calculus designed to avoid the commoner logical weaknesses without important loss of time or substantial increase in difficulty. The method involves some loss in generality but none in applicability.

14. *A class of functional equations*, by Dr. C. E. Langenhop and Dr. H. D. Block, Iowa State College.

This was a discussion of problem 4424 in this MONTHLY, and a generalization. The problem was to describe the most general continuous real-valued function $(-\infty \leqq x \leqq \infty)$ which satisfies the equation $f[1-f(x)]=1-f(x)$. If inf. $f(x)=L$, sup. $f(x)=U$, then it is necessary and sufficient that $f(x)=x$, $L \leqq x \leqq U$, and $L+U=1$. Similar analysis can be applied to the more general equation $f(h[f(x)])=h[f(x)]$ if $h(x)$ is continuous and monotone. Relations analogous to $L+U=1$ can be obtained, and in particular, if $h(x)$ is monotone increasing and $h(x)>x$ (or $h(x)<x$) for all x, no bounded solution exists. Such an equation would be $f[1+f(x)]=1+f(x)$.

15. *Extremals of velocity and distance in central force motion*, by Dr. H. D. Block, Iowa State College.

Let $f(r, \theta, t)$ be the force acting on a particle when it is at the point whose polar coordinates are

(r, θ) at time t. The force is assumed to be directed toward the origin so that $f>0$ is an attractive force, $f<0$ a repulsive force. We assume that the orbit parameter, t, can always be extended in both directions, that f is continuous and piecewise monotone in all of its arguments, and that the particle does not pass through the origin. Using only elementary calculus we can prove the following theorem. Let v_0 be the velocity at the point on the orbit (v_0, θ_0) at time t_0, P_0 the point (r_0, θ_0, t_0) and f_0 the force $f(P_0)$. Then (i) if r_0 is a maximum then v_0 is a minimum and $f_0>0$; (ii) if v_0 is a maximum then either r_0 is a minimum or r_0 is not an extremal and $f_0=0$; (iii) if r_0 is a minimum and $f_0>0$ (<0) then v_0 is a maximum (minimum); if r_0 is a minimum and f changes sign at P_0 then v_0 is not an extremal; if r_0 is a minimum and $f_0=0$ but does not change sign at P_0 then v_0 is a maximum (minimum) if $f(r, \theta, t)$ is $\geqq 0$ $(\leqq 0)$ for all (r, θ, t) sufficiently close to P_0. (iv) The statements in (iii) hold with v_0 and r_0 interchanged.

16. *Theorem on the zeros of a polynomial*, by Mr. W. J. Stoner, Iowa State College, introduced by the Secretary.

A simple method of finding limits for the moduli of the zeros of polynomials is found by means of Rouché's theorem. The moduli of the zeros of $a_0+a_1z+\cdots+a_nz^n$ are shown to be less than or equal to K, where $K=(|a_0|+|a_1|+\cdots+|a_{n-1}|)/|a_n|$ if $K\geqq 1$, or $\sqrt[n]{K}$ if $K<1$. The moduli are shown to be greater than or equal to $\sqrt[n]{M}$, where $M=|a_0|/(|a_1|+|a_2|+\cdots+|a_n|)$, if $M>1$.

Mr. W. J. Stoner won the award for the most meritorious paper presented in mathematics.

FRED ROBERTSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1–2, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.

ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.

INDIANA, Indiana University, Bloomington, Spring, 1952.

IOWA, Coe College, Cedar Rapids, April 18–19, 1952.

KANSAS, Bethany College, Lindsborg, March 29, 1929.

KENTUCKY, University of Kentucky, Lexington, April 19, 1952.

LOUISIANA-MISSISSIPPI, Northwestern State College, Natchitoches, Louisiana, February 15–16, 1952.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, Spring, 1952.

MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.

MINNESOTA, College of St. Catherine, St. Paul, May 10, 1952.

MISSOURI, Lindenwood College, St. Charles, May 2, 1952.

NEBRASKA, University of Nebraska, Lincoln, April 19, 1952.

NORTHERN CALIFORNIA

OHIO, April 19, 1952.

OKLAHOMA

PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.

PHILADELPHIA

ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May, 1952.

SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.

SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.

SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.

TEXAS, East Texas State Teachers College, Commerce, April, 1952.

UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May 10, 1952.

WISCONSIN, Milwaukee, May 10, 1952.

MACMILLAN

. . . from Macmillan

ELEMENTARY DIFFERENTIAL EQUATIONS

EARL D. RAINVILLE

University of Michigan

This text, containing all the material of the author's successful *Short Course in Differential Equations,* provides a complete introduction to elementary differential equations in a full year's course for students who have had standard calculus. It offers both a sound training in good techniques for obtaining solutions and an explanation of the basic theory behind the techniques. *Coming in March*

CALCULUS

JOHN F. RANDOLPH

University of Rochester

Maintaining a sound balance between formal problem material and a mathematically rigorous presentation, Professor Randolph's new text offers a flexible, well-proportioned, modern treatment of the theory and applications of calculus. The problems are numerous, practical, and well-illustrated. The author anticipates each topic gradually so that the student will be primed for its eventual definition and ready to master it. *Coming in March*

THE MACMILLAN COMPANY

60 FIFTH AVE., NEW YORK 11, N.Y.

A new college algebra text fitted to the needs of business majors

COMMERCIAL ALGEBRA

Robert M. Parker, Texas Technological College

A very important feature of this book is the emphasis on *demonstrating* a principle or procedure for the sake of clarity. More than 300 illustrative examples are included. Explanations are full and clear. Approximately 1900 problems throughout the book afford ample practice material. Because the business administration student may bring to this course only a very rusty knowledge of ninth-grade algebra, Professor Parker devotes the first two chapters to a review of fundamental principles. Cumulative review exercises are spaced after every third chapter. A large number of review exercises are at the end of the book.

A new book that minimizes the demands on memory

MATHEMATICS OF FINANCE

Albert E. May, Racine Extension Center, University of Wisconsin

A unique and striking feature of this text is its stress on *thinking* and *understanding* as opposed to blind substituting in formulas. It builds the fundamentals of investment theory on just eight simple important formulas. The student is trained in the basic principles so that he is able to adapt these few formulas to the many types of problems covered in the course. Whenever feasible, oral problems are used in introducing new symbols. This enables the student, unhampered by difficult computation, to grasp the mechanics of the new symbol while the instructor is present to correct initial misconceptions.

Outstanding features of three outstanding mathematics texts

by Frank M. Morgan

COLLEGE ALGEBRA
PLANE AND SPHERICAL TRIGONOMETRY
DIFFERENTIAL AND INTEGRAL CALCULUS

- Concrete examples illustrate each topic.
- Well-graded exercises are provided in abundance.
- Questions are interspersed throughout the text to make it read somewhat like a class discussion.
- Mastery tests are spaced at intervals throughout the book to give the student thorough periodic reviews.
- Answers to all exercises are available in separate pamphlets.

American Book Company

55 Fifth Avenue
New York 3, N.Y.
300 Pike Street
Cincinnati 2, Ohio
351 East Ohio St.
Chicago 11, Illinois

Smith's

THE MATHEMATICS OF FINANCE

This text for students of business administration emphasizes the use of basic principles rather than the use of special formulas. Contains a review of algebraic topics, 27 diagrams, many practical exercises, a new mortality table and an index. **356 pp., $4.00**

Stokes'

TEACHING THE MEANINGS OF ARITHMETIC

Text for teacher-training courses in the methods of teaching arithmetic is divided into four sections, consisting of discussions on the meanings of arithmetic, principles of number-learning, techniques of instruction, and a developmental program for grades one through six. **531 pp., $4.50**

Appleton-Century-Crofts, 35 West 32nd St., New York 1, N.Y.

NOTICE

A liberal arts college in the mid-west will have an opening next Fall for a young Ph.D. in Mathematics. Applicants should have sufficient training to supervise a small observatory and to teach elementary courses in astronomy in addition to courses in mathematics. The college emphasizes good teaching and encourages research and productive scholarship. Please address applications to

Advertising Manager

C/O MATHEMATICAL ASSOCIATION OF AMERICA

UNIVERSITY OF ROCHESTER, ROCHESTER 3, N.Y.

Choose an algebra text that meets *your* need from three by *W. L. Hart*

HEATH College Texts

BRIEF COLLEGE ALGEBRA, Revised

The most interesting and useful parts of college algebra presented concisely for well prepared freshmen. 339 pages. $3.00

COLLEGE ALGEBRA, 3rd edition

All the usual topics of college algebra, preceded by an extensive review of intermediate algebra. 424 pages. $3.25

INTRODUCTION TO COLLEGE ALGEBRA, Revised

Intermediate algebra at a mature level, with a simple treatment of selected topics from college algebra. 319 pages. $3.00

Be sure to see the new, excellent calculus text by *Tomlinson Fort*

CALCULUS

Beginning with the basic explanation of infinite series, the whole text is rigorous but not difficult. All the usual material and more is provided, allowing for selection appropriate to different classes. 576 pages. $4.75

D. C. HEATH and Company

Sales Offices: NEW YORK CHICAGO ATLANTA SAN FRANCISCO DALLAS Home Office: BOSTON

Prentice-Hall
COLLEGE TEXTS

BUSINESS MATHEMATICS

by WALTER F. CASSIDY and C. CARL RUBUSTO, St. John's University

Explanations in this text are simple, brief, and not too technical. Some facts and formulas have been stated without proof. Mathematical rigor has been avoided in the belief that it is neither desirable nor necessary on this particular level. Material proceeds from simple concepts in arithmetic to the more complex ones in life insurance. Many illustrations make subject matter concrete; problems are both interesting and practical.

464 pages 5½" x 8½" Published November 1951

MEANINGFUL MATHEMATICS

by H. S. KALTENBORN, Memphis State College

Designed to provide the mathematical part of a liberal education, the material in this text ranges from primitive number systems to calculus. As far as possible, a fresh start is made at the beginning of each chapter, encouraging the student to continue in spite of any possible failure to master some of the special techniques at an earlier stage. A large number of problems provide adequate drill on fundamental processes.

384 pages 5½" x 8½" Published October 1951

ANALYTIC GEOMETRY AND CALCULUS

by LYMAN M. KELLS, U. S. Naval Academy

This combination of calculus and analytic geometry in a single text results in (a) better assimilation of both subjects by students, (b) a saving of time, and (c) early equipment of students with basic analytic principles, and the powerful tools of calculus, derivatives, and integrals. A large number of carefully-graded problems provide drill work and numerous important applications.

608 pages 6" x 9" Published 1950

Send for your copies today

PRENTICE-HALL INC. 70 FIFTH AVE·NY 11·NY

Important McGraw-Hill Books

GENERAL COLLEGE MATHEMATICS

By W. L. AYRES, C. G. FRY, and H. F. S. JONAH, Purdue University. In press

A new approach to freshman mathematics, designed to help the student in his application of mathematical topics to the biological and social sciences. Keyed primarily for those students who will take only one year of college mathematics, this text examines such subjects as ratios and percentages, linear and quadratic equations, trigonometry, interest and its application to installment buying, laws of growth, statistics, etc.

COLLEGE ALGEBRA

By ROSS R. MIDDLEMISS, Washington University. Ready in March

The text covers all the topics usually taught in a standard course. The principal aim is to make the algebra course more valuable and more stimulating by emphasizing *reasoning and clear thinking*—and by thus combating the student's tendency toward mechanical operations unaccompanied by real thought.

TRIGONOMETRY

By CECIL T. HOLMES, Bowdoin College. 246 pages (with tables), $3.25

Here is a clearly understandable and thorough text that offers a closer correlation of trigonometry with other college mathematics than is done in most books of its kind. This is accomplished by introducing some very elementary notions from analytic geometry and using them consistently in proofs throughout the book. Emphasis throughout is upon understanding, rather than mechanical manipulation.

CALCULUS AND ANALYTIC GEOMETRY

By C. T. HOLMES, Bowdoin College. 416 pages, $4.75

Designed for a combination course in which the concepts and techniques of the calculus are the main objectives. Although calculus is emphasized, the essentials of analytic geometry are presented in sufficient detail for a subsequent major. The concept of integration is introduced early in the text.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, INC.

330 WEST 42ND STREET, NEW YORK 18, N. Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 59 NUMBER 3

CONTENTS

Integral Transforms and Boundary Value Problems . . . . . R. V. CHURCHILL 149

Euclid's Algorithm and the Least-Remainder Algorithm . . . . . A. W. GOODMAN AND W. M. ZARING 156

A Recurrence Relation for Three-Line Latin Rectangles . . . . . JOHN RIORDAN 159

On the Finite Difference Analogue of Rodrigues' Formula . . . . . MARIA WEBER AND A. ERDÉLYI 163

Mathematical Notes . . . . I. I. KOLODNER, J. D. NIBLETT 168

Classroom Notes . . . . . M. R. SPIEGEL, FURIO ALBERTI, J. P. BALLANTINE 175

Elementary Problems and Solutions 180

Advanced Problems and Solutions 186

Recent Publications 192

Clubs and Allied Activities 197

News and Notices 200

Mathematical Association of America 205

Annual Meeting 205

April Meeting of the Maryland-D.C.-Virginia Section . . . 209

April Meeting of the Minnesota Section 211

April Meeting of the Kentucky Section 213

April Meeting of the Metropolitan New York Section . . . 215

Calendar of Future Meetings 218

MARCH 1952

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	J. W. GIVENS	GEORGE SPRINGER
PATRICK DU VAL	L. M. KELLY	E. P. STARKE
HOWARD EVES	H. D. LARSEN	G. B. THOMAS
F. A. FICKEN	IVAN NIVEN	E. P. VANCE

EDITH R. SCHNECKENBURGER

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, SAUNDERS MACLANE, University of Chicago
Honorary President, W. D. CAIRNS, Oberlin College
First Vice-President, F. L. GRIFFIN, Reed College
Second Vice-President, JEWELL H. BUSHEY, Hunter College
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo
Editor, C. B. ALLENDOERFER, University of Washington

Additional Members of the Board of Governors: E. B. ALLEN, C. F. BARR, T. A. BICKERSTAFF, J. C. BRIXEY, R. H. CAMERON, G. R. CLEMENTS, T. F. COPE, H. S. M. COXETER, W. M. DAVIS, J. M. EARL, G. M. EWING, L. R. FORD, TOMLINSON FORT, J. S. FRAME, A. E. GAULT, R. E. GILMAN, E. H. HANSON, M. R. HESTENES, AUGHTUM S. HOWARD, C. G. JAEGER, B. W. JONES, M. S. KNEBELMAN, R. E. LANGER, D. H. LEHMER, C. C. MACDUFFEE, W. E. MILNE, H. P. PETTIT, J. C. POLLEY, G. E. RAYNOR, E. B. ROESSLER, R. G. SANGER, F. H. STEEN, EARL WALDEN, MARIE J. WEISS, F. B. WILEY

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

INTEGRAL TRANSFORMS AND BOUNDARY VALUE PROBLEMS*

R. V. CHURCHILL, University of Michigan

1. Introduction. The linear integral transformations of Laplace, Fourier and Hankel† can be used to solve certain classes of linear boundary value problems in partial differential equations that are not adapted to the method of separating variables and using Fourier expansions. When more than one method applies to a given problem, different methods may give different forms of the solution, and a variety of forms can be helpful in determining properties of the solution.

The particular transformations, if any exist, that will reduce a given boundary value problem to a simpler one depend not only upon the ranges of the independent variables but also upon the differential forms that are involved in the problem. It is the purpose of this paper to illustrate how appropriate integral transforms can be determined for a given problem.

The example used in this illustration was selected not because of its significance in applied mathematics but as a fairly simple one that leads to two different types of transforms, one a finite Hankel transform and the other a new modification of the infinite Fourier sine transform. To best illustrate the technique it seems wise to carry out the solution formally.

The method to be illustrated, of beginning with a general integral transformation and then determining its kernel so that the transformed problem is simpler than the original, has been presented during the past several years in one of the author's courses in partial differential equations. When the process is applied to quite general problems it leads to finite and infinite integral transformations whose kernels are characteristic functions of general Sturm-Liouville problems.‡

2. Steady temperatures in a long shaft. Let the positive x-axis be taken as the axis of a cylindrical shaft semi-infinite in length and let the cylindrical coordinate r denote distance from the axis of the shaft. The unit of length is taken as the radius of the cross section. Heat is generated within the shaft, possibly as a result of internal resistance when the shaft is transmitting a torque, at a steady rate per unit volume that is a prescribed function $F(r, x)$. The temperature of the surface $r=1$, $x>0$, is a prescribed function $G(x)$. We assume that the func-

* An address presented under the title "Integral transforms and differential equations" before the Joint Meeting of the Association with the American Society for Engineering Education at East Lansing, Michigan, on June 26, 1951.

† Applications of Hankel transforms are presented in a new book by I. N. Sneddon, Fourier Transforms, McGraw-Hill, New York, 1951. The kernels of Hankel transformations are Bessel functions.

‡ The finite integral transformations of this general type have been introduced in connection with ordinary differential equations in a thesis by P. Burgat, Resolution de problèmes aux limites au moyen de transformations fonctionnelles, Université de Neuchatel, Switzerland, 1950.

tions F and G approach zero as x tends to infinity. Let linear heat transfer take place from the base $x=0$ into surroundings at uniform and constant temperature. Then the steady-state temperatures in the shaft are represented by a function $W(r, x)$ that satisfies the equation of conduction

$$\frac{1}{r}(rW_r)_r + W_{xx} = F(r, x) \qquad (0 \leqq r < 1,\ x > 0), \tag{1}$$

where the subscripts denote partial derivatives, and the boundary conditions

$$W(1, x) = G(x) \qquad (x > 0), \tag{2}$$

$$W_x(r, 0) = hW(r, 0) - B \qquad (0 \leqq r < 1), \tag{3}$$

where h and B denote constants and $h \geqq 0$. Also we can require that

$$\lim_{x \to \infty} W(r, x) = 0. \tag{4}$$

The interpretation of the function $W(r, x)$ as the concentration of a substance diffusing within the shaft is also of some interest in the applications.

When the source function $F(r, x)$ involves both of its variables, and it may be a step function in either one or both variables here, the boundary value problem cannot be adapted to the method of separating variables. Methods of using Green's functions transfer the difficulties to the task of finding those functions.

The presence of the variable r in the coefficients of the derivatives in equation (1) is enough to prevent the effective use of either the Laplace or any ordinary Fourier transformation with respect to r. The Laplace transform of W_{xx} with respect to x involves both the functions $W(r, 0)$ and $W_x(r, 0)$; the linear combination of these functions that is prescribed in condition (3) leaves an unknown function of r in the transformed problem.* Similarly, condition (3) is not adapted to the use of the known Fourier transforms with respect to x.

The boundary value problem (1)–(4) may be solved by representing the known and unknown functions by the same type of series or integral in r or x so as to reduce the problem to one of determining the unknown coefficients. Unless the proper type of representation is selected, however, this method leads to awkward approximations. The method illustrated below determines appropriate types of representation that lead to exact solutions.

3. Transformation with respect to r. Let $w(\lambda, x)$ denote an integral transformation of the unknown function $W(r, x)$ of the type

$$T\{W\} = \int_0^1 W(r, x)p(r)\phi(\lambda, r)dr = w(\lambda, x). \tag{5}$$

---

* The unknown function can be determined in some cases, with persistence and care; this is illustrated for a fairly simple problem in Sec. 72 of the author's book, Modern Operational Mathematics, 1944.

The kernel $\phi(\lambda, r)$, the weight function $p(r)$ and the nature of the parameter λ are to be determined, if possible, so that the problem corresponding to the boundary value problem (1)–(4) will be one in ordinary differential equations with $w(\lambda, x)$ as the unknown function. The new problem should be fully determined by the conditions prescribed in the original problem and by the transformation itself so that, when the solution $w(\lambda, x)$ is found, the function $W(r, x)$ will be given as the inverse transform $T^{-1}\{w\}$. The inverse of our transformation is to be determined from the conditions on ϕ, p and λ.

The result of formally applying T to both members of our partial differential equation (1) can be written

$$T\left\{\frac{1}{r}(rW_r)_r\right\} + \frac{d^2}{dx^2}w(\lambda, x) = f(\lambda, x), \tag{6}$$

where $f(\lambda, x) = T\{F(r, x)\}$. Our transformation should be such that the first term here reduces to a linear function in $w(\lambda, x)$ involving only λ and prescribed boundary values of $W(r, x)$. Now after integrating twice by parts we find that

$$\begin{aligned} T\left\{\frac{1}{r}(rW_r)_r\right\} &= \int_0^1 \frac{1}{r}(rW_r)_r p(r)\phi(\lambda, r)dr \\ &= \int_0^1 \left[r\left(\frac{p}{r}\phi\right)'\right]' W dr + \left[p\phi W_r - \left(\frac{p}{r}\phi\right)' rW\right]_{r=0}^{r=1}, \end{aligned}$$

where the primes denote ordinary derivatives with respect to r. The last integral is proportional to $w(\lambda, x)$ if ϕ and p are such functions that

$$\left[r\left(\frac{p}{r}\phi\right)'\right]' = -\lambda^2 p\phi, \tag{7}$$

where the factor $-\lambda^2$ has been selected as the most convenient factor involving the parameter. Again, as a matter of convenience, the self-adjoint form of the differential equation (7) is preferred; hence we take

$$p(r) = r.$$

Since W and W_r are continuous functions in the interior of the shaft, the above transformation now reduces to

$$T\left\{\frac{1}{r}(rW_r)_r\right\} = -\lambda^2 w(\lambda, x) + \phi(\lambda, 1)W_r(1, x) - \phi'(\lambda, 1)W(1, x). \tag{8}$$

The function $W_r(1, x)$ is not prescribed in our boundary value problem; thus we shall require that its coefficient $\phi(\lambda, 1)$ in equation (8) be zero. If $\phi(\lambda, r)$ is to be a continuous function of r $(0 \leqq r \leqq 1)$ then, in view of equation (7), ϕ should satisfy the singular Sturm-Liouville system

$$(r\phi')' + \lambda^2 r\phi = 0, \quad \phi(\lambda, 1) = 0, \quad |\phi(\lambda, r)| < M \qquad (0 \leqq r \leqq 1), \tag{9}$$

where the constant M may depend on λ. The differential equation here is Bessel's equation with index zero, and the characteristic functions of this system are

$$\phi(\lambda, r) = J_0(\lambda_n r) \qquad (n = 1, 2, \cdots), \tag{10}$$

where the characteristic numbers λ_n are the positive roots of the equation

$$J_0(\lambda) = 0;$$

that is, the homogeneous system (9) has nonzero solutions satisfying the usual requirements as to continuity only when $\lambda = \lambda_n$, and those solutions are the functions (10) except for constant factors.* The numbers λ_n are listed in mathematical tables.

Our characteristic functions are orthogonal, with weight function r, on the interval $0 < r < 1$, and their norms are $J_1^2(\lambda_n)/2$; that is, if n and i are distinct positive integers then

$$\int_0^1 J_0(\lambda_n r) J_0(\lambda_i r) r dr = 0, \qquad \int_0^1 [J_0(\lambda_n r)]^2 r dr = \frac{1}{2}[J_1(\lambda_n)]^2.$$

Moreover any function of r defined on the interval $0 < r < 1$ that satisfies certain conditions of regularity is represented by its generalized Fourier series of those orthogonal functions. If for each fixed x our function $W(r, x)$ satisfies such conditions, then that expansion can be written

$$W(r, x) = 2\sum_{n=1}^{\infty} w(\lambda_n, x) \frac{J_0(\lambda_n r)}{J_1^2(\lambda_n)}, \tag{11}$$

where

$$w(\lambda_n, x) = \int_0^1 W(r, x) J_0(\lambda_n r) r dr = T\{W\}. \tag{12}$$

Since equation (11) expresses W in terms of w it represents the inverse of our transformation T. Our transformation T, now defined by equation (12), is a finite Hankel transformation.

The transformation (8) can now be written

$$T\left\{\frac{1}{r}(rW_r)_r\right\} = -\lambda_n^2 w(\lambda_n, x) + \lambda_n J_1(\lambda_n) W(1, x) \tag{13}$$

since $J_0'(y) = -J_1(y)$; thus T reduces the differential form in the variable r to an algebraic form in w and the prescribed boundary function $W(1, x)$. In view of condition (2) our equation (6), the transform of equation (1), can now be written

* For the elementary theory of Bessel functions used here see for instance Chapter 8 of the author's book, Fourier Series and Boundary Value Problems, 1941, and the references cited there.

$$\frac{d^2}{dx^2} w(\lambda_n, x) - \lambda_n^2 w(\lambda_n, x) = f(\lambda_n, x) - \lambda_n J_1(\lambda_n) G(x). \tag{14}$$

With the aid of the basic integration formula

$$T\{1\} = \int_0^1 J_0(\lambda_n r) r dr = \frac{J_1(\lambda_n)}{\lambda_n}$$

the transforms of boundary conditions (3) and (4) become, formally,

$$\frac{d}{dx} w(\lambda_n, x)\Big]_{x=0} - h w(\lambda_n, 0) = -\frac{B}{\lambda_n} J_1(\lambda_n), \qquad \lim_{x\to\infty} w(\lambda_n, x) = 0. \tag{15}$$

The problem (14)–(15) is one in ordinary differential equations with constant coefficients. It can be solved by elementary methods, or by using the Laplace transformation with respect to x. If $p_n(x)$ denotes the right-hand member of equation (14), one form of the solution can be written

$$\lambda_n w(\lambda_n, x) = \frac{B J_1(\lambda_n)}{h + \lambda_n} e^{-\lambda_n x} + \int_0^x p_n(y) \sinh [\lambda_n(x - y)] dy - \frac{h \sinh \lambda_n x + \lambda_n \cosh \lambda_n x}{h + \lambda_n} \int_0^\infty p_n(y) e^{-\lambda_n y} dy.$$

The formal solution of our problem (1)–(4) is obtained by substituting this expression for $w(\lambda_n, x)$ into our inversion formula (11).

4. Transformation with respect to x. We now solve the same boundary value problem (1)–(4) with the aid of an integral transformation of the type

$$S\{W(r, x)\} = \int_0^\infty W(r, x)\theta(\alpha, x) dx = w(r, \alpha). \tag{16}$$

The kernel $\theta(\alpha, x)$ is to be determined so that the problem in the transform $w(r, \alpha)$ is one in ordinary differential equations involving the parameter α and the prescribed boundary values of W. The formal application of S to the members of equation (1) gives the equation

$$\frac{1}{r}\frac{d}{dr}\left\{r \frac{dw}{dr}\right\} + S\{W_{xx}(r, x)\} = f(r, \alpha), \tag{17}$$

where $f(r, \alpha) = S\{F(r, x)\}$. This is an ordinary differential equation in $w(r, \alpha)$ if $S\{W_{xx}\}$ reduces to an algebraic form in w.

After integrating twice by parts we find that

$$S\{W_{xx}\} = \int_0^\infty \theta W_{xx} dx = \int_0^\infty W\theta'' dx + [\theta W_x - \theta' W]_{x=0}^{x=\infty},$$

where the primes indicate differentiation with respect to x. We now assume

that our prescribed functions $F(r, x)$ and $G(x)$ are such that $W_x(r, x)$, as well as $W(r, x)$, approaches zero as x tends to infinity, an assumption that seems justified in the physical problem. Let θ and θ' be bounded when $x \geqq 0$. According to condition (3), $W_x = hW - B$ when $x = 0$. Thus

$$S\{W_{xx}\} = \int_0^\infty W\theta'' dx + [\theta'(\alpha, 0) - h\theta(\alpha, 0)]W(r, 0) + B\theta(\alpha, 0). \tag{18}$$

Since the function $W(r, 0)$ is not prescribed in our boundary value problem the coefficient in the brackets should vanish. The integral is proportional to w if θ'' is proportional to θ. It is convenient to select $-\alpha^2$ as the factor of proportionality. Then θ satisfies the conditions

$$\theta''(\alpha, x) + \alpha^2\theta(\alpha, x) = 0 \qquad (x > 0), \tag{19}$$

$$\theta'(\alpha, 0) - h\theta(\alpha, 0) = 0, \qquad |\theta(\alpha, x)| < M \qquad (x \geqq 0), \tag{20}$$

where M is a constant.

Our problem in the kernel θ is an extension of the Sturm-Liouville type of problem to the semi-infinite interval. The characteristic values of the parameter α consist of all real positive numbers. A convenient form of the solution is

$$\theta(\alpha, x) = \sin [\alpha x + q(\alpha)]$$

where

$$\sin q(\alpha) = \frac{\alpha}{\sqrt{h^2 + \alpha^2}}, \qquad \cos q(\alpha) = \frac{h}{\sqrt{h^2 + \alpha^2}}.$$

Our transformation (16) is thus a modified Fourier transformation

$$S\{W(r, x)\} = \int_0^\infty W(r, x) \sin [\alpha x + q(\alpha)]dx = w(r, \alpha) \tag{21}$$

under which, according to equation (18), the image of W_{xx} is a linear combination of w and of the value of $W_x - hW$ at $x = 0$, namely,

$$S\{W_{xx}(r, x)\} = -\alpha^2 w(r, \alpha) + \frac{\alpha B}{\sqrt{h^2 + \alpha^2}}. \tag{22}$$

The expansion of an arbitrary function $H(x)$, defined on the interval $x \geqq 0$ and satisfying regularity conditions like those used in the ordinary Fourier integral representations, is given by the formula

$$H(x) = \frac{2}{\pi} \int_0^\infty \sin [\alpha x + q(\alpha)]d\alpha \int_0^\infty H(y) \sin [\alpha y + q(\alpha)]dy \quad (x > 0).$$

A method of establishing this extension of the Fourier integral formula is indi-

cated in the literature.* Since the second integral here represents $S\{H(x)\}$ the formula represents the inverse of our transformation S. For each fixed r then

$$W(r, x) = \frac{2}{\pi}\int_0^\infty \sin\,[\alpha x + q(\alpha)]w(r, \alpha)d\alpha. \tag{23}$$

In view of formula (22) the transform (17) of our partial differential equation (1) becomes

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dw}{dr}\right) - \alpha^2 w = f(r, \alpha) - \frac{\alpha B}{\sqrt{h^2 + \alpha^2}} \qquad (0 \leqq r < 1), \tag{24}$$

where we have used the boundary conditions (3) and (4). The transform of the remaining condition (2), and the condition that $W(r, x)$ is to be continuous when $0 \leqq r < 1$ can be written

$$w(1, \alpha) = g(\alpha), \qquad |\, w(r, \alpha)\,| < N \qquad (0 \leqq r < 1), \tag{25}$$

where $g(\alpha) = S\{G(x)\}$ and N is some constant. Equation (24) is the nonhomogeneous form of Bessel's equation with index zero. If $p(r, \alpha)$ denotes the right-hand member of equation (24), the solution of problem (24)–(25), obtained by the method of variation of parameters, can be written in the form

$$w(r, \alpha) = g(\alpha)\frac{I_0(\alpha r)}{I_0(\alpha)} - I_0(\alpha r)\int_r^1 \frac{dy}{yI_0^2(\alpha y)}\int_0^y p(t, \alpha)I_0(\alpha t)tdt, \tag{26}$$

where $I_0(y) = J_0(iy)$.

The solution of our boundary value problem (1)–(4) is obtained by substituting the expression (26) for $w(r, \alpha)$ into formula (23). This integral form of $W(r, x)$ is substantially different from the series representation of the same function found in Section 3.

Another representation of $W(r, x)$ in terms of infinite series and integrals can be obtained by using the Hankel transformation (12) and its properties (13) and (11) to solve problem (24)–(25).

* Page 210 of the author's book, Modern Operational Mathematics, 1944.

EUCLID'S ALGORITHM AND THE LEAST-REMAINDER ALGORITHM

A. W. GOODMAN and W. M. ZARING, University of Kentucky

1. Introduction. Euclid's algorithm* for finding the greatest common divisor (g.c.d.) of two positive integers $b_1 > b_2$ may be written as follows:

$$(E)\qquad \begin{aligned} b_1 &= b_2q_3 + b_3 \\ b_2 &= b_3q_4 + b_4 \\ &\cdots\cdots\cdots \\ b_{n-1} &= b_nq_{n+1} + b_{n+1} \\ b_n &= b_{n+1}q_{n+2}, \end{aligned}$$

where the conditions $b_2 > b_3 \cdots > b_{n+1} > 0$ determine the integers q_k and b_k uniquely. The g.c.d. of b_1 and b_2 is b_{n+1}.

The least-remainder algorithm for finding the g.c.d. of two positive integers $a_1 > a_2$ may be written as follows:

$$(L)\qquad \begin{aligned} a_1 &= a_2p_3 + \epsilon_3a_3 \\ a_2 &= a_3p_4 + \epsilon_4a_4 \\ &\cdots\cdots\cdots \\ a_{m-1} &= a_mp_{m+1} + \epsilon_{m+1}a_{m+1} \\ a_m &= a_{m+1}p_{m+2}, \end{aligned}$$

where $\epsilon_k = \pm 1$, $k = 3, 4, \cdots, m+1$, and is determined uniquely by the condition that the positive integers a_k, p_k be such that

$$(1)\qquad 0 < a_k < \frac{a_{k-1}}{2}, \qquad k = 3, 4, \cdots, m,$$

and

$$(2)\qquad 0 < a_{m+1} \leqq \frac{a_m}{2}.$$

If the equality sign occurs in (2) there is a choice for ϵ_{m+1}, and in this case we require that $\epsilon_{m+1} = +1$.

Kronecker has proved that for two given integers the least-remainder algorithm is not longer than any other algorithm for finding the g.c.d. In particular, if $a_1 = b_1$ and $a_2 = b_2$, then $n \geqq m$. If we restrict ourselves to Euclid's algorithm, and the least-remainder algorithm, this qualitative result may be put into a more

* Elementary Number Theory, Uspensky and Heaslet, McGraw-Hill, New York 1939, pp. 26–28 and pp. 45–51. All of the material required for reading this paper may be found in these few pages.

precise form, namely

THEOREM I. *If $a_1=b_1$ and $a_2=b_2$, then the number of equations in set E exceeds the number of equations in set L by just the number of negative signs in the sequence $\epsilon_3, \epsilon_4, \cdots, \epsilon_{m+1}$, that is*

$$n - m = \frac{1}{2}\sum_{k=3}^{m+1}\{|\epsilon_k| - \epsilon_k\}. \tag{3}$$

COROLLARY. *E and L have the same length if and only if they are identical.*

2. Proof of Theorem I. We construct a sequence of algorithms $E\equiv E_0$, $E_1, E_2, \cdots, E_r\equiv L$. In this sequence each E_k will have one less equation than E_{k-1}, and one more negative sign among the remainders $\epsilon_j a_j$. The theorem will then follow immediately from the fact that $E_0\equiv E$ has no negative signs.

Let us examine in E, the set of ratios $R_0=\{r_j\}$ where

$$0 < r_j = \frac{b_j}{b_{j-1}} < 1, \qquad j = 3, 4, \cdots, n. \tag{4}$$

Suppose that $r_j<1/2$ for $j=3, 4, \cdots, k\leqq n-1$, and that $r_{k+1}>1/2$. Then in equation sets E and L the first $k-2$ equations are identical. If no such k exists, then E and L coincide and since there are no negative signs in the sequence $\epsilon_3, \epsilon_4, \cdots, \epsilon_{m+1}$, in this case the theorem is established.

Suppose however that there is a $k\leqq n-1$ as described above. Consider the three equations

$$b_{k-1} = b_k q_{k+1} + b_{k+1}, \qquad 1 > \frac{b_{k+1}}{b_k} > \frac{1}{2}, \tag{5}$$

$$b_k = b_{k+1}q_{k+2} + b_{k+2}, \tag{6}$$

$$b_{k+1} = b_{k+2}q_{k+3} + b_{k+3}. \tag{7}$$

Clearly we have

$$b_{k-1} = b_k(q_{k+1} + 1) - (b_k - b_{k+1})$$

or

$$b_{k-1} = b_k p_{k+1} - a_{k+1}, \qquad 0 < \frac{a_{k+1}}{b_k} < \frac{1}{2}, \tag{8}$$

and this equation will now coincide with the $(k-1)$th equation of the set L. Further since $r_{k+1}>1/2$, $q_{k+2}=1$ and equation (6) must have the form

$$b_k = b_{k+1} + b_{k+2} \tag{9}$$

and hence

$$a_{k+1} = b_k - b_{k+1} = b_{k+2}. \tag{10}$$

Finally from (7) we have

$$b_{k+1} + b_{k+2} = b_{k+2}(q_{k+3} + 1) + b_{k+3}$$

which together with (9) and (10) gives

$$b_k = a_{k+1}(q_{k+3} + 1) + b_{k+3}. \tag{11}$$

The set E_1 is now obtained from E_0 by suppressing the equations (5), (6), and (7) and replacing them by equations (8) and (11). One equation has been lost and one minus sign has been added, namely the one in equation (8), and the first $k-1$ equations in E_1 and L are identical. For the set of equations E_1, there is a new set of ratios of remainders which we denote by R_1, consisting of the ratios

$$\frac{b_3}{b_2}, \frac{b_4}{b_3}, \cdots, \frac{b_k}{b_{k-1}}, \frac{a_{k+1}}{b_k}, \frac{b_{k+3}}{a_{k+1}}, \frac{b_{k+4}}{b_{k+3}}, \cdots, \frac{b_n}{b_{n-1}}. \tag{12}$$

If $k=n-1$ then $b_{k+3}=b_{n+2}=0$ and the sequence (12) has as its last term a_{k+1}/b_k.

The set of ratios R_1 is again examined and if one of these ratios exceeds one-half, we select the first such ratio and proceed to generate the set E_2 from the set E_1, by the same method used in obtaining E_1 from E_0. After a finite number of such steps, we arrive at an equation set E_ν with a set of ratios of remainders R_ν, none of which exceed one-half. Then $E_\nu \equiv L$ and Theorem 1 is proved.

3. Further results. The method developed in the preceding section can be used to prove two other theorems on the relative lengths of E and L.

THEOREM 2. *Let $a_1=b_1$ and $a_2=b_2$. Divide the set of ratios R_0 into blocks consisting only of those ratios which exceed one-half. A block may have only one member but each block is to be as long as possible. Number the ratios in each block consecutively beginning with the number one for each block. Let μ be the number of odd-numbered ratios. Then*

$$n - m = \mu. \tag{13}$$

Example: If $b_1=1065$ and $b_2=674$, the set R_0 is

$$\underset{\frac{391}{674}}{1}, \underset{\frac{283}{391}}{2}, \underset{\frac{108}{283}}{}, \underset{\frac{67}{108}}{1}, \underset{\frac{41}{67}}{2}, \underset{\frac{26}{41}}{3}, \underset{\frac{15}{26}}{4}, \underset{\frac{11}{15}}{5}, \underset{\frac{4}{11}}{}, \underset{\frac{3}{4}}{1}.$$

The blocks are indicated by heavy lines and the number of each ratio is indicated just above the ratio. In this case $\mu=5$.

Proof. Just as in the preceding sections we construct a sequence of algorithms $E \equiv E_0, E_1, \cdots, E_\nu = L$. The ratio set R_1 for E_1 is given by (12). In going from E_0 to E_1 the ratio b_{k+1}/b_k, which is an odd-numbered ratio, is replaced by $a_{k+1}/b_k < 1/2$, and one equation is lost. The ratio b_{k+2}/b_{k+1} disappears, but either

$b_{k+2}/b_{k+1}<1/2$ or it is an even-numbered ratio. Finally from equation (10) $a_{k+1}=b_{k+2}$ and hence all succeeding ratios are unchanged, and the next ratio exceeding one-half will be odd-numbered. This completes the proof of Theorem 2.

Although a little awkward, Theorem 2 has the merit of being a complement to Theorem 1. For given the set L one can determine by Theorem 1 how many equations will be in E, and conversely given the set E one can determine by Theorem 2, how many equations will be in L.

THEOREM 3. *Let $a_1=b_1$ and $a_2=b_2$. Then in E and L*

$$n+\sum_{j=3}^{n+2} q_j = m+\sum_{j=3}^{m+2} p_j, \tag{14}$$

that is, the number of equations plus the sum of the quotients is constant.

Proof. We only need to observe that in replacing equations (5), (6), and (7) by (8) and (11) the change in the quotient sum is $+1$. For $q_{k+2}=1$ is lost by the suppression of equations (6) and q_{k+1} and q_{k+3} are increased by 1 in equations (8) and (11). Theorem 3 is proved.

Theorem 3, and the corollary to Theorem 1, are no longer valid if other g.c.d. algorithms are considered. In fact Uspensky and Heaslet have already noted that a g.c.d. algorithm may be the same length as the Euclid algorithm for the same two positive integers, without the two algorithms being identical. It would be interesting to see in what way the above results can be modified to include other g.c.d. algorithms. In particular, it would be nice to have some expression such as (14) which would be invariant for all g.c.d. algorithms.

A RECURRENCE RELATION FOR THREE-LINE LATIN RECTANGLES

JOHN RIORDAN, Bell Telephone Laboratories

1. Introduction. This note gives a derivation of a new recurrence relation for the number of reduced three-line Latin rectangles. It may be recalled from earlier papers [2, 3] that a Latin rectangle is an array in which each row is a permutation of n elements and each column has unlike elements, and that it is reduced when the first row is in standard order.

The relation is "impure," in the sense that it contains an auxiliary function, but it is of lower order than the pure recurrence of Kerawala [1] and easier to compute. The structure of residues modulo p, a prime, is an almost immediate consequence. Finally it serves to simplify the derivation of the elegant asymptotic formula of Yamamoto [4, 5].

2. Derivation of recurrence relation. Write $L(3, n)$ for the number of three-line Latin rectangles and $K(3, n)$, abbreviated to K_n, for the corresponding reduced number ($L(3, n)=n!\,K(3, n)$). Then the basic formula used in the first of Yamamoto's papers, of which I have an English translation through the kindness of the author, may be written as follows.

$$K_n = \sum_0^n \binom{n}{r} 2^r v_{n-r} = (v+2)^n \tag{1}$$

with the last a symbolic form and

$$v_n = n! \sum_0^n (-1)^s (n-s)! \binom{3n-2s+2}{s} \tag{2}$$
$$= n!\, u_n, \text{ say.}$$

Using the identity

$$\frac{3n}{3n-2s}\binom{3n-2s}{s} = \binom{3n-2s+2}{s} - 3\binom{3n-2s}{s-2} + 2\binom{3n-2s-1}{s-3}$$

multiplied throughout by $(-1)^s(n-s)!$ and summed on s, it follows that

$$u_n = nu_{n-1} + 3u_{n-2} + 2u_{n-3} + 3(-1)^n - 2\delta_{0n} \tag{3}$$

with δ a Kronecker symbol ($\delta_{00}=1$, $\delta_{0n}=0$, $n\neq 0$).

Hence

$$v_n = n^2 v_{n-1} + 3(n)_2 v_{n-2} + 2(n)_3 v_{n-3} + 3n!(-1)^n - 2\delta_{0n}, \tag{4}$$

$(n)_k = n(n-1)\cdots(n-k+1)$ being a factorial symbol.

It is now convenient to use the exponential generating function:

$$\exp vt = \sum_0^\infty v_n t^n/n!$$

with the left-hand side symbolic. By (1)

$$\exp Kt = \sum_0^\infty K_n t^n/n! = \exp (v+2)t \tag{5}$$

and by (4)

$$[1 - t - t^2(v+3) - 2t^3] \exp vt = 3(1+t)^{-1} - 2, \tag{6}$$

so that

$$[1 - t - t^2(K+1) - 2t^3] \exp Kt = e^{2t}[3(1+t)^{-1} - 2] \tag{7}$$
$$= \exp kt, \text{ say,}$$

and

$$K_n = n^2K_{n-1} + (n)_2K_{n-1} + 2(n)_3K_{n-3} + k_n, \tag{8}$$

the new recurrence formula.

Since

$$(1 + t) \exp kt = e^{2t}(1 - 2t)$$

it follows at once that

$$k_n + nk_{n-1} = -(n - 1)2^n \tag{9}$$

which may be used to calculate k_n. The following short table is introduced for numerical concreteness.

n	0	1	2	3	4	5	6	7
k_n	1	−1	−2	−10	−8	−88	208	−2,224
K_n	1	0	0	2	24	552	21,280	1,073,760

Recurrence (9) may be used to eliminate k_n in (8); the further elimination of $(n-1)2^n$ yields Kerawala's recurrence which I do not take space to write.

3. Asymptotic expansion. As $k_n = O(n!)$ it may be ignored for asymptotic results, and substitution of

$$n!^{-2}e^3K_n = 1 + \frac{a_1}{n} + \frac{a_2}{(n)_2} + \cdots + \frac{a_s}{(n)_s} + \cdots \tag{10}$$

into recurrence (8) and use of the identity

$$\frac{1}{(n-1)_s} = \frac{1}{(n)_s} + \frac{s}{(n)_{s+1}}$$

shows that

$$(s + 1)a_{s+1} + a_s + 2a_{s-1} = 0, \tag{11}$$

so that

$$a_s = H_s(-1/2)/s! \tag{12}$$

with H_s an Hermite polynomial, which is Yamamoto's result.

4. Arithmetical structure. By (9) and Fermat's theorem, with p an odd prime,

$$k_p \equiv 2 \ (p). \tag{13}$$

Since $k_0 = 1$ and $(p+n-1)2^{p+n-1} \equiv 2(n-1)2^n$, this and (9) show that

$$k_{n+p} \equiv 2k_n, \tag{14}$$

$$k_{n+xp} \equiv 2^x k_n. \tag{15}$$

Hence the period of these numbers is xp where $2^x \equiv 1 \ (p)$; x is a divisor of $p-1$.

The structure of the K_n is exactly the same, because by (8)

$$K_p \equiv k_p \equiv 2 \equiv 2K_0 \ (p) \tag{16}$$

and

$$K_{p+n} \equiv n^2 K_{p+n-1} + (n)_2 K_{p+n-2} + 2(n)_3 K_{p+n-3} + k_{p+n} \ (p)$$

which with (16) entails

$$K_{p+n} \equiv 2K_n. \tag{17}$$

It is interesting to notice that $K_{p+2} \equiv K_{p+1} \equiv 0 \ (p)$ is concordant with the relation $K_n = (n-1)(n-2)J_n$ where J_n is the number of rectangles with first column as well as first row standardized, and hence an integer. Thus the numbers $K_0 = 1$, $K_1 = K_2 = 0$, which are not defined combinatorially because there are no such rectangles, may be taken as defined by this circumstance, as appears naturally in the development.

A table of residues, which is easy to make for small moduli, then serves to check the tables of K_n (Kerawala [1], $n = 2(1)15$, Yamamoto [5], $n = 2(1)20$) and any extensions, like the one I have made to $n = 25$, but I do not take space for these large numbers (50 digits for $n = 25$).

From such a residue table curiosities like the following are read off at once:

$$K_n \equiv 0 \ (19), \qquad n \equiv 6, 18 \ (19);$$
$$K_n \equiv 0 \ (23), \qquad n \equiv 5, 10, 11 \ (23).$$

The complete factorization of K_n, and the arithmetical structure, yield results like

$$K_n \equiv 0 \ (2237), \qquad n \equiv 7 \ (2237);$$
$$K_n \equiv 0 \ (26{,}153), \qquad n \equiv 8 \ (26{,}153).$$

Bibliography

1. S. M. Kerawala, The enumeration of the Latin rectangle of depth three by means of a difference equation, Bull. Calcutta Math. Soc., vol. 33, pp. 119–127, 1941.

2. J. Riordan, Three-line Latin rectangles, this MONTHLY, vol. 51, pp. 450–452, 1944.

3. J. Riordan, Three-line Latin rectangles—II, this MONTHLY, vol. 53, pp. 18–20, 1946.

4. Koichi Yamamoto, Asymptotic number of Latin rectangles and the symbolic method, Sugaku, vol. 2, 1949.

5. Koichi Yamamoto, An asymptotic series for the number of three-line Latin rectangles, Jour. Math. Soc. of Japan, vol. 1, pp. 226–241, 1950.

ON THE FINITE DIFFERENCE ANALOGUE OF RODRIGUES' FORMULA*

MARIA WEBER and A. ERDÉLYI, California Institute of Technology

1. It is well known that several important systems of orthogonal polynomials can be obtained in the form

$$\frac{1}{\rho(x)}\frac{d^n}{dx^n}\{\rho(x)[X(x)]^n\}; \qquad n = 0, 1, 2, \cdots, \tag{1}$$

where $\rho(x)$ is an infinitely differentiable function and $X(x)$ is a polynomial, both independent of n. One may ask for all functions $\rho(x)$ and polynomials $X(x)$ for which (1) is a polynomial of exact degree n for each n. As far as we know this question was first put, slightly differently, and completely answered, by Tricomi [8]. By a slight modification of Tricomi's argument one sees that the degree of $X(x)$ cannot exceed 2, and that all polynomials generated in the manner of (1) may be reduced to one of the "classical" orthogonal polynomials by a linear change of the independent variable.

There are also some systems of polynomials generated similarly by finite differences in the form:

$$p_n(x) = \frac{1}{\rho(x)}\Delta^n[\rho(x-n)X(x)X(x-1)\cdots X(x-n+1)]; \qquad n = 0, 1, 2, \cdots, \tag{2}$$

where $\rho(x)$ is any function, $X(x)$ is a polynomial, both are independent of n, and Δ is the operator of finite differences defined by

$$\Delta f(x) = f(x+1) - f(x), \qquad \Delta^{n+1}f(x) = \Delta(\Delta^n f(x)), \qquad n = 0, 1, 2, \cdots. \tag{3}$$

One may again ask for all functions $\rho(x)$ and all polynomials $X(x)$ for which (2) is a polynomial of exact degree n for each n. The answer is surprisingly simple. $X(x)$ must be of degree $\leqq 2$, $\rho(x)$ is a combination of gamma functions, and all systems of polynomials capable of a representation in the form (2) can be enumerated.

Actually, W. Hahn [3], has discussed the corresponding problem for the operator $\mathfrak{d}$ defined by

$$\mathfrak{d}f(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

(with a restriction on the degree of $X(x)$). Both d/dx and Δ are limiting cases of $\mathfrak{d}$. Since Hahn's work introduces of necessity basic hypergeometric series, and since the limiting process $\mathfrak{d}\to\Delta$ is quite involved, it seemed worth while to give a simple and elementary discussion of (2).

* Prepared under contract N6onr-244, Task Order XIV of the Office of Naval Research with the California Institute of Technology.

2. We proceed to determine all the sets of polynomials (2), each $p_n(x)$ of exact degree n. In the following we exclude tacitly all values of the parameters which make the formulas meaningless. We shall first prove that if the degree k of X exceeds 2, we do not obtain a required set, because for some n the degree of $p_n(x)$ is larger than n.

Let $k \geqq 2$. We write (2) for $n=1$ and get

$$p_1(x) = [\rho(x)]^{-1}\Delta[\rho(x-1)X(x)],$$

which gives

$$\frac{\rho(x-1)}{\rho(x)} = \frac{X(x+1) - p_1(x)}{X(x)}. \tag{4}$$

Factoring the polynomials on the right we have

$$\frac{\rho(x-1)}{\rho(x)} = \prod_{j=1}^{k} \frac{(x+\alpha_j)}{(x+\beta_j)}. \tag{5}$$

A solution of this equation is

$$\rho(x) = \prod_{j=1}^{k} \frac{\Gamma(x+\beta_j+1)}{\Gamma(x+\alpha_j+1)}.$$

Since (2) does not depend on the particular solution of (5) chosen, we have

$$p_n(x) = \prod_{j=1}^{k} \frac{\Gamma(x+\alpha_j+1)}{\Gamma(x+\beta_j+1)} \Delta^n \left\{ \prod_{l=1}^{k} \frac{\Gamma(x+\beta_l+1)}{\Gamma(x+\alpha_l-n+1)} \right\} \tag{6}$$

and therefore

$$p_1(x) = X(x+1) - \prod_{j=1}^{k} (x+\alpha_j). \tag{7}$$

Combining (6) for $n=2$ with (7) we get

$$p_2(x) = X(x+1)\Delta^2 X + p_1(x)\Delta X + X(x+1)\Delta p_1 + p_1(x)p_1(x-1).$$

If $k>2$ this clearly makes $p_2(x)$ of degree larger than 2. We shall therefore only consider $k \leqq 2$.

Case 1: $k=2$.

From (6) we obtain

$$p_n(x) = \frac{\Gamma(x+\alpha_1+1)\Gamma(x+\alpha_2+1)}{\Gamma(x+\beta_1+1)\Gamma(x+\beta_2+1)} \Delta^n \left[\frac{\Gamma(x+\beta_1+1)\Gamma(x+\beta_2+1)}{\Gamma(x+\alpha_1-n+1)\Gamma(x+\alpha_2-n+1)} \right] \tag{8}$$

and

$$p_n(x) = \sum_{p=0}^{n} \binom{n}{p} (-1)^p \times \frac{\Gamma(x+\beta_1+1+n-p)\Gamma(x+\beta_2+1+n-p)\Gamma(x+\alpha_1+1)\Gamma(x+\alpha_2+1)}{\Gamma(x+\beta_1+1)\Gamma(x+\beta_2+1)\Gamma(x+\alpha_1+1-p)\Gamma(x+\alpha_2+1-p)}.$$

Using the identity

$$\Gamma(z-p) = (-1)^p \frac{\Gamma(z)\Gamma(1-z)}{\Gamma(p+1-z)}, \qquad p \text{ an integer},$$

we obtain

$$p_n(x) = \frac{\Gamma(x+\beta_1+1+n)\Gamma(x+\beta_2+1+n)}{\Gamma(x+\beta_1+1)\Gamma(x+\beta_2+1)} \times \sum_{p=0}^{n} \binom{n}{p} (-1)^p \frac{\Gamma(-x-\beta_1-n)\Gamma(-x-\beta_2-n)}{\Gamma(-x-\beta_1-n+p)\Gamma(-x-\beta_2-n+p)} \times \frac{\Gamma(-x-\alpha_1+p)\Gamma(-x-\alpha_2+p)}{\Gamma(-x-\alpha_1)\Gamma(-x-\alpha_2)},$$

which can be written

$$p_n(x) = \frac{\Gamma(x+\beta_1+1+n)\Gamma(x+\beta_2+1+n)}{\Gamma(x+\beta_1+1)\Gamma(x+\beta_2+1)} \times {}_3F_2(-n,\ x-\alpha_1,\ -x-\alpha_2;\ -x-\beta_1-n,\ -x-\beta_2-n;\ 1). \tag{9}$$

A particular case of (9) are the polynomials investigated by Tchebichef (cf. Szegö [7])

$$\alpha_1 = \beta_1 = 0; \qquad \alpha_2 = \beta_2 = -N; \qquad \rho(x) = 1.$$

In order to prove that (9) defines polynomials of degree n and that these are identical with a class of polynomials mentioned by Hahn, we use a transformation formula to be established in section 3. According to (18)

$$p_n(x) = \frac{\Gamma(\alpha_1-\beta_1)\Gamma(\alpha_1-\beta_2)}{\Gamma(\alpha_1-\beta_1-n)\Gamma(\alpha_1-\beta_2-n)} \times {}_3F_2(-n,\ -x-\alpha_1,\ 1+\beta_1+\beta_2-\alpha_1-\alpha_2+n;\ \beta_1-\alpha_1+1,\ \beta_2-\alpha_1+1;\ 1). \tag{10}$$

In this form it is easily seen that the Bateman [2] polynomials are a particular case of (9): put

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = -1.$$

A recurrence relation can be given for these polynomials if we use the known contiguous function relations for ${}_pF_q$ (cf. Rainville [6]). The derivation is elementary, but messy, so we only give the result:

$$a_n p_{n+1} + b_n p_{n-1} + (c_n x + d_n) p_n = 0 \tag{11}$$

where

$$a_n = -(s + n + 1)(s + 2n)$$
$$b_n = n(s + 2n + 2)(\beta_1 + n - \alpha_1)(\beta_1 + n - \alpha_2)(\beta_2 + n - \alpha_1)(\beta_2 + n - \alpha_2)$$
$$c_n = (s + 2n)(s + 2n + 1)(s + 2n + 2)$$
$$d_n = n(s + 2n + 2)[(\alpha_1 + \alpha_2)(s + 2n) + \alpha_1\alpha_2 - (\beta_1 + n)(\beta_2 + n)]$$
$$- (s + 2n)(s + n + 1)[\alpha_1\alpha_2 - (\beta_1 + n + 1)(\beta_2 + n + 1)]$$

and $s = \beta_1 + \beta_2 - \alpha_1 - \alpha_2$.

A recurrence formula given for the Tchebichef polynomials by Jordan [4] is a particular case of our formula (11).

Assuming that $\alpha_2 - \alpha_1$ is a positive integer, we obtain orthogonality relations for our polynomials with a weight-function which has jumps

$$\frac{\Gamma(x + \beta_1 + 1)\Gamma(x + \beta_2 + 1)}{\Gamma(x + \alpha_1 + 1)\Gamma(x + \alpha_2 + 1)}$$

at the points $x = -\alpha_1 - n$, $n = 0, 1 \cdots$.

Case 2: $k = 1$

From equation (4) we obtain two subcases

$$\text{(a)} \qquad \frac{\rho(x)}{\rho(x - 1)} = c\frac{x + \beta}{x + \alpha}$$

$$\text{(b)} \qquad \frac{\rho(x)}{(x - 1)} = c(x + \gamma).$$

(a) gives

$$\rho(x) = c^x \frac{\Gamma(x + \beta + 1)}{\Gamma(x + \alpha + 1)} \tag{12}$$

and

$$p_n(x) = \frac{\Gamma(x + \beta + n + 1)}{\Gamma(x + \beta + 1)} {}_2F_1\left(-n, -x - \alpha; -x - \beta - n; \frac{1}{c}\right) \tag{13}$$

which can also be written as

$$p_n(x) = (-1)^n \frac{\Gamma(\alpha - \beta)}{\Gamma(\alpha - \beta - n)} {}_2F_1\left(-n, -x - \alpha; \beta - \alpha + 1; 1 - \frac{1}{c}\right). \tag{14}$$

These are polynomials derived from the Jacobi polynomials and were studied by M. Krawtchouk (cf. Szegö [7]) and J. Meixner [4].

(b) gives

$$\rho(x) = c^x\Gamma(x+\gamma+1)$$

and

$$p_n(x) = \frac{\Gamma(x+\gamma+n+1)}{\Gamma(x+\gamma+1)}\,{}_1F_1\left(-n;\ -x-n-\gamma;\ -\frac{1}{c}\right). \tag{15}$$

These polynomials are derived from the Laguerre polynomials and were first studied by Charlier (cf. Szegö [7]). The case $k=0$ does not yield anything new. We obtain

$$\rho(x) = \frac{c^x}{\Gamma(x+1)}$$

and

$$p_n(x) = c_n L_n^{x-n}(c)$$

which are again the Charlier polynomials.

We collect in a table all the polynomials represented by (2), where k is the degree of $X(x)$.

k	Polynomials
$k>2$	none
2	${}_3F_2$; special cases: Tchebichef, Bateman
1	${}_2F_1$; Krawtchouk, Meixner
1	${}_1F_1$; Charlier
0	${}_1F_1$; Charlier

3. In this section we shall derive a transformation formula for a terminating ${}_3F_2$ of unit argument. This formula is one of a large group of known transformations (cf. Bailey [1]), but as we have used the result in deriving (10), we shall give the proof here.

We write

$${}_3F_2(-n,\alpha,\beta;\gamma,\delta;1) = \frac{\Gamma(\delta)}{\Gamma(\beta)\Gamma(\delta-\beta)}\int_0^1 {}_2F_1(-n,\alpha;\gamma;t)t^{\beta-1}(1-t)^{\delta-\beta-1}dt \tag{16}$$

which can be checked easily by expanding both sides in power-series; but we have the known identity

$${}_2F_1(-n,\alpha;\gamma;t) = \frac{\Gamma(\gamma)\Gamma(\gamma+n-\alpha)}{\Gamma(\gamma+n)\Gamma(\gamma-\alpha)}\,{}_2F_1(-n,\alpha;\alpha-n-\gamma+1;1-t).$$

We substitute this in (16), replace the variable t by $1-t$, expand and integrate term by term. We thus obtain the identity

$$
\begin{aligned}
&{}_3F_2(-n, \alpha, \beta; \gamma, \delta; 1) \\
&\quad = \frac{\Gamma(\gamma)\Gamma(\gamma + n - \alpha)}{\Gamma(\gamma + n)\Gamma(\gamma - \alpha)} {}_3F_2(-n, \alpha, \delta - \beta; 1 + \alpha - \gamma - n, \delta; 1).
\end{aligned} \tag{17}
$$

Using (17) over again, with the roles of γ and δ interchanged, to transform the right side of (17), we finally obtain

$$
\begin{aligned}
{}_3F_2(-n, \alpha, \beta; \gamma, \delta; 1) = {} & \frac{\Gamma(\delta)\Gamma(\gamma)\Gamma(\delta + n - \alpha)\Gamma(\gamma + n - \alpha)}{\Gamma(\delta + n)\Gamma(\gamma + n)\Gamma(\delta - \alpha)\Gamma(\gamma - \alpha)} \\
& \times {}_3F_2(-n, \alpha, 1 + \alpha + \beta - \gamma - \delta - n; \\
& \qquad 1 + \alpha - \delta - n, 1 + \alpha - \gamma - n; 1).
\end{aligned} \tag{18}
$$

References

1. Bateman, H., Tôhoku Math. J. 1933, vol. 37, pp. 24–38.
2. Bailey, W. N., Generalized hypergeometric series. Cambridge Tracts in Math. No. 32, Chapter III.
3. Hahn, W., Math. Nachr. 1949, vol. 2, pp. 4–34, 263–278, 340–379.
4. Jordan, Proc. London Math. Soc. (2) 1920, vol. 20, pp. 297–325.
5. Meixner, J., J. London Math. Soc. 1934, vol. 9, pp. 6–13.
6. Rainville, E. D., Bull. Amer. Math. Soc. 1945, vol. 51, pp. 268–271.
7. Szegö, G., Orthogonal polynomials. Amer. Math. Soc. Colloquium Publ. vol. 23, p. 32-36.
8. Tricomi, F., Serie ortogonali di funzioni, Torino, 1948.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

ON THE APPLICABILITY OF A CERTAIN FORMULA IN THE THEORY OF LINEAR DIFFERENTIAL EQUATIONS

I. I. KOLODNER, Institute for Mathematics and Mechanics, New York University

Consider the second order linear differential equation for $y(x)$,

$$
Ly \equiv (p\dot{y})^{\cdot} + qy = 0 \tag{1}
$$

where p is continuously differentiable and positive, and q is continuous in the open interval Δ: $a<x<b$. (A dot denotes differentiation with respect to the argument.) A well known theorem asserts the existence of a unique solution to the initial value problem $Ly=0$, $y(c)=y_0$, $\dot{y}(c)=\dot{y}_0$, $(a<c<b)$, which is continuously twice differentiable and bounded in every closed subinterval of Δ. It follows then that two linearly independent solutions of $Ly=0$ having the same properties exist.

Let $y_1(x)=u(x)$ be a nontrivial solution of equation (1) in the interval Δ. It is often taught, without further qualifications, that the expression

$$y_2(x) = u(x)\int_{\alpha_1}^{x} \frac{d\xi}{p(\xi)u^2(\xi)} \qquad (a < \alpha_1 < b), \tag{2}$$

yields the second solution linearly independent of y_1. If, however, x_0 and x_1 are two zeros of u such that $x_0<\alpha_1<x_1$ and $u(x)\neq 0$ for $x_0<x<x_1$, it may easily be verified that the formula (2) does not represent a function outside the interval $x_0<x<x_1$. Indeed,

$$\lim_{x\to x_1}\int_{\alpha_1}^{x}\frac{d\xi}{pu^2}=\infty$$

(and likewise for $x\to x_0$), since otherwise we would have $y_2(x_1)=0$ contradicting the independence of y_2 and y_1. The integrand in (2) being positive, the integral clearly diverges for $x>x_1$ which proves our assertion. A similar conclusion follows for $x<x_0$.

As an example, consider the simplest case when $p=1$ and $q=0$: $y_1=x$ is a solution and equation (2) yields the second solution $y_2=-1+x/\alpha_1$ whenever sg $(x)=$sg (α_1). When sg $(x)=-$sg (α_1), the expression (2) does not represent a function.

The object of the present note is to extend the definition of y_2 as given by (2) outside the interval $x_0<x<x_1$. Denote by x_i the zeros of $u(x)$ arranged in increasing sequence, and let $x_{i+}(x_{i-})$ be the closest zero of $\dot{u}(x)$ to the right (left) of x_i (u and $\dot{u}$ cannot vanish simultaneously). Let α_i and β_i be numbers such that

$$x_{i-} < \alpha_i < x_i < \beta_i < x_{i+}.$$

Equation (2) represents y_2 in the interval $x_0<x<x_1$. Now, integrating by parts the integral in (2), one gets

$$y_2(x) = -\frac{1}{p(x)\dot{u}(x)} + \frac{u(x)}{p(\alpha_1)u(\alpha_1)\dot{u}(\alpha_1)} + u(x)\int_{\alpha_1}^{x}\frac{q\,d\xi}{(p\dot{u})^2}. \tag{3}$$

Clearly, (3) is identical with (2) in the interval $x_{1-}<x<x_1$. Yet, it represents a twice differentiable solution of (1) in the larger interval $x_{1-}<x<x_{1+}$, hence an extension of (2) to the right of x_1.

To extend the validity of (3) we split the above integral in two parts, first over the interval (α_1, β_1), second, over the interval (β_1, x) and integrate the latter integral by parts, obtaining

$$y_2 = u(x)\left[\frac{1}{p(\alpha_1)u(\alpha_1)\dot{u}(\alpha_1)} - \frac{1}{p(\beta_1)u(\beta_1)\dot{u}(\beta_1)} + \int_{\alpha_1}^{\beta_1} \frac{q\,d\xi}{(p\dot{u})^2} + \int_{\beta_1}^{\alpha_2} \frac{d\xi}{pu^2}\right] + u(x)\int_{\alpha_2}^{x} \frac{d\xi}{pu^2}. \tag{4}$$

The expression (4) matches with (3) for $x_1 < x < x_{1+}$, yet it represents the solution of (1) for $x_1 < x < x_2$. Following Hadamard [1], the expression in the brackets above may be conveniently termed the finite part (FP) of

$$\int_{\alpha_1}^{\alpha_2} \frac{d\xi}{pu^2}.$$

Without further difficulty one obtains now

$$y_2 = u(x)\left[\mathrm{FP}\int_{\alpha_1}^{\alpha_k} \frac{d\xi}{pu^2} + \int_{\alpha_k}^{x} \frac{d\xi}{pu^2}\right], \qquad x_{k-1} < x < x_k, \tag{5}$$

or, simply,

$$y_2 = u(x)\left(FP\int_{\alpha_1}^{x} \frac{d\xi}{pu^2}\right), \qquad a < x < b. \tag{6}$$

Coming back to the simple example considered above, with $\alpha_1 < 0$,

$$\mathrm{FP}\int_{\alpha_1}^{\alpha_2} \frac{d\xi}{\xi^2} = \frac{1}{\alpha_1} - \frac{1}{\alpha_2}, \qquad \alpha_2 > 0,$$

so that (5) yields

$$y_2 = -1 + \frac{x}{\alpha_1} \qquad \text{for } x > 0,$$

as expected.

Our result could have been anticipated on the basis of work of Hadamard [1]; yet, to prove it, one would have to supply the derivation just carried out. There exists a more elegant method of proving our result which uses, however, the recent and still little known notion of distributions. For those familiar with the work of L. Schwartz [2] the matter may be settled in a few words. Indeed, if one considers the equation (1) as an equation for an unknown distribution y, one finds two linearly independent distributions y_1 and y_2, the latter given by (2). As a distribution, the expression (2) has a meaning in the full interval Δ, and furthermore, there corresponds to it a real function, namely the function (6).

References

1. J. Hadamard, Lectures on Cauchy's problem in linear partial differential equations, Yale Univ. Press, 1921.

2. L. Schwartz, Theorie des Distributions, Hermann et Cie Ed., Paris, 1950.

A THEOREM OF NESBITT

J. D. NIBLETT, University of Nottingham

Muir in his *Contributions to the History of Determinants*, 1900–1920, page 316, mentions a result announced by Nesbitt, but unproved, that the n-rowed determinant

$$\left| {}^{n+1}C_{2s-r} \right| = 2^{\frac{1}{2}n(n+1)},$$

where s, r denote column and row respectively, and ${}^{n}C_{r}$ has its usual meaning for $0 \leqq r \leqq n$ and is 0 in all other cases.

The form of this result suggests that it is related to the binomial expansions of $(x+y)^n \pm (x-y)^n$, and this is the basis of the two proofs given here.

Proof I.

In this proof the result is obtained entirely by "elementary" manipulations of the binomial expansion.

Let $A = P + y$ and $B = P - y$ where P, $y \neq 0$. Then

$$\tfrac{1}{2}(A^{2n+1} - B^{2n+1}) = R = \sum_{r=0}^{n} {}^{2n+1}C_{2r+1}P^{2n-2r}y^{2r+1} \tag{1}$$

and

$$\tfrac{1}{2}(A^{2n+1} + B^{2n+1}) = Q = \sum_{r=0}^{n} {}^{2n+1}C_{2r}P^{2n+1-2r}y^{2r}. \tag{2}$$

These equations can be written in the form

$$\left({}^{2n+1}C_1 - \frac{R}{yP^{2n}}\right)P^{2n+1} + \sum_{r=1}^{n} {}^{2n+1}C_{2r+1}P^{2n-2r+1}y^{2r} = 0$$

and

$$\left({}^{2n+1}C_0 - \frac{Q}{P^{2n+1}}\right)P^{2n+1} + \sum_{r=1}^{n} {}^{2n+1}C_{2r}P^{2n-2r+1}y^{2r} = 0.$$

Multiplying these by $(y^2P^{-2})^s$ for $s = 0, 1, 2, \cdots, n-1$ and re-arranging we obtain the $2n$ equations

$$\begin{array}{l}
({}^{2n+1}C_1 - Ry^{-1}P^{-2n})P^{2n+1} + {}^{2n+1}C_3P^{2n-1}y^2 + {}^{2n+1}C_5P^{2n-3}y^4 + \cdots + {}^{2n+1}C_{2n+1}Py^{2n} = 0 \\
({}^{2n+1}C_0 - QP^{-2n-1})P^{2n+1} + {}^{2n+1}C_2P^{2n-1}y^2 + {}^{2n+1}C_4P^{2n-3}y^4 + \cdots + {}^{2n+1}C_{2n}Py^{2n} = 0 \\
(-RyP^{-2n-2})P^{2n+1} + {}^{2n+1}C_1P^{2n-1}y^2 + {}^{2n+1}C_3P^{2n-3}y^4 + \cdots + {}^{2n+1}C_{2n+1}P^{-1}y^{2n+2} = 0 \\
(-Qy^2P^{-2n-3})P^{2n+1} + {}^{2n+1}C_0P^{2n-1}y^2 + {}^{2n+1}C_2P^{2n-3}y^4 + \cdots + {}^{2n+1}C_{2n}P^{-1}y^{2n+2} = 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(-Qy^{2n-2}P^{-4n+1})P^{2n+1} + {}^{2n+1}C_0P^3y^{2n-2} + {}^{2n+1}C_2Py^{2n} + \cdots + {}^{2n+1}C_{2n}P^{-2n+3}y^{4n-2} = 0.
\end{array}$$

If we eliminate the $2n$ functions P^{2n+1}, $P^{2n-1}y^2, \cdots \cdots P^{-2n+3}y^{4n-2}$ from these equations we obtain an identity in P, y which can be written as

$$\Delta_1 = \begin{vmatrix} {}^{2n+1}C_1 - Ry^{-1}P^{-2n} & {}^{2n+1}C_3 & {}^{2n+1}C_5 \cdots {}^{2n+1}C_{2n+1} \cdots & 0 \\ {}^{2n+1}C_0 - QP^{-2n-1} & {}^{2n+1}C_2 & {}^{2n+1}C_4 \cdots {}^{2n+1}C_{2n} \cdots & 0 \\ -RyP^{-2n-2} & {}^{2n+1}C_1 & {}^{2n+1}C_3 \cdots {}^{2n+1}C_{2n-1} \cdots & 0 \\ -Qy^2P^{-2n-3} & {}^{2n+1}C_0 & {}^{2n+1}C_2 \cdots {}^{2n+1}C_{2n-2} \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ -Qy^{2n-2}P^{-4n+1} & 0 & 0 \cdots {}^{2n+1}C_2 \cdots & {}^{2n+1}C_{2n} \end{vmatrix} \equiv 0$$

where R, Q are defined as in (1), (2). Expanding by the first column, we obtain

$$(3) \quad N_{2n} = y^{-1}P^{1-4n} \begin{vmatrix} RP^{2n-1} & {}^{2n+1}C_3 & {}^{2n+1}C_5 \cdots {}^{2n+1}C_{2n+1} \cdots & 0 \\ QP^{2n-2}y & {}^{2n+1}C_2 & {}^{2n+1}C_4 \cdots {}^{2n+1}C_{2n} \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ Qy^{2n-1} & 0 & 0 \cdots {}^{2n+1}C_2 \cdots & {}^{2n+1}C_{2n} \end{vmatrix}$$

$$= y^{-1}P^{1-4n}\Delta_2,$$

where N_{2n} is Nesbitt's determinant of $2n$ rows.

This result is true for any values of P, y not equal to 0. Write $P=y$, then

$$A^{2n+1} = 2^{2n+1}y^{2n+1}, \qquad B^{2n+1} = 0$$

and

$$R = Q = 2^{2n}y^{2n+1}.$$

With these substitutions each element of the first column of Δ_2 is $2^{2n}y^{4n}$. The other elements in Δ_2 are all ${}^{2n+1}C_{2s-r}$ and can be replaced by ${}^{2n}C_{2s-r} + {}^{2n}C_{2s-r-1}$.

If we denote rows by r and columns by c, and perform the successive operations on Δ_2 (as now re-arranged),

(1) remove the factor $2^{2n}y^{4n}$ from the first column

(2) $r_s - r_{s-1}$ for $s = 2n, \cdots 2$

(3) $c_s + c_{s+1}$ for $s = 2n-1, 2n-2, \cdots 2$

(4) expand by the first column,

we obtain

$$N_{2n} = 2^{2n}N_{2n-1}.$$

Similarly it can be shown that

$$N_{2n-1} = 2^{2n-1}N_{2n-2}.$$

Hence

$$N_n = 2^nN_{n-1} = 2^pN_2 \qquad \text{where} \qquad p = \sum_3^n r$$

But

$$N_2 = 2^3 = 2^{1+2}.$$

Hence

$$N_n = 2^q \quad \text{where} \quad q = \sum_1^n r = \tfrac{1}{2}\, n(n+1).$$

Proof II.

I am indebted to Dr. T. W. Chaundy for suggesting that this result can also be deduced from Sylvester's eliminant E, as follows:—We can show* that if

$$A(x) \equiv a_0 x^m + a_1 x^{m-1} + \cdots + a_m$$
$$B(x) \equiv b_0 x^n + b_1 x^{n-1} + \cdots + b_n,$$

then

$$E = \begin{vmatrix} a_0 & a_1 \cdots a_m & 0 & \cdots 0 \\ 0 & a_0 \cdots a_{m-1} & a_m & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_m \\ b_0 & b_1 \cdots \cdots b_n & & \cdots 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & b_n \end{vmatrix} = a_0^n \prod_{r=1}^{m} B(\alpha_r)$$

where $\alpha_1, \alpha_2, \cdots \alpha_m$ are the zeros of $A(x)$.

To prove the required result we evaluate E when $A(x)$, $B(x)$ are particular polynomials of the form $(y+1)^n \pm (y-1)^n$, and show that E is numerically equal to N. We have to consider the odd and even cases separately.

Let

$$A(x) = \frac{(\sqrt{x}+1)^{2n+1} + (\sqrt{x}-1)^{2n+1}}{2\sqrt{x}},$$
$$B(x) = \frac{(\sqrt{x}+1)^{2n+1} - (\sqrt{x}-1)^{2n+1}}{2}.$$

Then if

$$\theta^{2n+1} = 1 \quad (\theta \neq 1), \qquad \frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1} = -\theta$$

gives the zeros of $A(x)$. Then

$$B(\alpha) = \frac{2^{2n+1}}{(\theta+1)^{2n+1}}.$$

But θ, θ^{-1} give the same α. Thus taking all the $2n$ complex θ, we obtain E^2 and not E.

* This is a simple extension of the proof given for the case $m=3$, $n=2$, by Professor E. T. Whittaker, Proc. Edinburgh Math. Soc. Series 1, Vol. 40 (1922), pp. 62–63, and quoted by Professor H. W. Turnbull, Theory of Equations, Oliver and Boyd, 1939, pp. 141–142.

i.e. $$E^2 = \prod_\theta \frac{2^{2n+1}}{(\theta+1)^{2n+1}} = 2^{2n(2n+1)}, \quad \text{since} \quad \prod_\theta (\theta+1) = 1.$$

With these values of $A(x)$, $B(x)$, $E=(-1)^{\frac{1}{2}n(n+1)}N_{2n}$ and hence $N_n = \pm 2^{\frac{1}{2}n(n+1)}$ when n is even.

To determine the sign, since the values of θ which give just n roots α are

$$\theta_r = \exp\left(\frac{2ri\pi}{2n+1}\right), \qquad r = 1, 2, \cdots, n,$$

$$B(\alpha_r) = \frac{2^{2n+1}}{(e^{2i\phi}+1)^{2n+1}} \qquad \text{where} \quad \phi = \frac{r\pi}{2n+1}$$

$$= \frac{1}{(e^{i\phi}\cos\phi)^{2n+1}}.$$

But $0<\phi<\frac{1}{2}\pi$ so $\cos\phi>0$, and the sign of $B(\alpha_r)$ is the sign of $\exp(-ir\pi)$, *i.e.* $(-1)^r$. Hence the sign of $\pi B(\alpha_r)$ is $(-1)^{\frac{1}{2}n(n+1)}$ and $N_n = 2^{\frac{1}{2}n(n+1)}$, when n is even. The case when n is odd is similar. Let

$$A(x) = \frac{(\sqrt{x}+1)^{2n} + (\sqrt{x}-1)^{2n}}{2}, \qquad B(x) = \frac{(\sqrt{x}+1)^{2n} - (\sqrt{x}-1)^{2n}}{2\sqrt{x}},$$

then

$$\frac{\sqrt{\alpha}+1}{\sqrt{\alpha}-1} = \theta, \qquad \text{where} \quad \theta^{2n} = -1$$

and

$$B(\alpha) = \frac{-2^{2n}}{(\theta+1)(\theta-1)^{2n-1}}.$$

This gives $E^2 = 2^{2n(2n-1)}$, since $\Pi_\theta(\theta-1) = \Pi_\theta(\theta+1) = 2$. Also $E=(-1)^{\frac{1}{2}n(n-1)}N_{2n-1}$, as $B(x)$ is only of degree $n-1$ and

$$B(\alpha_r) = \frac{-2^{2n}}{(e^{2i\phi}+1)(e^{2i\phi}-1)^{2n-1}}, \quad 2\phi = \frac{(2r+1)\pi}{2n}, \quad r = 0, 1, \cdots, n-1$$

$$= \frac{-2^{2n}}{(e^{i\phi})^{2n}(2i\sin\phi)^{2n-1}2\cos\phi}.$$

Hence the sign of $\Pi B(\alpha_r)$ is the same as that of $\Pi\{-e^{-(r+n)i\pi}\}$ and

$$E = (-1)^{\frac{1}{2}n(n-1)}N_{2n-1} = (-1)^{n^2+n+\frac{1}{2}n(n-1)}2^{n(2n-1)}$$

i.e. $N_n = 2^{\frac{1}{2}n(n+1)}$, when n is odd, since $n(n+1)$ is even.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

RECIPROCAL QUADRATIC EQUATIONS

M. R. SPIEGEL, Rensselaer Polytechnic Institute

A reciprocal polynomial equation of order n given by $f(x)=0$ is one for which $f(x)=\pm x^n f(1/x)$. Thus if the polynomial equation is

$$f(x) = \sum_{k=0}^{n} a_k x^k \qquad a_n \neq 0 \tag{1}$$

it follows that $f(x)=0$ is a reciprocal equation if and only if

$$a_r = a_{n-r} \quad \text{or} \quad a_r = -a_{n-r}. \tag{2}$$

It is well-known that such equations may be reduced to equations of lower degree by the substitution

$$u = x + \frac{1}{x}. \tag{3}$$

Thus for example, the reciprocal equation

$$2x^4 - 5x^3 + 6x^2 - 5x + 2 = 0 \tag{4}$$

may be written as

$$2\left(x^2 + \frac{1}{x^2}\right) - 5\left(x + \frac{1}{x}\right) + 6 = 0, \tag{5}$$

which in turn may be reduced by the substitution (3) to the quadratic

$$2u^2 - 5u + 2 = 0. \tag{6}$$

Upon solving the quadratic (6) and using (3) the equation (4) may be solved completely.

The reciprocal equation of least order (other than the monomial equation $x \pm 1 = 0$) is a quadratic of the form

$$x^2 - rx + 1 = 0. \tag{7}$$

It is not surprising that no one (at least so far as the author knows) has applied the methods of reciprocal equations to the equation (7) since this equation is solved so simply. Nevertheless it is of interest that the method with slight variation applies.

We write (7) in the form

$$x + \frac{1}{x} = r. \tag{8}$$

Adding 2 and subtracting 2 from both sides of the equation (8) we have

$$\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 = r + 2 \tag{9}$$

and

$$\left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 = r - 2. \tag{10}$$

Hence

$$\sqrt{x} + \frac{1}{\sqrt{x}} = \pm\sqrt{r + 2} \tag{11}$$

and

$$\sqrt{x} - \frac{1}{\sqrt{x}} = \pm\sqrt{r - 2}. \tag{12}$$

Multiplication of (11) and (12) yields

$$x - \frac{1}{x} = \pm\sqrt{r^2 - 4}. \tag{13}$$

Adding (8) and (13) we find

$$x = \frac{r \pm \sqrt{r^2 - 4}}{2} \tag{14}$$

as we should.

It is interesting to note that any quadratic equation can be transformed into a reciprocal quadratic equation. The general quadratic equation is given by

$$ax^2 + bx + c = 0. \tag{15}$$

Letting $x = \sqrt{(c/a)}u$ this equation becomes

$$cu^2 + b\sqrt{\frac{c}{a}}u + c = 0$$

or

$$u^2 + \frac{b}{\sqrt{ac}}u + 1 = 0,$$

a reciprocal quadratic equation of the form (7) with $r = -(b/\sqrt{ac})$. Hence by (14),

$$u = \frac{-b \pm \sqrt{b^2 - 4ac}}{2\sqrt{ac}}, \tag{16}$$

and since $x = \sqrt{(c/a)}u$ we have finally for the solutions of (15),

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{17}$$

as we should.

A THEOREM ON EVOLUTES

FURIO ALBERTI, University of Illinois

In this note is given what seems to be a new theorem on the notion of generating evolutes of a given twisted curve. We shall prove the following:*

THEOREM. *If a circle of second order contact at P with a twisted curve C makes a variable angle $\omega = \int_{s_0}^{s}(ds/\tau)$ with the osculating plane of C at P, as P moves along C, then the locus of the centers of these circles is an evolute.*

Proof. Consider the circle represented by $\xi^2 + (\eta - r)^2 = r^2$ and $\zeta = 0$. Rotating the axes about the ξ-axis through an angle ω, we obtain $\xi^2 + (\eta \cos \omega + \zeta \sin \omega)^2 - 2r(\eta \cos \omega + \zeta \sin \omega) = 0$. Substituting $\xi = s - \cdots$, $\eta = (s^2/2\rho) + \cdots$, $\zeta = -(s^3/6\rho\tau) - \cdots$ and demanding second order contact, we find that $r = \rho/\cos \omega$. Hence we find that the coordinates of the center of the circle are $(0, \rho, -\rho \tan \omega)$. But these are the coordinates of the points on the evolute† of the curve C, and the theorem is proved.

This theorem is, in a way, an extension of the theorem in plane geometry concerning the locus of the centers of the osculating circles to a given curve. It is also of interest to note that the theorem of this note implies that the tangents of any one evolute and the principal normal of the curve C form the angle ω, since the radius of the circle of second order contact must also be perpendicular to C and pass through P. Also note, as a corollary to the theorem, that if C is a plane curve, $1/\tau = 0$, then $\omega =$ constant; hence we obtain the twisted evolutes of the plane curve C by means of the circle of second order contact moving at a constant angle to the plane of C.

* The notation and methods follow those given in E. P. Lane's Metric Differential Geometry of Curves and Surfaces, University of Chicago Press, 1940, Chapter II.

† *Ibid.*, p. 52.

INVERSE MATRICES IN INTERPOLATION

J. P. BALLANTINE, University of Washington

1. Introduction. The standard problem of polynomial interpolation is to pass a curve, $y=P_n(x)$ through $n+1$ points. We shall take the case when $n=3$, but the reader will soon see that the method holds for any value of n. To pass a curve

$$y = A_0 + A_1x + A_2x^2 + A_3x^3 \tag{1}$$

through the four points (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), it becomes necessary to solve the system of equations:

$$\begin{aligned} A_0 + A_1x_1 + A_2x_1^2 + A_3x_1^3 &= y_1, \\ A_0 + A_1x_2 + A_2x_2^2 + A_3x_2^3 &= y_2, \\ A_0 + A_1x_3 + A_2x_3^2 + A_3x_3^3 &= y_3, \\ A_0 + A_1x_4 + A_2x_4^2 + A_3x_4^3 &= y_4. \end{aligned} \tag{2}$$

The usual procedure is to establish the non-vanishing of the determinant of the matrix of coefficients:

$$X = \begin{Vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{Vmatrix}. \tag{3}$$

Then some method is given for determining the parameters A_0, A_1, A_2, A_3. Both steps can be accomplished at once by finding the inverse of the matrix X.

It is the purpose of this note to show that the desired inverse can be written by inspection.

2. The inverse. First, one new notation will be introduced.

$$(a; b, c, d, e) \quad \text{means} \quad (a-b)(a-c)(a-d)(a-e).$$

Use will be made of the well known identity:

$$x^3 - (r_1 + r_2 + r_3)x^2 + (r_2r_3 + r_3r_1 + r_1r_2)x - r_1r_2r_3 = (x; r_1, r_2, r_3). \tag{4}$$

Let any row of the matrix X, namely $(1, x_i, x_i^2, x_i^3)$, be multiplied (*i.e.* by the vector dot product) by the vector

$$-x_2x_3x_4, \qquad x_3x_4 + x_4x_2 + x_2x_3, \qquad -(x_2 + x_3 + x_4), \qquad 1.$$

The result, in view of equation (4), is simply $(x_i; x_2, x_3, x_4)$. This will vanish for $i=2$, 3, and 4, but not for $i=1$ unless one of the numbers x_2, x_3, or x_4 equals x_1.

In this manner, it is apparent that $XY=Z$, where

$$Y=\begin{Vmatrix} -(x_2x_3x_4) & -(x_3x_4x_1) & -(x_4x_1x_2) & -(x_1x_2x_3) \\ (x_3x_4+x_4x_2+x_2x_3) & (x_4x_1+x_1x_3+x_3x_4) & (x_1x_2+x_2x_4+x_4x_1) & (x_2x_3+x_3x_1+x_1x_2) \\ -(x_2+x_3+x_4) & -(x_3+x_4+x_1) & -(x_4+x_1+x_2) & -(x_1+x_2+x_3) \\ 1 & 1 & 1 & 1 \end{Vmatrix}$$

and

$$Z=\begin{Vmatrix} (x_1;\, x_2,\, x_3,\, x_4) & 0 & 0 & 0 \\ 0 & (x_2;\, x_3,\, x_4,\, x_1) & 0 & 0 \\ 0 & 0 & (x_3;\, x_4,\, x_1,\, x_2) & 0 \\ 0 & 0 & 0 & (x_4;\, x_1,\, x_2,\, x_3) \end{Vmatrix}.$$

Y requires only a little modification to be converted into the inverse of X. Its first column must be divided by the first element of Z, namely $(x_1;\, x_2,\, x_3,\, x_4)$, which is short for $(x_1-x_2)(x_1-x_3)(x_1-x_4)$, its second column is divided by $(x_2;\, x_3,\, x_4,\, x_1)$, and so forth.

The inverse of X is now in form to apply to the solution of the system of equations (2), and immediately gives the required formula for $P_n(x)$. It is interesting to note that when the inverse was obtained, it was considered as a right inverse, but when it is used, it is a left inverse. The algebra to show that it is a left inverse is not so simple, so it is fortunate that we have the theorem that a right inverse is also a left inverse.

3. Applications. When $P_n(x)$ is written as a linear combination of y_1, y_2, y_3, and y_4, the coefficients of the latter are called Lagrange coefficients. It is easily seen that the Lagrange coefficients are precisely the columns of the inverse of X.

Also with the inverse of X in mind, the coefficients of any desired integration formula can be readily obtained. Suppose the desired formula is:

$$\int_a^b f(x)dx = A_1f(x_1) + A_2f(x_2) + A_3f(x_3) + A_4f(x_4).$$

The values of a, b, x_1, x_2, x_3, and x_4 are assigned. The formula is useless until the coefficients, A_1, A_2, A_3, and A_4 are known. To find the coefficients, note that the formula must integrate correctly the functions $f(x)=1$, x, x^2, and x^3. Then

$$\begin{aligned} A_1 + A_2 + A_3 + A_4 &= (b-a), \\ A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 &= \tfrac{1}{2}(b^2-a^2), \\ A_1x_1^2 + A_2x_2^2 + A_3x_3^2 + A_4x_4^2 &= \tfrac{1}{3}(b^3-a^3), \\ A_1x_1^3 + A_2x_2^3 + A_3x_3^3 + A_4x_4^3 &= \tfrac{1}{4}(b^4-a^4). \end{aligned}$$

Here the matrix of coefficients of the unknowns is the transpose of X, so its inverse is the transpose of the inverse of X. Hence A_1 is the product of the first column of the inverse of X by the right members of the above system, namely

$$A_1 = [-(x_2x_3x_4)(b-a) + (x_3x_4 + x_4x_2 + x_2x_3)\tfrac{1}{2}(b^2 - a^2) - (x_2 + x_3 + x_4)\tfrac{1}{3}(b^3 - a^3) + \tfrac{1}{4}(b^4 - a^4)] \div (x_1; x_2, x_3, x_4).$$

As x_1 is any of the assigned values of x, the same formula gives all the coefficients.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1006. *Proposed by Leo Moser, Texas Technological College*

A wholesale jobber received radios from the manufacturer packed in two types of cartons. Each type held a different number of radios. The shipping clerk found that by judiciously selecting the right number of one or both kinds, he could fill almost any size order without opening a carton. Indeed, there were exactly six orders possible which would require a carton to be opened and repackaged.

One day the manufacturer discontinued the smaller carton and sent a new size, containing a different number of radios. The shipping clerk calculated that with the new set up there were ten different orders which would be impossible to fill without repacking. How many radios were contained in this new carton?

E 1007. *Proposed by Harvey Berry, University of Kentucky*

Prove that the G.C.D. of two numbers is equal to the G.C.D. of their sum and their L.C.M.

E 1008. *Proposed by R. H. Moorman, Tennessee Polytechnic Institute*

Through a given point draw a line cutting the sides of a given angle so as to form a triangle of given perimeter.

E 1009. *Proposed by Arthur Danzl, St. John's University*

Rectify the loop of the folium of Descartes by means of elementary functions either finitely, if possible, or approximately.

E 1010. *Proposed by Michael Goldberg, Washington, D. C.*

Given n equal unit squares, dissect each of these in exactly the same way with straight cuts into $p(n)$ parts such that the n $p(n)$ pieces may be assembled to form a square of edge $\sqrt{n}$. Investigate $k(n)$, the minimal values of $p(n)$, and show that $k(n) \leqq 5$ for all n.

SOLUTIONS

The Wine Cask Problem

E 962 [1951, 260]. *Proposed by B. H. Brown, Dartmouth College*

A cask of unit volume is full of wine. A man withdraws from the cask an amount $a(0<a<1)$, and then adds amount a of *water*, which is assumed to mix perfectly with the wine. He continues this process b times. He then withdraws amount a and adds amount a of *wine*, continuing this process b times.

(1) Derive a general formula for the amount of wine in the cask after the $2b$ operations.

(2) If $1/a=A$ is an integer, and if B is that corresponding (integral) value of b for which the eventual wine-content is a minimum, so determine A and B that the wine-content is (a) the minimum minimorum (b) the maximum minimorum.

(3) If $A=51$, find B and the wine-content.

(4) Find $\lim_{A\to\infty}(B/A)$.

Solution by A. H. Payne, University of North Carolina. It follows from an induction on b that

(1) *The wine-content, W, after $2b$ operations, is given by*

$$W = 1 - (1 - a)^b + (1 - a)^{2b},$$

where use is made of the fact that after b operations (of removing wine) the wine-content is equal to $(1-a)^b$. This last relation itself follows from an easy induction.

Now

$$\partial W/\partial b = (1 - a)[2(1 - a)^b - 1] \log (1 - a).$$

Since $0<a<1$, a zero of $\partial W/\partial b$ requires $(1-a)^b=1/2$. If b is considered a real continuous variable this last equation determines, for each a (or A) the b which enables W to attain its minimum value of $3/4$. Denote this critical value of b by $b(A)$. Then

$$b(A) = -\frac{\log 2}{\log (1 - a)} = \frac{\log 2}{\log A - \log (A - 1)}.$$

But only for $A=2$ is $b(A)$ an integer. Therefore, for integral b,

(2a) *The minimum minimorum of wine-content is* 3/4 *and occurs at* $(A, B)=(2, 1)$.

Since W decreases with increasing b for $0<b<b(A)$, and increases for increasing $b>b(A)$, B must be one of the integers adjacent to $b(A)$. Set $p=b(A)-[b(A)]$. Then the integers

$$B' = b(A) - p \quad \text{and} \quad B' + 1$$

are the two possibilities for B. Now denote the wine-content at (A, b) by $W(A, b)$, and let $D(A, b)$ denote the difference between the wine-content at b, for fixed A, and the attainable (for real b) minimum of 3/4. That is, $D(A, b)=W(A, b)-3/4$. In this notation

$$W(A, B') = 1 - (1 - a)^{b(A)-p} + (1 - a)^{2b(A)-2p}, \qquad (0 < p < 1)$$

and, since $(1-a)^{b(A)}=1/2$,

$$D(A, B') = 1/4 - (1/2)(1 - a)^{-p} + (1/4)(1 - a)^{-2p}.$$

Similarly,

$$D(A, B' + 1) = 1/4 - (1/2)(1 - a)^{1-p} + (1/4)(1 - a)^{2-2p}.$$

Now $D(A)=\min\{D(A, B'), D(A, B'+1)\}$ is the difference between the minimum obtainable with integral b (*i.e.* B) and 3/4, obtainable with real b, for a given A.

(2b) *The maximum minimorum of wine-content is* 0.75390625 *and occurs at* $(A, B)=(4, 2)$.

Proof: By calculation, $D(4)=1/256$. It will be shown that for $A\neq 4$, $D(A)<1/256$. This is accomplished by showing that, for $A\neq 4$, both

$$D(A, B') - 1/256 < 0 \quad \text{and} \quad D(A, B' + 1) - 1/256 < 0.$$

In the first case we wish to show that

$$D(A, B') - 1/256 = 63/256 - (1/2)(1 - a)^{-p} + (1/4)(1 - a)^{-2p} < 0.$$

Denote by $U(p)$ the result of multiplying the left member by $256(1-a)^{2p}$. Then

$$U(p) = 63(1 - a)^{2p} - 128(1 - a)^{p} + 64,$$

and it is sufficient to show that $U(p)<0$. This follows at once from: (i) $U(0)=-1$; (ii) $U(1)<0$ if $a<1/9$; (iii) $dU/dp>0$ (hence $U(p)$ is increasing as p increases from 0 to 1). This means that

$$D(A, B') < 1/256 \quad \text{if} \quad A > 9.$$

Similarly

$$D(A, B' + 1) < 1/256 \quad \text{if} \quad A > 8.$$

In order to complete the proof of (2*b*) it is necessary only to test $D(A)$ by numerical calculation for $A = 3, 5, 6, 7, 8, 9$.

Since $D(A, B') \leqq D(A, B'+1)$ implies

$$p = b(A) - [b(A)] \leqq \frac{\log (2 - a) - \log 2}{\log (1 - a)},$$

it follows that

$$[b(A)] \geqq \frac{\log (2A - 1) - \log A}{\log A - \log (A - 1)}.$$

Therefore B can be determined as follows:

$$B = B', \qquad B = \text{either } B' \text{ or } B' + 1, \qquad B = B' + 1,$$

according as

$$[b(A)] \gtreqless \frac{\log (2A - 1) - \log A}{\log A - \log (A - 1)}.$$

(3) *If* $A = 51$, *the above criterion gives* $B = 35$; *and, therefore,*

$$W(51, 35) = 1 - (50/51)^{35} + (50/51)^{70} \doteq 0.750000000762.$$

(4) *Finally,*

$$\lim_{A\to\infty} (B/A) = \lim_{A\to\infty} (b(A)/A) = \lim_{A\to\infty} \frac{(\log 2)/A}{\log A - \log (A - 1)} = \log 2.$$

Also solved by A. L. Epstein, Prasert Na Nagara, W. O. Pennell, and the proposer.

A Minimum Right Triangle

E 976 [1951, 491]. *Proposed by R. D. Stalley, University of Arizona*

What is the least area of a right triangle which can be placed on a rectangular coordinate system so that no side is parallel to a coordinate axis and so that the coordinates of its vertices and the lengths of its sides are integral?

Solution by R. Z. Vause, Jr., Clemson Agricultural College. Let a and b denote the legs of the triangle and c the hypotenuse. Let a_x, a_y and b_x, b_y denote the projections of a and b on the coordinate axes. Then

$$a_y/b_x = a_x/b_y = A/B, \qquad (A, B) = 1,$$

and there exist non-zero m and n such that

$$a_y = mA, \qquad b_x = mB, \qquad a_x = nA, \qquad b_y = nB.$$

The area of the triangle is given by

$$K = AB(m^2 + n^2)/2.$$

Since

$$a^2 = A^2(m^2 + n^2), \qquad c^2 = (A^2 + B^2)(m^2 + n^2),$$

it follows that A, B, m, n must be chosen so that m^2+n^2 and A^2+B^2 are both squares. There is no solution smaller than $A=m=3$, $B=n=4$, whence $K=150$. Such a triangle is the one having its vertices at (0, 9), (12, 0), (−12, −7).

Also solved by Leon Bankoff, J. H. Braun, F. F. Dorsey, R. T. Hood, Sidney Kravitz, Josef Langr, Prasert Na Nagara, Leola Odland, C. S. Ogilvy, Elijah Swift, G. W. Walker, and the proposer.

A Family of Ellipses

E 977 [1951, 491]. *Proposed by Jose Gallego-Diaz, Madrid, Spain*

Let OA be a radius of a circle with center O. Produce OA its own length to B. Through A draw a horizontal line and on it mark off, in each direction from A, $AM=AN=OA$. If OB and MN are taken as the equal conjugate diameters of an ellipse, find the loci of the vertices and the foci of this ellipse as the radius OA varies. Also find the envelope of the major axis of the ellipse as OA varies.

Solution by Roscoe Woods, State University of Iowa. Without loss of generality, let the fixed circle be taken as the unit circle. Let the point A be given by the coordinates (cos 2θ, sin 2θ). The equations of the bisectors of the angles formed by the lines OB and MN are

$$x \sin\theta - y\cos\theta + \sin\theta = 0 \tag{1}$$

and

$$x\cos\theta + y\sin\theta - \cos\theta = 0. \tag{2}$$

The major axis of the ellipse through O, B, M, N which has OB and MN as equal conjugate diameters will lie on line (1) when A is any point on the unit circle in the first and fourth quadrants, and on line (2) when A lies in the second and third quadrants. The lengths of the semimajor and semiminor axes of the ellipse are readily found to be $\sqrt{2}\cos\theta$ and $\sqrt{2}\sin\theta$ respectively. The equation of the ellipse is then readily found to be

$$\frac{(x\cos\theta + y\sin\theta - \cos\theta)^2}{2\cos^2\theta} + \frac{(x\sin\theta - y\cos\theta + \sin\theta)^2}{2\sin^2\theta} = 1. \tag{3}$$

It should be pointed out that the locus of (3) is a circle when $\theta=45°$ or 135°, and a degenerate or flat ellipse when $\theta=0°$ or 90°.

If the parameter θ be eliminated between (1) and (3), and between (2) and (3), the equations of the loci of the extremities of the axes are readily found to be the four circles

$$x^2 + y^2 \pm \sqrt{2}x \pm \sqrt{2} - 1 = 0.$$

The foci of the system of ellipses in (3) are the intersections of line (1) and the circle

$$(x - \cos 2\theta)^2 + (y - \sin 2\theta)^2 = 2 \cos 2\theta$$

when A lies in the first and fourth quadrants. When A lies in the second and third quadrants, the foci of the system are the intersections of line (2) and the circle

$$(x - \cos 2\theta)^2 + (y - \sin 2\theta)^2 = -2 \cos 2\theta.$$

The result of eliminating θ between the two pairs of equations gives, as the loci of the foci, the two circles

$$x^2 + y^2 \mp 2x - 1 = 0.$$

Obviously the envelopes of the lines on which the major axes of the ellipses lie are the two points $(\pm 1, 0)$.

Also solved by Pompeys Alonso.

A Lemma Used by Hansraj Gupta

E 978 [1951, 491]. *Proposed by J. M. Kingston, University of Washington*

If $[x]$ denotes the integral part of x and if j is any positive integer, show that

$$\sum_{i=0}^{[j/3]} [(j - 3i)/2] = [(j^2 + 2j + 4)/12].$$

Solution by S. T. Thompson, Plattsburg, N. Y. Set

$$F(j) = \sum_{i=0}^{[j/3]} [(j - 3i)/2] - [(j^2 + 2j + 4)/12].$$

One easily verifies that $F(j) = 0$ for $j = 0, 1, 2, 3, 4, 5$. Now suppose $F(n) = 0$. Then

$$\begin{aligned} F(n+6) &= \sum_{i=0}^{[n/3]+2} [(n + 6 - 3i)/2] - [\{(n+6)^2 + 2(n+6) + 4\}/12] \\ &= \sum_{i=-2}^{[n/3]} [(n - 3i)/2] - [(n^2 + 2n + 4)/12] - n - 4 \\ &= 4 + [n/2] + [(n+1)/2] + \sum_{i=0}^{[n/3]} [(n - 3i)/2] \\ &\quad - [(n^2 + 2n + 4)/12] - n - 4 = 0. \end{aligned}$$

This proves the theorem.

Also solved by J. H. Braun, C. V. Fronabarger, Sidney Kravitz, Prasert Na Nagara, C. F. Pinzka, R. Z. Vause, Jr., W. D. Wood, and the proposer.

Scoring a Test

E 979 [1951, 491]. *Proposed by E. S. Keeping, University of Alberta*

A student is asked on a test to match n given historical events against n different given dates, assigning a date to each event. He is sure of k dates, but is completely ignorant of the others and assigns them at random. Show that a fair method of scoring would be to deduct 1 from the number of correct matches.

Solution by Frank Herlihy, Comstock, Michigan. Let $n-k=r$ be the number of dates to be assigned at random. Let the score of the guesses be

$$s = x_1 + x_2 + \cdots + x_r,$$

where each x_i has the value 0 if it is not a coincidence and 1 if it is. The mathematical expectation of the score will then be

$$E(s) = E(x_1) + E(x_2) + \cdots + E(x_r).$$

The probability of a coincidence in the ith place will be $1/n$, whence $E(x_i) = (1/n)1 = 1/n$. Therefore $E(s) = (1/n)n = 1$. Thus the mathematical expectation gives us an extra score of 1 point from the random guesses. If the student knew all but one coincidence he would be graded correctly, but if he knew all he would be cheated of one point.

This solution is shown on page 169 of Uspensky's *Introduction to Mathematical Probability.*

Also solved by J. H. Braun, F. F. Dorsey, T. L. Jordan, Jr., C. F. Pinzka, W. R. Van Voorhis, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey.

PROBLEMS FOR SOLUTION

4474 [1952, 109]. *Correction.*

Equation (3) should be changed to read:

$$|Z_1 Z_2 \cdots Z_k|^2 \leqq \tfrac{1}{2}(B^2 + \sqrt{B^4 - 4|a_0|^2}), \quad (1 \leqq k \leqq n). \tag{3}$$

4478. *Proposed by Albert Wilansky, Lehigh University*

A basis in a Banach space B is a set of elements $\{x_n\}$ such that every element of the space is a unique, finite or infinite, linear combination of elements of the set.

(1) Prove that the x_n are isolated.

(2) If $\sum x_n$ is an element of B, show that there is a divergent sequence $\{s_n\}$ of numbers such that $\sum s_n x_n$ is an element of B.

4479. *Proposed by D. J. Newman, Harvard University*

Suppose that all the series

$$\sum_{n=0}^{\infty} \alpha_n, \quad \sum_{n=0}^{\infty} \alpha_n^2, \quad \sum_{n=0}^{\infty} \alpha_n^3, \cdots$$

converge to zero, where the α_n are complex numbers. Does it follow that all the $\alpha_n = 0$?

4480. *Proposed by Rufus Isaacs, the Rand Corporation, Santa Monica, California*

Given a closed solid polyhedron with faces which are equilateral triangles, we can construct another by the following process. Join the mid-edge points of each face, dividing it into four triangles. On each face place externally a regular tetrahedron having the central triangle for a base.

If we start with a regular tetrahedron and repeat this process, show that the limiting figure is a cube less a certain point set which is non-enumerable, yet of measure zero.

4481. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a system of numeration with base B, there are n one-digit numbers less than B whose cubes have $B-1$ as the unit's digit. Determine the relation between B and n.

4482. *Proposed by Harold Shapiro, Massachusetts Institute of Technology*

Let $n_1 < n_2 < \cdots$ be any sequence of positive integers (finite or infinite), and let $T_{n_1 n_2 \ldots} x$ denote a transformation which alters precisely the n_1, n_2, $\cdots$ digits in x, where $x (0 < x < 1)$ is expanded in dyadic notation. Show that $T_{n_1 n_2 \ldots}$ is measure-preserving on subsets of (0, 1).

SOLUTIONS

Generalization of Euler's Differential Equation

4406 [1950, 490]. *Proposed by H. J. Zimmerberg, Rutgers University*

Show that the most general linear differential equation

$$\sum_{i=0}^{n} a_i [p(x)]^i D^i y = f(x),$$

where $p(x)$ is of class C' and the a_i are constants with $a_n \neq 0$, which is reducible to the case of constant coefficients by a transformation of the independent variable is the generalized form of Euler's (sometimes called Cauchy's) equation, wherein $p(x)$ is linear in x.

Solution by the Proposer. Consider the change of independent variable $x=\phi(u)$, where ϕ is of class $C^{(n)}$ and $\phi'(u)\neq 0$. If $D^i\equiv d^i/dx^i$ and $\mathcal{D}^i\equiv d^i/du^i$, it then follows at once by mathematical induction that

$$D^r = (1/\phi')^r(\mathcal{D} - (r-1)\phi''/\phi')(\mathcal{D} - (r-2)\phi''/\phi') \cdots (\mathcal{D} - \phi''/\phi')\mathcal{D}$$

for arbitrary positive integers r. Substituting and equating the coefficients of $\mathcal{D}^n$ and $\mathcal{D}^{n-1}$ to constants, we obtain

$$\frac{a_n p^n[\phi(u)]}{\phi'^n} \equiv c_n \neq 0, \tag{1}$$

$$\frac{a_{n-1}p^{n-1}[\phi(u)]}{\phi'^{n-1}} - \frac{n(n-1)}{2}\frac{a_n p^n[\phi(u)]\phi''}{\phi'^{n+1}} \equiv c_{n-1}. \tag{2}$$

As $\phi'(u)=(a_n/c_n)^{1/n}p[\phi(u)]$ from (1), relation (2) reduces to $\phi''/\phi'=$constant. However, $\phi''(u)=(a_n/c_n)^{1/n}((d/dx)p[\phi(u)])\phi'(u)$ and, therefore, $p'(x)=$constant. Hence $p(x)$ is linear in x.

Also solved by M. S. Klamkin.

Conics Circumscribed about the Faces of a Tetrahedron

4407 [1950, 490]. *Proposed by N. A. Court, University of Oklahoma*

If four conics are circumscribed about the faces of a tetrahedron (T) in such a way that at each vertex of (T) the three tangents to the three conics passing through that vertex are coplanar, then the four conics lie on a quadric surface.

Solution by the Proposer. Let (q_a), (q_b), (q_c), (q_d) be four conics circumscribed about the faces BCD, CDA, DAB, ABC of $(T)\equiv ABCD$ so that the triads of tangents to these conics at the points A, B, C, D lie in the planes α, β, γ, δ, respectively. The three tangents to (q_d) at the vertices A, B, C of triangle ABC meet the respectively opposite sides BC, CA, AB in the three points P, Q, R which are collinear. (The line PQR may be called the Lemoine axis of the triangle ABC for the conic considered.) The tangent AP lies in the plane α, whence P lies on the line of intersection (α, DBC) of the two planes α and DBC. Similarly for the points Q, R. Thus the lines of intersection of the three pairs of planes α, DBC; β, DCA; γ, DAB are met by the line PRQ which lies in the plane ABC and therefore also meets the line (δ, ABC).

Considering the other faces of (T) in the same manner we conclude that the lines of intersection of corresponding faces of the two tetrahedrons (T) and $\alpha\beta\gamma\delta$ are met by the Lemoine axes of the triangles BCD, CDA, DAB, ABC for the conics (q_a), $\cdots$. Hence the two tetrahedrons are skewly perspective and, therefore, a quadric surface (Q) may be inscribed in the tetrahedron $\alpha\beta\gamma\delta$ touching its faces at the points A, B, C, D, respectively (H. F. Baker, *Principles of Geometry*, Cambridge, 1923, vol. 3, p. 53, §14.)

The plane ABC cuts the quadric (Q) along a conic (p) passing through the points A, B, C and having for tangents at these points the traces of the planes α, β, γ in the plane ABC. (p) is therefore identical with (q_d), that is, (q_d) lies on the quadric (Q). Similarly for the other three conics. Hence the proposition.

Note. The four Lemoine axes considered form a hyperbolic group supplementary to the hyperbolic group formed by the lines of intersection of the corresponding faces of the two tetrahedrons. By reversing the steps of the above proof we obtain the theorem: *If four conics are circumscribed about the faces of a tetrahedron so that the Lemoine axes of the triangles for the respective conics form a hyperbolic group, the four conics lie on a quadric surface.*

Sign of $\Gamma^{(n)}(1)$

4408 [1950, 490]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute*

Prove that $\Gamma^{(n)}(1)$ is positive or negative according as the integer n is even or odd.

Solution by C. F. Pinzka, Xavier University, Cincinnati, Ohio. Defining

$$\Gamma(x) = \int_0^\infty e^{-y}y^{x-1}dy, \qquad x > 0,$$

the conditions for differentiating under the integral sign are satisfied and

$$\begin{aligned}\Gamma^{(n)}(1) &= \int_0^\infty e^{-y}(\log y)^n dy \\ &= \int_0^1 e^{-y}(\log y)^n dy + \int_0^1 (-1)^n e^{-1/y}y^{-2}(\log y)^n dy \\ &= \int_0^1 [y^2 + (-1)^n e^{y-1/y}]e^{-y}y^{-2}(\log y)^n dy.\end{aligned}$$

When n is even, the integrand is always positive, whence the integral is positive. When n is odd, the integrand has a sign opposite to that of the expression in brackets, which becomes $y^2 - e^{y-1/y}$, and which can be proved positive for $0<y<1$ as follows:

$$y^2 = e^{2\log y} = e^{2[(y-1)-(y-1)^2/2+\cdots]} > e^{-y^2+4y-3} > e^{y-1/y}.$$

Also solved by M. S. Klamkin, J. G. Millar, C. S. Ogilvy, O. E. Stanaitis, R. E. Wild, A. G. Clark's Class in Probability Theory, and the proposer.

Convergent and Divergent Series

4409 [1950, 490]. *Proposed by Fritz Herzog, Michigan State College*

Given a power series $f(x)=a_1x+a_2x^2+a_3x^3+\cdots$ with real coefficients. It

is assumed that at least one of the coefficients a_n, $n \geqq 2$, is different from zero and that the radius of convergence R is positive or infinite. Show that there exist constants u_n, $0 < u_n < R$, such that $\sum_n (-1)^n u_n$ converges but $\sum_n (-1)^n f(u_n)$ diverges.

Solution by G. G. Lorentz, University of Toronto. Let a_p be the first coefficient a_n, $n \geqq 2$, with $a_n \neq 0$. Define

$$u_{4k+j} = C_j k^{-1/p}, \qquad k = K, K+1, \cdots; j = 0, 1, 2, 3;$$
$$C_0 = 1, \quad C_1 = 2, \quad C_2 = 3, \quad C_3 = 2;$$

and K is an integer such that $K > (3/R)^p$. Then

$$\sum_{j=0}^{3} (-1)^j u_{4k+j} = k^{-1/p}(C_0 - C_1 + C_2 - C_3) = 0,$$

and therefore $\sum_n (-1)^n u_n$ converges with sum 0.

On the other hand, for small x, $f(x) = a_1 x + a_p x^p + O(x^{p+1})$, and we obtain for large k,

$$\sum_{j=0}^{3} (-1)^j f(u_{4k+j}) = a_p k^{-1}(C_0^p - C_1^p + C_2^p - C_3^p) + O(k^{-(p+1)/p}).$$

Since $C_0^p - C_1^p + C_2^p - C_3^p \geqq 1$ for $p = 2, 3, \cdots$, and the series with the term $k^{-(p+1)/p}$ is convergent, we see that $\sum_n (-1)^n f(u_n)$ behaves like the series $\sum k^{-1}$, that is, it is divergent.

Also solved by the proposer, and by another contributor who did not give his name.

Tchebychef Polynomials

4410 [1950, 564]. *Proposed by C. D. Olds, San Jose State College, California*

Show that, for every positive integer n, there is a polynomial $f(x)$ of degree n and a related polynomial $\phi(x)$ which satisfy

$$[f(x)]^2 - 1 = (x^2 - 1)[\phi(x)]^2.$$

I. *Solution by W. V. Parker, Alabama Polytechnic Institute.* Write

$$(x + \sqrt{x^2 - 1})^n \equiv f(x) + \sqrt{x^2 - 1}\,\phi(x),$$

where $f(x)$ and $\phi(x)$ are uniquely defined polynomials. Then

$$(x - \sqrt{x^2 - 1})^n \equiv f(x) - \sqrt{x^2 - 1}\,\phi(x),$$

and since the product of the two left members is 1, we have

$$1 = [f(x)]^2 - (x^2 - 1)[\phi(x)]^2 \tag{1}$$

which is the required relation. Explicit expressions for $f(x)$ and $\phi(x)$ are

$$f(x) = \frac{(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n}{2},$$

$$\phi(x) = \frac{(x + \sqrt{x^2 - 1})^n - (x - \sqrt{x^2 - 1})^n}{2\sqrt{x^2 - 1}}.$$

A polynomial $f(x)$ of degree n which satisfies (1) may be readily shown to be unique except for sign. Since $f(x)$ and $\phi(x)$ are relatively prime we obtain by differentiation that $n\phi(x) = f'(x)$. The polynomial $f(x)$ satisfies the differential equation

$$(x^2 - 1)f''(x) + xf'(x) - n^2 f(x) = 0$$

for positive integral values of n.

II. *Solution by J. B. Kelly, Institute for Advanced Study.* Set $x = \cos\theta$. (It is not necessary to restrict x by limiting θ to real values.) Put $f(x) = \cos n\theta$, $\phi(x) = \sin n\theta / \sin\theta$. It is easily established by induction that $f(x)$ and $\phi(x)$ are polynomials. Further, the proposed relation is satisfied identically.

III. *Solution by A. B. Boggs, Michigan College of Mining and Technology.* In his article, *Sur les questions de minima qui se rattachement à la représentation approximative des fonctions*, Tchebychef considers the equation

$$[f(x)]^2 - L^2 = (x^2 - h^2)[\phi(x)]^2.$$

See pp. 297–300 of *Oeuvres de Tchebychef*, v. I, 1899. The solution to the present problem is obtained at once with $L = h = 1$.

Also solved by H. L. Alder and D. A. Norton, Norman Anning, A. B. Boggs (an independent solution), D. H. Browne, L. Carlitz, S. H. Eisman, J. E. Freund, P. M. Hummel, M. S. Klamkin, Roger Lessard, G. Lumer, Fred Marer, S. T. Parker, C. F. Pinzka, G. F. Rose, R. D. Schafer, O. E. Stanaitis, Hans Stetter, Morgan Ward, and the Proposer.

Editorial Note. If f_n and ϕ_n are polynomials having positive leading coefficient and satisfying the given relation, so are

$$\begin{aligned} f_{n+1} &= xf_n + (x^2 - 1)\phi_n \\ \phi_{n+1} &= f_n + x\phi_n, \end{aligned} \tag{2}$$

as is easily verified directly. Upon solving (2) we have

$$\begin{aligned} f_n &= xf_{n+1} - (x^2 - 1)\phi_{n+1} \\ \phi_n &= -f_{n+1} + x\phi_{n+1}, \end{aligned} \tag{3}$$

and also

$$f_n + f_{n+2} = 2xf_{n+1}, \qquad \phi_n + \phi_{n+2} = 2x\phi_{n+1}. \tag{4}$$

Suppose now that f_{n+1} and ϕ_{n+1} are any pair of polynomials which satisfy (1) with $f_{n+1} = ax^{n+1} + bx^n + cx^{n-1} + \cdots$, $a > 0$. Then from (1)

$$\phi_{n+1} = ax^n + bx^{n-1} + (c + \tfrac{1}{2}a)x^{n-2} + \cdots .$$

Now (3) gives $f_n = \frac{1}{2}ax^n + \cdots$, $\phi_n = \frac{1}{2}ax^{n-1} + \cdots$ which satisfy (1), and by repeated application of (3) there are obtained pairs of polynomials for each lower value of the subscript, until we reach for $n = 1$, $f_1 = x$ and $\phi_1 = 1$, since these are the only positive solutions, with positive leading coefficient, with f_1 of the first degree. It follows that all polynomial solutions are obtained, except for sign, from $f_1 = x$ and $\phi_1 = 1$ by repeated application of (2).

With the substitutions suggested in the second solution above, equations (2), (3), (4) are immediately equivalent to familiar trigonometric identities, as are also the following:

$$\phi_n f_{n-1} - f_n \phi_{n-1} = 1, \qquad f_n f_{n-1} - x = (x^2 - 1)\phi_n \phi_{n-1}.$$

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

1. *Essentials of College Algebra.* By J. B. Rosenbach and E. A. Whitman. New York. Ginn and Company, 1951. 10+322+30 pages. $3.00.
2. *College Algebra.* By H. L. Rietz, A. R. Crathorne, and J. W. Peters. New York. Henry Holt and Company, 1951. 17+387 pages.
3. *College Algebra.* By H. K. Fulmer and Walter Reynolds. New York. Ginn and Company, 1951. 6+204+14 pages.
4. *Intermediate College Algebra.* By E. M. J. Pease. New York. Prentice-Hall, Inc., 1950. 7+420+36 pages. $2.85.
5. *Intermediate Algebra.* By P. K. Rees and F. W. Sparks. New York. McGraw-Hill Book Company, 1951. 8+328 pages. $3.25.
6. *Elements of Algebra.* By L. C. Peck. New York. McGraw-Hill Book Company, 1950. 13+230 pages. $2.75.
7. *Practical Mathematics, Part II, Algebra with Applications.* By C. I. Palmer and S. F. Bibb. New York. McGraw-Hill Book Company, Inc., 1950. 12+252 pages. $2.20.
8. *Algebra for Commerce and Liberal Arts.* By A. K. Bettinger and W. A. Dwyer. New York. Pitman Publishing Corporation, 1951. 11+225 pages.

The content material of these books varies from the older habit of including partial fractions, limits, and infinite series as in (2) to the modern habit of more careful explanations with sacrifice of content, as in (6). (1), (2), (3) and (4) give certain coverage to such topics as simultaneous quadratics, progressions,

logarithms, binomial theorem, theory of equations, and complex numbers, while (1), (2) and (3) also include determinants and inequalities. Mathematical induction is covered by (2) and (3) while (8) includes a brief useful chapter on the slide rule. (4) has the unique feature of certain clever tests at the end of the chapters.

An impartial reader notes that (1), (2) and (3) are strong in content material of the classical type. (4) and (5) are slanted more for the intermediate (as their titles clearly imply) algebra student, while (6) is a jewel for the belated college student. (7) and (8) fulfill different purposes (see titles).

(1) "is considerably shorter than the authors' *College Algebra*, Third Edition, the same standards in fullness of explanation have been maintained," according to the preface. The usual topics in college algebra for the past two decades are found in this text. Common student mistakes are highlighted by 'Warnings' distributed throughout the book showing both the correct and incorrect procedures, such as $f(x)$ does not mean f times x. Answers to odd problems, as well as certain appropriate tables, combine to make the book a useful educational asset.

First published in 1909, and revised every ten years except 1949, (2) is well known for the breadth of topics covered and for its careful exposition of the facts. The first chapter has been rewritten to emphasize the number system. The preface reminds the reader that "most of the exercises and problems have been changed and an effort has been made to arrange them in order of increasing difficulty." Brief tables of powers, roots, reciprocals, and logarithms are noted in the back of the book.

(3) reminds one of the scholarly books of the last two decades as far as content is concerned, with commendable effort to define for the student his vocabulary of integer, term, monomial, binomial, product, factor, coefficient, and exponent in the first chapter. Certain elementary concepts have been amplified, as "Thus, in the treatment of fractions the idea of simple indeterminate forms is explained," (preface). Even mathematical induction appears in this text, which is a small book, as far as exterior is concerned. Answers to all problems, plus the brief logarithm table combine to make this book one of interest to the reader.

The author of (4) says that "the fundamentals of arithmetic and algebra are covered to such an extent that a good student may use this text for his first course in algebra." One must admit that the tests arranged at the end of the chapters should speed the student along so that he might also cover a great many of the solid topics, such as simultaneous quadratics, theory of equations, and binomial theorem. While no tables appear, yet a good index and answers to even problems assure the student of an acceptable text for its purpose.

(5) "is designed for the student who has had only one year of high school algebra" and "the problems are arranged in groups of four on about the same level of difficulty", the authors state. Timely definitions of such terms as integer, rational number, algebraic expression, term, and the stating of axioms concern-

ing equations help to remind present day mathematics teachers that their students have limited mathematical heritage. On page 40 one notices a pleasant effort to explain that zero times a finite number is zero. Answers to all even problems and half of the odd problems appear. Topics such as systems of quadratic equations in two variables, logarithms, progressions, and binomial theorem, make this text suitable for a large group of students.

(6), "written primarily for college students who have had no algebra in high school and thus presupposes no mathematical background other than ordinary arithmetic," according to the author, accomplishes its purpose, as far as explanations are concerned. Even though the identity sign is not used, great emphasis is placed upon such operations; later the conditional equation appears on page 63. Elementary work on translations from English to algebra on page 76 of Chapter V, interesting facts concerning the presence of zero in the denominator in next chapter, and careful illustrations on page 97 concerning reducing a fraction to its lowest terms help to interest the reader. The unusual practice of omitting a numerator or a denominator in some problems reminds one of examination questions instead of text book problems. Factoring and graphs get painstaking explanations while the exhibit of the square root of 2 is a welcome addition for an introduction to quadratic equation study. Answers to odd problems, as well as additional answers in the last chapter are given. Including a chapter on complex numbers in such a book increases the stature of this publication.

Like (2), (7) is the fifth edition. It is a useful book known since its first edition in 1912. It is the second part of a series of four books in practical mathematics. While this text does include progressions (this latter a new chapter), yet its mission is emphasized in the more elementary processes of algebra. Hence, no tables are available, but answers to most all problems, as well as a brief index, appear.

The inclusion of brief chapters in (8) on curve fitting, extraction of square roots (it does have a logarithm chapter also), and slide rule, in addition to most of the usual topics found in good college algebra books is noted. Appropriate topics of percentage, markup, and markdown are sandwiched into the chapter on ratio, proportion, and variation. Answers to odd problems are given.

W. R. HUTCHERSON
University of Florida

The Teaching of Mathematics. By D. R. Davis. Cambridge, Mass., Addison-Wesley Press, Inc., 1951. xv+415 pp.

Into this text, intended for a one semester course, the author has attempted to put what he considers the essentials of effective teaching; namely, (1) teaching techniques, procedures, and trends discussed in relation to the subject matter of mathematics and with especial emphasis on means of creating interest, (2) a systematically organized knowledge of the subject matter, plus (3) practical suggestions for the supervision of mathematics instruction and for the

preparation and out-of-class duties of mathematics teachers.

The distribution of time to these topics is as follows: topic (1) is treated in chapters 1–5, (2) in chapters 7–14, (3) in chapters 15, 16. This leaves Chapter 6 "Organization and Treatment of Subject Matter" to serve in part a unifying role and in part as the place where the author re-emphasizes his own themes in and philosophy of mathematical instruction. These seem to be (a) the importance of sequence, order, and logic in the organization and teaching of mathematics, (b) the need for a teacher to understand and present in a broad and general way the postulational nature of the foundations of mathematics, but (c) to realize simultaneously that complete rigor is not suited to elementary instruction where a psychological organization must also be planned. In this connection the author stresses (d) the importance of the use of *analysis*, and the role of intuition and discovery in both mathematics and teaching, and (e) the importance of generalization, of stressing fundamental concepts rather than memorized formulas, and relationships and fusion rather than watertight compartmentalization.

All of these "themes" are excellent in the reviewer's opinion. Among the strongest points of the book are its detailed examples of the analysis of theorems and of the construction of teaching units ending with generalizations of earlier theorems. These are in line with the author's assertion that what most teachers want is "practical assistance" in making teaching theories concrete. Similarly concrete and practical are many of his suggestions for rationalizing processes and enriching teaching, such as the graphical interpretation of the solution of simultaneous linear equations by addition and the discussion of Pythagorean numbers. However, the reviewer wondered if the rather lengthy syllabi or course outlines for all grades found in chapters 7–12 were not in the first place arbitrary and unnecessarily detailed while on the other hand the discussion of such topics as the conic sections appears too sketchy for persons with inadequate background in freshman mathematics and too elementary for well trained upper classmen. Perhaps this latter material could have been replaced by more of the genuine additional enrichment material which is scattered through the book particularly in Chapter 13, "Algebraic Operations and our Number System" where, for example, Fermat's theorem and Euler's ϕ function are used in a discussion of repeating decimals.

Weaknesses of the book lie in the omission of adequate discussions of several significant trends and emphases in mathematical education such as: the teaching of geometry without a text, stressing its structure and the nature of proof, and the use of teaching aids, field work, and laboratory methods. Further, intuitive geometry receives little attention, problem solving as a general objective is not adequately discussed and the work of the Post War Plans Committee of the National Council of Teachers of Mathematics is ignored (except for a brief mention of its *Guidance Pamphlet*), as are several other significant committee reports.

One suspects that these omissions were the result of combining the neces-

sity for limiting the text with the author's preferences in mathematical educational philosophy. However, teachers should be aware of and evaluate conflicting views, different procedures and important reports. The author's own classroom program from whence the book stems seems also to be reflected in the style which is occasionally telegraphic and occasionally repetitive as if he had written up at separate times the notes for a number of units without fully surveying the whole to eliminate unnecessary repetition and to correlate the parts. For example, section 15-6 "Professional Activities" in the chapter on *Professional Duties and Teacher Preparation* is almost identical with section 16-9 "Professional Growth of Teachers" in the chapter on *Supervision and Improvement of Instruction.*

The book is remarkable for a much more complete discussion of the foundations of mathematics than is customarily found in such a text as well as for its discussion of junior college and advanced high school courses. However, granting that complete rigor is not desirable on an elementary level, the reviewer would have liked to see more care displayed in the definitions of a *function*, *continuity*, *locus*, and *conditional equation.* The postulates for algebra on page 129 are far from complete but no reference is made either to this fact or to a source for a complete set. Further, the discussion of approximate computation ignores completely the difference between decimal places and significant figures and between the rules for addition and multiplication of approximate numbers. The "odd-add" rule is treated well as is scientific notation.

The author's repeated insistence on teaching for meaning (understanding of basic principles) leaves one surprised that he advocates the use of the term and process "transposition" (after it has been rationalized) and that he uses the ":" symbolism for ratios in secondary school geometry although he stresses teaching proportion from the algebraic equation viewpoint. These were frowned upon as far back as the 1923 report of the Joint Committee on the *Reorganization of Mathematics in Secondary Education*, an important document to which the author does refer several times.

Only a few minor typographical errors were noted. On page 291 the cosine law is named where the sine law is intended. The format is pleasing, the headings and outlines are helpful, but the shiny paper makes the book hard to read in some lights.

It is a large task to put into one book all that the author proposes. It requires selection and condensation. The author has done this on the avowed basis of his experience in teaching both mathematics and mathematics instructors, and implicitly on the basis of his personal beliefs and tastes in regard to desirable emphases in mathematics teaching. Not everyone will agree with the omissions, but few will debate the main themes and emphases of the book.

In general, it is a book to parts of which the reviewer will refer his students and is one of the few usable texts now in print in this field.

P. S. JONES
University of Michigan

NEW BOOKS RECEIVED

Mathematics of Investment. By P. R. Rider and C. H. Fischer. New York, Rinehart and Company, 1951. 12+359 pages. $5.00.

Finite Matrices. By W. L. Ferrar. New York, Oxford University Press, 1951. vi+182 pages. $4.00.

Tensor Analysis, Theory and Applications. By I. S. Sokolnikoff. New York, John Wiley and Sons, Inc., 1951. ix+335 pages. $6.00.

Spinoza Dictionary. By R. D. Dagobert. New York, Philosophical Library, Inc., 1951. xiv+309 pages. $5.00.

Advanced Engineering Mathematics. By C. R. Wylie, Jr. New York, McGraw-Hill Book Company, 1951. xiii+640 pages. $7.50.

Philosophical Problems of Mathematics. By Baron v. Freytag gen Loringhoff Bruno. New York, Philosophical Library, 1951. 88 pages. $2.75.

Intermediate Algebra for Colleges. By J. B. Rosenbach and E. A. Whitman. New York, Ginn and Company, 1951. x+219+xxii pages+a four-place common logarithm table of numbers (in pocket on back cover). $3.00.

Calculus. Revised Edition. By J. V. McKelvey. New York, The Macmillan Company, 1951. 7+405 pages. $4.50.

Mathematics for Technical Students, Book III. By J. D. N. Gasson. Cambridge, The University Press, 1951. xii+451 pages. $3.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to H. D. Larsen, Albion College, Albion, Michigan.

Pi Mu Epsilon, Iowa State College

During the school year 1950–51, the following papers were presented at meetings of the *Iowa Alpha* chapter of *Pi Mu Epsilon*:

Paradoxes simple and complex, by Dr. Joe Foote

The use of matrix notation for transfer functions of physical systems, by Mr. A. J. Bradt

Some equations for adsorption of gases, by Dr. Robert S. Hansen.

Eight meetings were held during the year, including two initiation banquets in the fall and spring. The Pi Mu Epsilon prize, which is given each year to an outstanding mathematics student, was awarded to Robert H. Tweedy.

Officers elected for 1951–52 are: Director, Richard E. Johnson; Vice-Director, John Druyor; Secretary, Caroline Iverson; Treasurer, Donald Klippenstein; Faculty Advisor, Ralph M. Robinson.

Mathematics Club, Iowa State College

The *Iowa State College Mathematics Club* is sponsored by the Iowa Alpha chapter of Pi Mu Epsilon, the fraternity Vice-Director acting as student-director of the club. Richard E. Johnson is the retiring and John Druyor is the new student-director. Three meetings were held during 1950–51 at which the following programs were presented:

Statistics: what it is and what it can do, by Dr. Bernard Ostle

Movie: The Pythagorean theorem

Prizes in math competition, by Miss Lieberknecht.

Mathematics Club, Los Angeles City College

The *Los Angeles City College Mathematics Club*, which was reorganized in September of 1950 after a lapse of several years, held bi-weekly meetings during 1950–51. City College staff members delivered the following lectures:

Continued fractions, by Mr. Samuel Skolnik

Rabbits and rectangles: Fibonacci series, by Mr. Benjamin K. Gold

The four-color problem, by Mr. Robert B. Herrera

The theory of relativity, by Mr. Samuel Skolnik

Napier's logarithms, by Mr. Fred Marer

Electronic computing machines, by Mr. George Mapes.

The following lectures by visiting speakers were presented for the students and faculties of both Los Angeles City College and Los Angeles State College. The first, third and fifth lectures were sponsored by L. A. State, the second and fourth lectures by L. A. City:

Foundations of mathematics, by Professor Hans Reichenbach

Pythagorean number triples, by Dr. Tobias Dantzig

Interplanetary navigation, by Professor Robert Herrick

Mathematical logic, by Professor Leon M. Henkin

Properties of zero and infinity, by Professor Samuel E. Urner.

The officers for 1950–51 were: President, Leroy K. Barickman; Vice-President, Arthur Freed; Secretary-Treasurer, Henry P. Thompson; Faculty Sponsor, Robert B. Herrera.

Pi Mu Epsilon, University of Richmond

During 1950–51, the *Virginia Alpha* chapter of *Pi Mu Epsilon* held nine meetings at which the following papers were presented:

Mathematical side of astronomy, by Mr. Ogburn

Topology, by Miss Marion Jeffries

Opportunities in the field of mathematics, by Dr. C. H. Wheeler III, Samuel Smith, and Barbara McGeehee

A glimpse of time, by Miss Isabel Harris.

Officers elected for 1951–52 are: Director, Dr. C. H. Wheeler III; Vice-Director, Willard E. Meador, Jr.; Secretary, James L. Judson; Treasurer, William F. Herget.

Pi Mu Epsilon, Hunter College

During 1950–51, the following papers were presented at meetings of the *Hunter College* chapter of *Pi Mu Epsilon*:

Cerva's theorem and applications, by Hannah Stein

Menelaus' theorem and applications, by Ursel Kramer

Miquel theorem, pedal triangle, Simson line, by Marie Barbiere

Brocard configuration, by Dorothy Freudenberger

Euler's line, by Virginia Cacich

Nine-point circle, by Muriel Teitell

Inscribed and escribed circles of a triangle, by Stanley Gaffen

Theorem by Feuerbach, by Rita Goldstein

History of cartography, by Jane Luhks

The terrestrial sphere, by Norma Lackow

Equidistant azimuthal projection and UN flag, by Alice Gersh

Stereographic projection, by Mildred Spiewak

Equations of transformation by cartesian coordinates; applications of theory of inversion to analytic geometry, by Evelyn Horvath

Geometric construction of inverse points by compasses alone; applications of theory of inversion to modern geometry, by Margot Henlein

Gerardus Mercator and Mercator projection, by Elaine Gross

The celestial sphere, by Mildred Spiewak.

Professor Gustav A. Hedlund of Yale University was guest speaker at the May initiation; his topic was *Clifford Klein space forms or small geometries.*

Officers for the second semester were: Director, Prof. Laura Guggenbuhl; President, Mildred Spiewak; Corresponding Secretary, Norma Lackow; Recording Secretary, Rita Goldstein; Treasurer, Margot Henlein.

Ricci Mathematics Academy, Boston College

The *Ricci Mathematics Academy of Boston College* held regular semi-monthly meetings during 1950–51 at which the following papers were presented:

The applications of visual aids in mathematics, by Professor H. W. Syer

Introduction to plane geometry, by Mr. E. J. Montella

The duties and responsibilities of the actuary, by Mr. W. L. O'Connor

Electronic digital computers, by Mr. N. Cedrone

Engineering work and sales liaison, by Mr. P. J. Equi.

Also presented were films on algebra, geometry, navigation, and nautical astronomy. The *Ricci Mathematical Journal*, official publication of the Academy, appeared in November, February, and April.

The spring meeting of the Greater Boston Intercollegiate Mathematics Clubs Association was held on April 26 at Boston College, at which Prof. Hans G. Haefeli presented a paper on *The use and development of geometry.*

Officers of the Academy were: President, Charles W. Hunter; Vice-President, William J. Kelly; Secretary, Charles L. Niles, Jr.; Treasurer, William J. Broderick.

Mathematics Club, Brown University

The thirty-seventh annual program of the *Mathematics Club of Brown University* consisted of six regular meetings and the club picnic. The papers presented at the meetings in 1950–51 were:

Models of vibrating systems, by Professor Erastus Henry Lee
Cubic equations, by James Gray
Boolean algebra, by Simone Matteodo
Women versus mathematics, by Dean Nancy Duke Lewis, Pembroke College
What is wrong with this picture? by Dora Bucco
Planetary motion, by Paul Duclos
Generalising the factorial, by William Youden
How big is the factorial? by Barbara Webb
Theory of games, by Professor David Gale.

The Committee on Program and Arrangements was: Chairman, Simone Matteodo; Secretary, Barbara Webb; Treasurer, James Gray; Faculty Advisor, Professor Frank Stewart; and Martin Badoian, Edwin Boynton, Louise O'Donnell, Patricia Wandelt, Robert Warren, William Youden.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Assistant Professor P. B. Burcham of the University of Missouri represented the Association at the inauguration of President M. E. Collins of Missouri Valley College on December 10, 1951.

Boston University announces: Dr. J. B. Giever and Dr. F. J. Scheid have been promoted to assistant professorships; Dr. G. E. Noether of New York University has been appointed Assistant Professor of Mathematical Statistics; Dr. Felix Browder of Massachusetts Institute of Technology has been appointed to an instructorship; Assistant Professor Ottam Chand has returned to India; Instructor N. G. Blomquist has returned to the Institute of Mathematical Statistics, Stockholm, Sweden.

Georgia Institute of Technology reports the following: Instructor L. R. Daniel has been promoted to an assistant professorship; Mr. M. B. Sledd has been appointed Research Associate Professor; Dr. R. A. Willoughby has been appointed to an assistant professorship; Dr. W. B. Evans has been called to active service with the United States Air Force.

Louisiana State University announces the following: Assistant Professor E. S. Elyash of the University of Pittsburgh and Assistant Professor E. G.

Kundert of the University of Tennessee have been appointed to assistant professorships; Dr. Eugenio Calabi is on leave of absence for the academic year 1951–52 and is at Princeton University; Dr. H. C. Wang is also on leave for the year 1951–52 and is at the Institute for Advanced Study.

Massachusetts Institute of Technology makes the following announcements: Dr. G. F. D. Duff and Dr. J. F. Nash, Jr., have been appointed to C. L. E. Moore instructorships; Dr. D. E. Richmond has been appointed Lecturer.

Southern Methodist University announces that Mr. John Derr and Mrs. I. D. Russell have been appointed to instructorships.

St. Bonaventure University reports: Reverend M. V. Perry, formerly of Siena College, has been appointed to an instructorship; Reverend Joseph Ruther, formerly an instructor, is now at St. Joseph Seminary, Callicoon, New York; Assistant Professor Robert Smith and Instructors Donald Smith and Philip Wolf have accepted positions in industry.

At Texas Christian University: Assistant Professor L. A. Colquitt has been promoted to an associate professorship; Miss Margaret Welch has been appointed to a graduate fellowship.

The University of Alabama announces: Dr. O. R. Ainsworth, previously a graduate assistant at the University of California, has been appointed to an assistant professorship; Professor J. D. Mancill has been granted a leave of absence in order to accept a position as Mathematics Consultant at Redstone Arsenal, Huntsville, Alabama.

The University of California at Los Angeles makes the following announcements: Associate Professor Clifford Bell has been promoted to a professorship; Dr. R. M. Redheffer has been promoted to an assistant professorship; Dr. Edward Stiefel of Eidenossische Technische Hochschule, Zurich, Zwitzerland, has been appointed Visiting Professor; Dr. Enrique Cansado of the Bureau of the Budget, Washington, D. C., has been appointed Visiting Assistant Professor; Dr. Maria Weber, formerly Bateman research associate at California Institute of Technology, has been appointed Lecturer; Dr. Barrett O'Neill, previously a graduate student at California Institute of Technology, has been appointed to an instructorship; Professor G. E. F. Sherwood has retired with the title of Professor Emeritus.

The University of Connecticut announces that Dr. G. B. Robison and Dr. Henry Zatzkis have been appointed to instructorships and that Miss Ann M. Curran and Miss Cecilia T. Welna have been appointed to assistant instructorships.

At the University of Illinois: Associate Professor Harry Levy has been promoted to a professorship; Assistant Professor Herman Chernoff has been promoted to the position of Associate Professor of Mathematical Statistics; Assistant Professors L. L. Steimley and H. E. Vaughn have been promoted to associate professorships; Professor Beno Eckmann of Zurich, Switzerland, was Visiting Professor during the first semester of 1951–52; Professor W. J. Trjitzinsky is on sabbatical leave during the second semester; Associate Professor G. P.

Hochschild is on leave for the academic year and is at Yale University; Associate Professor Herman Chernoff and Assistant Professor M. E. Munroe were on leave during the first semester; Assistant Professor F. E. Hohn has received a Ford Foundation award for 1951–52.

The University of Kansas announces the following: Professor G. W. Smith has retired as Chairman of the Department of Mathematics but is continuing to teach in the Department; Professor G. B. Price has been appointed Chairman of the Department; Professors Nachman Aronszajn and A. H. Diamond of Oklahoma Agricultural and Mechanical College have been appointed to visiting professorships; Dr. S. S. Shrikhande of the College of Science, Nagpur, India, has been appointed Visiting Assistant Professor; Dr. W. K. Moore, previously a graduate student at the University of Kansas, and Mr. A. J. Zeichner, formerly an instructor at Oklahoma Agricultural and Mechanical College, have been appointed to instructorships.

At the University of Maryland: Dr. G. S. S. Ludford, previously Choate fellow at Harvard University, has been appointed to an assistant professorship; Mr. N. Z. Wolfsohn, formerly teaching and A. E. C. fellow at Harvard University, has been appointed to an assistant professorship; Mr. G. L. Spencer II, who has been a teaching and research assistant at the University of Michigan, has been appointed to an instructorship; Associate Professor J. L. Vanderslice is on leave of absence; Professor M. H. Martin has been appointed Chairman of the Committee on Applied Mathematics of the American Mathematical Society for a three year term beginning January 1, 1951 and Editor of the Proceedings of the Fourth Symposium on Applied Mathematics.

The University of Mississippi reports the following: Dr. L. L. Scott, previously a graduate student at the University of Illinois, has been appointed to an assistant professorship; Assistant Professor R. D. Sheffield is on leave of absence for the academic year 1951–52 for the purpose of graduate study at the University of Tennessee.

The University of Pennsylvania announces: Dr. H. E. Rauch, formerly of the Institute for Advanced Study, has been appointed Associate; Professor I. J. Schoenberg is on leave of absence and is at the Institute for Numerical Analysis.

The University of Rochester announces the following: Assistant Professor Dorothy L. Bernstein has been promoted to an associate professorship; Mr. Robert MacDowell, previously a graduate student and part-time instructor at the University of Michigan, has been appointed to an instructorship; Dr. Arthur Dutton, formerly an instructor and research assistant in Statistics at Iowa State College, has been appointed Assistant in Atomic Energy Project and Lecturer in Mathematical Statistics; Mr. W. A. Small, who has been a graduate student at the University, has been appointed to a part-time instructorship.

The University of Saskatchewan reports that Dr. James Sanders, previously a graduate student at Syracuse University, and Mr. Nathan Shklov, formerly a graduate student at the University of Indiana, have been appointed to instructorships.

At the University of Utah: Dr. Cecil Burgess, formerly a part-time instructor at the University of Texas, Dr. Henry Hiz of the Department of Philosophy of the University of Pennsylvania, and Dr. Eugene Parker, previously a graduate assistant at California Institute of Technology, have been appointed to instructorships; Professor I. O. Horsfall has retired.

Wayne University makes the following announcements: Associate Professor D. C. Morrow has been promoted to a professorship; Assistant Professor Arvid Jacobson has been promoted to an associate professorship; Instructor Samuel Conte has been promoted to an assistant professorship; Mr. George Johnson, who has been with the Kaiser-Frazer Corporation, Mr. Charles Kraft, a graduate student of the University of California at Berkeley, and Mr. Martin Wechsler, a graduate student at the University of Michigan, have been appointed to instructorships.

West Virginia University announces the following: Assistant Professor Margaret B. Cole has been promoted to an associate professorship; Mr. C.N. Cochran, Mr. F. E. Forsythe and Mr. P. S. Null have been appointed to instructorships; Professor C. N. Reynolds has retired with the title of Professor Emeritus.

Mr. L. G. Arnold, previously a part-time instructor at the University of Rochester, is now with the Navy Ordnance Division of Eastman Kodak Company.

Dr. Elsie T. Church has been appointed to an associate professorship at Northwestern State College of Louisiana.

Instructor Robert Coke of Southern Methodist University is now an instructor at Sheppard Field, Wichita Falls, Texas.

Associate Professor G. N. Conwell of the University of Georgia has retired.

Instructor J. A. Englund, formerly of Massachusetts Institute of Technology, has been appointed to an instructorship at Creighton University.

Associate Professor Howard Eves of Oregon State College has been appointed Professor and Head of the Department of Mathematics of Champlain College.

Instructor D. O. Gray, who has been promoted to an assistant professorship at the University of Houston, is on military leave with the United States Navy.

Assistant Professor G. P. Loweke of the College of Engineering of Wayne University has been promoted to an associate professorship.

Associate Professor Kenneth May of Carleton College has been promoted to the position of Head of the Department of Mathematics.

Professor W. T. Reid is on leave of absence from Northwestern University in order to serve as a staff member of the Sandia Corporation, Albuquerque, New Mexico.

Professor L. S. Shively of Ball State Teachers College, Indiana, has retired with the title of Professor Emeritus.

Associate Professor J. H. Simester of the Speed Scientific School, University of Louisville, has been promoted to a professorship.

Sister M. Tarcisius, formerly of Mt. Mercy and St. Mary's High School, Pittsburgh, Pennsylvania, is now at St. Peter's Convent, McKeesport, Pennsylvania.

Dr. M. M. Slotnick has been promoted to the position of Chief Mathematician, Humble Oil and Refining Company, Houston, Texas.

Professor L. W. Stark has accepted a position at Redstone Arsenal, Huntsville, Alabama.

Assistant Professor H. W. Syer of the School of Education, Boston University, has been promoted to an associate professorship.

Mr. C. M. Terry has a position as Analyst with the Department of Defense, Washington, D. C.

Lecturer F. B. Thompson of the University of California at Berkeley has accepted a position as Mathematician with the Rand Corporation, Santa Monica, California.

Mr. W. E. Timon, Jr., formerly a graduate fellow at Tulane University, is employed by the Esso Standard Oil Company Refinery, Baton Rouge, Louisiana, and has a part-time teaching position at Louisiana State University.

Mr. N. D. Vlachos, who has been an instructor at Drexel Institute of Technology, has a position as Mathematician at the Kellett Aircraft Corporation, Camden, New Jersey.

Instructor B. T. Wade of Kent State University has been appointed to an assistant professorship at Clemson Agricultural College.

Mr. O. L. Wadkins, Jr., formerly at White Sands Proving Ground, has a position as Research Engineer with North American Aviation, Los Angeles, California.

Mr. Alan Wayne, who has been teaching at Automotive High School, Brooklyn, has been appointed Chairman of the Department of Mathematics and Science, Williamsburg Vocational High School, Brooklyn, New York.

Mr. Edmund Wesolowski, previously a student at Elmhurst College, has a position as a spectrographer at Pure Oil Research and Development Laboratories, Crystal Lake, Illinois.

Assistant Professor R. H. Wilson of Temple University has been appointed to an assistant professorship at the University of Louisville.

Miss Laura Blakeley of Armstrong College died on October 19, 1951.

Assistant Professor Emeritus R. D. Daugherty of Kansas State College died on December 1, 1951.

Professor Emeritus H. L. Hall, retired head of the Department of Mathematics of Northwestern State College, Oklahoma, died on November 18, 1951.

Mr. J. F. Hubbard of J. H. Goddard and Company, Boston, Massachusetts, died on September 24, 1951.

Professor Emeritus N. J. Lennes of Montana State University died on November 21, 1951. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-FIFTH ANNUAL MEETING OF THE ASSOCIATION

The thirty-fifth annual meeting of the Mathematical Association of America was held at Brown University, Providence, Rhode Island, on Saturday, December 29, 1951, in conjunction with the annual meeting of the American Mathematical Society. A total of five hundred and sixty persons were registered, including the following two hundred and seventy-two members of the Association:

C. R. Adams, J. B. Adkins, R. P. Agnew, L. V. Ahlfors, M. I. Aissen, R. G. Albert, C. B. Allendoerfer, R. D. Anderson, R. C. Archibald, H. E. Arnold, L. G. Arnold, Max Astrachan, W. E. Barnes, Grace E. Bates, J. D. Baum, W. R. Baum, Helen P. Beard, Ralph Beatley, W. R. Beck, E. G. Begle, H. A. Bender, A. A. Bennett, H. G. Bergmann, Brother Bernard Alfred, S. D. Bernardi, Barbara B. Betts, Garrett Birkhoff, D. W. Blackett, Shirley A. Blackett, S. G. Bourne, Julia W. Bower, Robert Breusch, H. W. Brinkmann, Bailey Brown, H. D. Brunk, J. A. Bullard, R. S. Burington, R. A. Burrows, L. J. Burton, Jewell H. Bushey, E. A. Cameron, Mildred E. Carlen, L. Virginia Carlton, T. C. Carson, W. B. Carver, W. F. Cheney, Jr., Sarvadaman Chowla, G. R. Clements, R. N. Cobb, L. W. Cohen, Nancy Cole, E. M. Cook, A. H. Copeland, Richard Courant, R. R. Coveyou, Robert Davies, W. M. Davis, A. H. Diamond, H. L. Dorwart, M. P. Dossey, F. G. Dressel, G. R. Dube, Nelson Dunford, Ruth B. Eddy, Samuel Eilenberg, Benjamin Epstein, D. H. Erkiletian, Jr., A. B. Farnell, W. E. Ferguson, H. E. Fettis, F. A. Ficken, C. D. Firestone, J. R. Foote, L. R. Ford, Tomlinson Fort, J. S. Frame, H. D. Friedman, Orrin Frink, J. W. Gaddum, R. W. Gardner, A. E. Gault, H. M. Gehman, Irving Gerst, K. G. Getman, W. M. Gilbert, R. E. Gilman, R. D. Glauz, A. M. Gleason, H. E. Goheen, W. O. Gordon, W. H. Gottschalk, Arthur Grad, H. E. H. Greenleaf, J. B. Gregg, Florence N. Greville, T. N. E. Greville, Emil Grosswald, Seymour Haber, Franklin Haimo, P. C. Hammer, Frank Harary, B. T. Harris, W. L. Hart, Frank Hawthorne, G. A. Hedlund, M. H. Heins, Melvin Henriksen, Fritz Herzog, T. H. Hildebrandt, Einar Hille, W. M. Hirsch, S. P. Hoffman, D. L. Holl, L. Aileen Hostinsky, E. Marie Hove, Aughtum S. Howard, C. C. Hsiung, M. Gweneth Humphreys, W. A. Hurwitz, L. C. Hutchinson, R. N. Johanson, L. W. Johnson, R. E. Johnson, Roberta F. Johnson, B. W. Jones, F. B. Jones, O. C. Juelich, G. K. Kalisch, L. H. Kanter, Irving Kaplansky, J. H. B. Kemperman, D. E. Kibbey, H. S. Kieval, V. L. Klee, S. C. Kleene, J. R. Kline, Morris Kline, J. C. Knipp, R. L. Korgen, H. W. Kuhn, C. E. Langenhop, E. H. Larguier, J. A. Larrivee, R. D. Larsson, E. H. Lee, P. J. Leonard, Tadeusz Leser, C. J. Lewis, F. W. Light, Jr., Sally I. Lipsey, Lee Lorch, R. C. Lyndon, L. A. MacColl, Saunders MacLane, H. M. MacNeille, H. P. Manning, R. J. Marcou, Morris Marden, M. H. Martin, W. T. Martin, V. O. McBrien, N. H. McCoy, A. E. Meder, Jr., D. F. Mela, Karl Menger, W. E. Mientka, W. H. Mills, Deane Montgomery, T. W. Moore, R. K. Morley, A. J. Mortola, B. H. Mount, Jr., Pauline E. Mount, W. L. Murdock, Paul Nesbeda, C. V. Newsom, E. N. Nilson, C. O. Oakley, L. F. Ollmann, Oystein Ore, E. F. O'Shea, J. C. Oxtoby, D. K. Pease, W. O. Pennell, P. M. Pepper, I. E. Perlin, H. P. Pettit, C. R. Phelps, J. C. Polley, Walter Prenowitz, G. B. Price, M. H. Protter, Tibor Rado, J. F. Randolph, G. E. Raynor, C. J. Rees, Mina S. Rees, Eric Reissner, C. N. Reynolds, Harris Rice, D. E. Richmond, G. deB. Robinson, Selby Robinson, G. B. Robison, I. H. Rose, Alex Rosenberg, P. C. Rosenbloom, Arthur Rosenthal, Jane S. Rothe, H. L. Royden, Evelyn C. Rusk, Charles Saltzer, Arthur Sard, J. A. Schatz, M. A. Scheier, Edith R. Schneckenburger, I. J. Schoenberg, Augusta L. Schurrer, Seymour Schuster, Abraham Schwartz, B. L. Schwartz, C. H. W. Sedgewick, Esther Seiden, Sister Mary Felice, Sister Mary Petronia, E. C. Smith, Jr., Robert Edward Smith, Andrew Sobczyk, Elizabeth S. Sokolnikoff, Joseph Spear, Marion E. Stark, E. P. Starke, J. R. K. Stauffer, F. M. Stewart, Ruth W. Stokes, J. S. Stubbe, R. L. Swain, Otto Szasz, Gabor Szego, R. C. Taliaferro, William Clare Taylor, G. B. Thomas, Jr., J. M. Thomas, D. L. Thomsen, R. M. Thrall, W. J.

Thron, John Todd, Marian M. Torrey, J. I. Tracey, A. W. Tucker, E. P. Vance, Helen E. Van Sant, C. H. Vehse, John von Neumann, T. L. Wade, Jr., R. W. Wagner, S. E. Walkley, William Wallace, J. L. Walsh, Mother Mary E. Walsh, W. H. Warner, S. E. Warschawski, J. V. Wehausen, Marie J. Weiss, P. M. Whitman, G. T. Whyburn, D. V. Widder, W. L. Williams, D. M. Young, Jr., Oscar Zariski, J. L. Zemmer, Jr., J. A. Zilber.

Sessions of the Association were held on Saturday morning and afternoon in the Auditorium of the Metcalf Chemical Laboratory, Brown University. President Saunders MacLane presided at both sessions. The Program Committee for the meeting consisted of N. H. McCoy, Chairman, Bailey Brown, and Moses Richardson.

FIRST SESSION OF THE ASSOCIATION

"Partially Ordered Sets," by Professor Orrin Frink, Jr., Pennsylvania State College.

"Trends in Algebraic Topology," by Professor N. E. Steenrod, Princeton University.

"Cardano's Book on Gambling and Probability," by Professor Oystein Ore, Yale University.

SECOND SESSION OF THE ASSOCIATION

Symposium on the Teaching of Calculus

"Variables in Calculus," by Professor Karl Menger, Illinois Institute of Technology.

"Insight and Understanding in the Calculus," by Professor Walter Prenowitz, Brooklyn College.

"Concepts Versus Rigor," by Professor Herbert Federer, Brown University.

"Notion and Definition of a Function," by Professor J. F. Randolph, University of Rochester.

"Combining Analytic Geometry with Calculus," by Professor A. W. Tucker, Princeton University.

MEETING OF THE BOARD OF GOVERNORS

The Board met on Friday morning in the Faunce Memorial Room of Faunce House. Nineteen members of the Board were present. Among the more important items of business transacted were the following:

The Board voted to approve the appointment by President MacLane of the following Nominating Committee for 1952: W. L. Duren, Chairman, E. P. Northrop, and R. J. Walker.

To fill the vacancy in the Board caused by the death of Professor J. B. Rosenbach, the Board voted to elect Professor F. H. Steen of Allegheny College as Governor from the Allegheny Mountain Section for a term expiring on June 30, 1954.

To succeed Dr. C. V. Newsom as a representative of the Association on the

Policy Committee for Mathematics, the Board elected Professor J. S. Frame of Michigan State College.

On the recommendation of the Executive Committee, the Board voted to re-elect Professor H. M. Gehman of the University of Buffalo as Secretary-Treasurer of the Association for a second term of five years, beginning on January 1, 1953.

The Board voted to approve the recommendation of the Committee on Expository Lectures that Professor Tibor Rado be invited to deliver a series of three expository lectures at the 1952 Summer Meeting.

On the recommendation of the Committee to Plan a Symposium on Teacher Education in Mathematics, the Board voted (subject to parallel action by the National Council of Teachers of Mathematics) that the Committee be authorized to determine the general plan and place of meeting for the Symposium and to appoint a Director. The Director will, by conference with the Committee, select consultants and put the plan for the Symposium into operation.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Saturday, December 29, 1951, at 2:00 P.M. in the Auditorium of the Metcalf Chemical Laboratory of Brown University, Providence, Rhode Island.

The Secretary announced the results of the balloting for officers, in which 1152 votes were cast. F. L. Griffin of Reed College was elected First Vice-President for the two-year term, 1952–1953. D. H. Lehmer of the National Bureau of Standards, Los Angeles, California and W. E. Milne of Oregon State College were elected Governors for the three-year term, 1952–1954.

The Secretary read the following resolution adopted by the Board of Governors:

> "The Board of Governors wish to express their deep sense of appreciation to Dr. Carroll V. Newsom for his long service as an editor of the American Mathematical Monthly. Beginning in 1943, when he started a widely useful section on War Information, and ending at the present time after a five year span as Editor-in-Chief, he has given freely of his time and his talents. We commend him for his steady devotion and his fine sense of duty."

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on Wednesday, December 26, and continued through Friday afternoon. The Josiah Willard Gibbs Lecture was delivered by Dr. Kurt Gödel of the Institute for Advanced Study. The title of his address was "Some Basic Theorems on the Foundations of Mathematics and Their Philosophical Implications." Invited addresses were given by Professor Shizuo Kakutani of Yale University on "Brownian Motion" and by Professor G. P. Hochschild of the University of Illinois and Yale University on "Cohomology in Algebraic Number Fields."

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of F. M. Stewart, Chairman, C. R. Adams, A. A. Bennett, L. W. Cohen, H. M. Gehman, and E. H. Lee.

Registration headquarters was in the west wing of Faunce House on Waterman Street. Rooms in the University dormitories were available from Wednesday morning to Sunday morning. Meals were served in the University Refectory.

The ladies of the Mathematics Department, of the Graduate Division of Applied Mathematics and of the Department of the History of Mathematics entertained at tea in the Art Gallery of Faunce House on Wednesday afternoon.

The movie, "Ruggles of Red Gap," was shown in the Faunce House Theatre on Thursday evening.

On Thursday afternoon there was held a tour of Pendleton House, an example of an eighteenth-century Providence house. Tours through other similar houses were also held on Friday afternoon.

A number of special collections and exhibits were shown in the libraries of Brown University. These included exhibits of early mathematical books including the first printed edition of Euclid's "Elements."

The new offices of the American Mathematical Society and of Mathematical Reviews at 80 Waterman Street were open for inspection by those attending the meetings.

A joint dinner for the members of the mathematical organizations and their guests was held at 6:30 P.M. on Friday in the University Refectory. Professor C. R. Adams of Brown University acted as toastmaster. President John von Neumann of the American Mathematical Society extended the greetings of the Society. Mrs. Jewell H. Bushey, Second Vice-President of the Association, spoke about the past and present status of women mathematicians, stating certain theorems on this subject. President Henry M. Wriston of Brown University expressed the gratitude of the University at being able to entertain the mathematicians at this meeting.

Professor C. B. Allendoerfer presented a motion, which was adopted by a rising vote, expressing the deep appreciation of the members of the mathematical organizations for the hospitality which had been tendered to them at this meeting by Brown University. Especial thanks are due to the administrative staff, to the mathematicians of Brown University and to the local members of the Committee on Arrangements for their painstaking efforts in caring for the wants of those attending the meetings.

H. M. GEHMAN
Secretary-Treasurer

THE APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the United States Naval Academy at Annapolis, Maryland, on Saturday, April 28, 1951. The Chairman of the Section, Professor O. J. Ramler of Catholic University, presided at both the morning and afternoon sessions.

A total of one hundred three persons registered their attendance at the meeting including the following sixty-nine members of the Association:

J. C. Abbott, R. P. Bailey, N. H. Ball, W. E. Barnes, T. J. Benac, B. H. Buikstra, L. E. Carville, Randolph Church, C. R. Clark, G. R. Clements, A. E. Currier, R. L. P. Eiseman, F. D. Faulkner, C. H. Frick, B. C. Getchell, R. A. Good, J. R. Gorman, E. S. Grable, E. C. Gras, Donald Greenspan, R. E. Greenwood, T. N. E. Greville, D. W. Hall, J. R. Hammond, S. B. Jackson, Walter Jennings, L. M. Kells, H. L. Kinsolving, A. E. Landry, W. W. Leutert, B. J. Lockhart, G. A. Lyle, C. G. Maple, K. F. McLaughlin, D. St. C. Melvin, Joseph Milkman, J. F. Milos, T. W. Moore, Paul Nesbeda, N. O. Niles, W. H. Norris, J. F. Paydon, A. J. Pejsa, J. W. Popow, F. M. Pulliam, O. J. Ramler, V. J. Rathbun, C. H. Rawlins, R. W. Rector, J. N. Rice, R. E. Root, C. A. Rupp, J. B. Scarborough, Paul Shapiro, W. F. Shenton, E. R. Sleight, Robert Edward Smith, C. A. Spicer, M. F. Stilwell, H. C. Stotz, E. G. Swafford, O. M. Thomas, J. A. Tierney, C. C. Torrance, Marian M. Torrey, John Tyler, J. H. White, P. M. Whitman, R. H. Wilson.

At the business meeting in the afternoon session the following officers were elected for the coming year: Chairman, Professor E. R. Sleight of the University of Richmond; Secretary, Professor C. H. Frick, Mary Washington College; Members of the Executive Committee, Professor Florence Mears, George Washington University, and Professor V. L. Klee, University of Virginia. In connection with the celebration of its fiftieth anniversary during 1951, the National Bureau of Standards has invited the section to hold its next meeting there. This invitation has been accepted, and the next meeting will therefore be held at the Bureau of Standards in Washington on December 8, 1951.

The morning session was scheduled to consist of five contributed papers as follows, though the fifth was actually postponed to the afternoon session in order that lunch could be served at 1:00 P.M.

1. *The method of enclosures for nonlinear differential equations*, by Professor A. R. Craw, United States Naval Academy, introduced by Professor R. P. Bailey.

The Poincaré-Bendixson Theorem was used to establish sufficient conditions for the existence of periodic solutions of non-linear differential equations. A demonstration was then given of the existence of periodic solutions of the van der Pol equation for certain values of the parameter.

2. *Some curves associated with laws of growth*, by Professor C. H. Frick, Mary Washington College.

If a quantity y is given as a function of some parameter t, the equation $y^* = y(t+1)$, the corresponding curve $y^* = F(y)$ in the y, y^*-plane is a curve related to this law of growth. Since the situation $y^* = y$ indicates the limiting situation or state of full growth, the intersections of the curve with the line $y^* = y$ are of particular interest. It was shown in a number of examples how the geometry of this curve is related to finding such limits. The method gives some results for summing divergent series.

3. *Napier and his logarithms*, by Miss Stirling Clark, University of Richmond, introduced by Professor E. R. Sleight.

This paper contained a brief history of the development of mathematics before Napier's time, together with the educational qualifications of this great Scotsman. Napier's rods as an aid to multiplication were discussed. Also the method was explained which was used in formation of logarithmic tables, Napier's best known accomplishment.

4. *Closure and complementation as operators*, by Mr. R. L. Eiseman, University of Maryland.

It has been shown by Donal Adelman that the operators of closure and complementation, applied successively to any set in a space satisfying Hausdorff's first three axioms, can produce at most fourteen distinct sets. R. A. Good discovered a set in the euclidean plane which actually yields all fourteen sets. It is possible to find sets which produce exactly each even number of sets from two to fourteen. The null set in the null space produces exactly one. There are no sets which will produce exactly any odd number of sets greater than one.

5. *An application of elliptic functions to plane flow*, by Professor J. O. Tierney, United States Naval Academy.

A steady, irrotational, sentropic plane flow governed by Frankl's equation $\psi_{\sigma\sigma}+k(\sigma)\psi_{\theta\theta}=0$ is determined by the choice of the function $k(\sigma)$. With suitable units, for a polytropic gas it is found that $k(\sigma)=\sigma-\frac{1}{2}\delta^2\sigma^2+\cdots$. Employing σ and the stream function ψ as independent variables, M. H. Martin has shown that Frankl's equation is equivalent to the equation.

$$\theta_{\sigma\sigma}=\left(\frac{K(\sigma)+\theta_\sigma^2}{\theta_\psi}\right)_\psi. \tag{1}$$

By taking $K(\sigma)=\sigma$ as a first approximation, Frankl's equation reduces to Tricomi's equation which has been used by various authors in studying transonic flow.

To obtain a better approximation, consider $K(\sigma)=\sigma-\delta^2\sigma^2/2$. By assuming θ as a quadratic function of σ, substitution in (1) and use of the method of undetermined coefficients leads to a system of ordinary differential equations and thence to the following solution of (1).

$$\theta=(2\delta^3)^{-1}\{\sqrt{2}\ \ln\ [2^{-1/2}\ \mathrm{nc}\ \delta\psi(\mathrm{sn}\ \delta\psi+\sqrt{2}\ \mathrm{dn}\ \delta\psi]-\mathrm{sn}\ \delta\psi\ \mathrm{dn}\ \delta\psi(1-\delta^2\psi)^2\}.$$

This is a generalization of the solution $\theta=\sigma\psi+\psi^3/3$ studied by Martin for the case $K(\sigma)=\sigma$.

6. *On truncation error and stability* by Professor W. W. Leutert, University of Maryland.

This was an invited address. If a partial differential equation is replaced by a partial difference equation, the truncation error is defined as the difference between the exact solution u of the boundary or initial value problem for the differential equation and an exact solution v of a corresponding boundary or initial value problem for the difference equation.

Two simple examples, heat conduction in a slab and the vibrating string, were considered and it was shown that for any positive value of the mesh ratio $r=\Delta t/\Delta x^2$ and $r=\Delta t/\Delta x$ respectively, there exist infinitely many exact solutions of an initial value problem for the difference equation which converge to the exact solution of the differential equation as the size of the mesh is decreased.

If the mesh ratio r exceeds $\frac{1}{2}$ or 1 respectively, random round off errors in a numerical computation will in general increase indefinitely, i.e. the numerical process is no longer stable.

These results contradict statements in a recent paper by O'Brien, Hyman, Kaplan, *A Study of the Numerical Solution of Partial Differential Equations*, J. Math. Phys. 29, pp. 223–251 (1951),

especially the assertion on page 228 that for the second problem and $r>1$ convergence is "clearly impossible" and that "$r \leqq 1$ is a necessary condition for convergence."

S. B. JACKSON, *Secretary*

THE APRIL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Benedict in St. Joseph, Minnesota, on Saturday, April 28, 1951. Sessions were held in the forenoon, at luncheon and in the afternoon. Professors Kenneth May, W. R. McEwen and J. M. H. Olmsted (Chairman of the Section) presided at the respective sessions.

Sixty-seven persons attended the meeting, including the following forty-three members of the Association:

H. M. Anderson, F. J. Arena, H. S. Berg, C. R. Bonnell, K. H. Bracewell, R. W. Brink, L. E. Bush, W. H. Bussey, R. H. Cameron, E. J. Camp, Elizabeth Carlson, W. B. Fulks, C. B. Germain, Gladys Gibbens, H. W. Godderz, Ruby M. Grimes, J. E. Hafstrom, K. L. Hankerson, A. G. Hill, Hildegarde Horeni, G. H. Jaeger, E. R. Johnston, Reverend W. C. Kalinowski, Karlis Kaufmanis, P. G. Kirmser, Fulton Koehler, W. S. Loud, Kenneth May, W. H. McBride, W. R. McEwen, Sigurd Mundhjeld, W. D. Monro, M. J. Norris, J. M. H. Olmsted, P. A. Rognlie, L. W. Sheridan, Sister M. Joanne, F. C. Smith, A. G. Swanson, P. Y. Tani, F. J. Taylor, Takashi Terami, J. W. Wegner.

At the business meeting the following officers were elected for the coming year: Chairman, H. D. Colson, Bemidji State Teachers College; Secretary, F. C. Smith, College of St. Thomas; Executive Committee, C. S. Carlson, St. Olaf College, A. G. Hill, North Dakota State College, J. M. H. Olmsted, University of Minnesota.

By invitation of the Executive Committee, Professor M. D. Donsker of the University of Minnesota delivered an address at the morning session entitled *Probability Limit Theorems.*

The following short papers were presented:

1. *A Note on special solutions of partial differential equations related to Laplace's equation*, Professor O. E. Stanaitis, St. Olaf College, introduced by Professor C. S. Carlson.

Some polynomial solutions of the partial differential equation

$$\frac{\partial^2 w}{\partial \alpha^2} - \frac{\partial^2 w}{\partial \beta^2} - k^2 n(n+1)(\mathrm{sn}^2 \alpha - \mathrm{sn}^2 \beta) w = 0$$

have been established by E. T. Whittaker. In a different approach it is shown here that there exists a complete linearly independent system of polynomial solutions which is easily obtained explicitly by simple transformation. The paper concludes with remarks on the applicability of the method to related differential equations.

2. *Some relations among the roots of cubic equations*, Professor W. S. Loud, University of Minnesota.

If the roots of a cubic equation, r_1, r_2, and r_3 are distinct, there is a linear fractional transformation $f(z)$ which takes r_1, r_2, and r_3 into r_2, r_3, and r_1 respectively. The transformation $f(z)$ can be found directly from the cubic equation without calculation of the roots. If the equation

is $x^3+ax+b=0$, $f(z)=(Az+B)/(Cz+D)$, where $A=9b+(-4a^3-27b^2)^{1/2}$, $B=-2a^2$, $C=-6a$, and $D=-9b+(-4a^3-27b^2)^{1/2}$. If the coefficients of the cubic are rational, the coefficients of $f(z)$ are all rational if and only if the discriminant is a square. In this latter case when in addition the three roots are all irrational, $f(z)$ is the only linear fractional transformation among the roots with rational coefficients. In the case of distinct irrational roots, and rational coefficients in $f(z)$, with A, B, C, and D relatively prime integers, if it also happens that $AD-BC=\pm 1$, it is also true that the continued fraction expansions of the three roots are identical after finitely many terms. The equation $x^3-3x+1=0$, for which $f(z)=(z-1)/z$, has all the above special properties.

3. *Vibration of a string with periodic tension*, Mr. H. W. Godderz, University of Minnesota.

The problem of the vibrating string with periodic tension was discussed. The equation

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\mu}(1 - 2\gamma \cos 2\omega t)\frac{\partial^2 y}{\partial x^2}$$

was transformed into Mathiew's equation of canonical form

$$\frac{d^2 f}{dz^2} + (z - 2q \cos 2z)f = 0.$$

For periodic, and non-periodic oscillatory bounded solutions there are definite restrictions placed on a which is a complicated function of q. The results of three special cases having stable solutions were given.

4. *Derivation by rotation of the distance from a line to a point*, Professor Kenneth May, Carleton College.

The familiar proof of the formula for the distance from a line to a point involves drawing an auxiliary line and consideration of two possible cases for its location. These complications are avoided by using the following proof based on rotation of the axes. Let the line and point be given by $x \cos \alpha+y \sin \alpha-p=0$ and (x_0, y_0). Rotate the axes through the angle α. The line and point are now given by $x'=p>0$ and (x_0', y_0'). The directed distance from the line to the point is in all cases $d=x_0'-p$. But $x_0'=x_0 \cos \alpha+y_0 \sin \alpha$, and the desired result follows. Since trigonometry must precede study of the normal form, rotations can easily be introduced before or at this point. This is only one of many simplifications which result from the full exploitation of transformations.

5. *A new mnemonic for* π *(mod 10)*, Professor F. J. Arena, North Dakota State College.

Most mnemonics give the value of π to about 31 or 32 digits. The thirty-third digit in the value of π is zero. The reason why mnemonics do not represent π beyond 32 digits may be because mathematicians do not know how to get past this zero. The following mnemonic has been written in order to show how we can get around this difficulty. By counting the number of letters in each word and reducing that number modulus 10, the following mnemonic gives the-value of π to more than 33 digits:

See, I have a great invention
To assist eager geometricians
Whose problems implicate circle's periphery
And if you required that number pi
Useful transcendental and distinguished constant
For to readily ascertain
Count successive hieroglyphic lettered ciphered word.

$\therefore \quad \pi = 3.14159\ 26535\ 89793\ 23846\ 26433\ 83279\ 50288\ 4\cdots.$

6. *A note on functions harmonic in the unit circle*, Professor W. B. Fulks, University of Minnesota.

The author presents a proof, believed to be new, of a result which is contained in more general theorems in the literature. However, the theorem presented seems not to have been explicitly stated. It is as follows: If $u(r, \alpha)$ is representable in the unit circle by a Poisson-Stieltjes integral and has radial boundary values everywhere zero, then $u(r, \alpha)$ is identically zero in the unit circle.

7. *The inversion of Brillouin zones for thermodynamic solutions of solids*, Mr. C. R. Bonnell, Minneapolis-Honeywell Regulator Company.

The paper demonstrated a method of translating Brillouin zones into reciprocal lattice unit cells by cyclic changes of subscripts of the associated wave vector for an oscillating lattice. The translation was shown to be necessary for the calculation of the normal modes of vibration of the system. Some simple rules were given to predict whether or not the partition function was invariant to the translation.

The general equation of state was derived and the translation carried out for the square lattice, the cubic lattice, the body-centered lattice, and the face-centered lattice.

8. *Nicollet was a mathematician*, Professor W. H. Bussey, University of Minnesota.

The name of Joseph N. Nicollet is well known in Minnesota. His name has been given to Nicollet Avenue and Nicollet Island in Minneapolis, to a county in Southern Minnesota, and to the "Infant Mississippi" which flows into Lake Itasca. Nicollet came to Fort Snelling, Minnesota, in 1836 primarily for the purpose of exploring the upper Mississippi Valley. He completed with scientific accuracy the work of Henry R. Schoolcraft who had recently named Lake Itasca as the "true source" of the Mississippi River.

What is not well known in Minnesota is that Nicollet, before he came to the U. S. A., was an accomplished mathematician, astronomer, and educator, and that his mathematical interests and accomplishments were in such diverse fields as mathematical astronomy and the mathematics of life insurance and probability.

Mr. Bussey's paper told about Nicollet's life in France and how it happened that, at the age of 42 years, he gave up a very successful and distinguished career there for an altogether different career in this country.

FRANKLIN C. SMITH, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at Eastern Kentucky State College, Richmond, Kentucky, on Saturday, April 28, 1951. Dr. R. S. Park, Chairman of the Section, presided at the morning and afternoon sessions.

Fifty-one members were present at the meeting, including the following twenty-nine members of the Association:

H. H. Berry, M. C. Brown, R. C. Brown, Jr., Elsie T. Church, Esther A. Compton, V. F. Cowling, H. H. Downing, Clarence Ford, A. E. Foster, A. W. Goodman, Charles Hatfield, Sr., Aughtum S. Howard, Tadeusz Leser, A. G. McGlasson, W. L. Moore, R. S. Park, Sallie E. Pence, D. W. Pugsley, Sara L. Ripy, G. G. Roberts, W. J. Robinson, W. C. Royster, W. S. Snyder, D. E. South, A. L. Stamper, Guy Stevenson, J. T. Vallandingham, J. A. Ward, and W. M. Zaring.

Officers elected at the meeting were: Chairman, J. A. Ward, University of Kentucky; Secretary, G. G. Roberts, Berea College. The annual meeting in

1952 is to be held at the University of Kentucky, Lexington, Kentucky.

The following papers were presented:

1. *Three dimensional representation of complex roots*, by Professor J. A. Ward, University of Kentucky.

Denote three mutually perpendicular axes in euclidean space by X, Y_r, and Y_i. Let $(a, b+ci)$, with a, b, and c real and $i^2=-1$, satisfy $f(x, y)=0$. The set of points (a, b, c) are called the "real-complex" graph of $f(x, y)=0$. The graphs of plane analytic geometry are identified with that portion of the "real-complex" graph which lies in the X Y_r-plane. Mr. Ward discussed these graphs for several elementary functions and pointed out that for a conic section the real-complex graph will always lie in two perpendicular planes.

2. *On a joint life probability*, by Mr. Sherman Vanaman, Jr., University of Kentucky, introduced by the Secretary.

Standard probability notation with certain extensions was used to prove the following joint life probability theorem: The probability that the joint existence of the lives (x) and (y) will fail in the $(n+1)$st year is given by

$$(A_n \cdot B_n) - (A_{n+1} \cdot B_{n+1})$$

where the symbol $(A_n \cdot B_n)$ represents the probability that (x) and (y) will both survive n years. The result was then used in developing the expression

$$A_{xy} = va_{xy} - a_{xy}$$

for the present value of a \$1 joint whole life insurance.

3. *A study of word frequencies*, by Mr. Strawn Taylor, University of Louisville, introduced by Professor Guy Stevenson.

This study is essentially a pilot study regarding large populations which are subdivided into many subsets, the written language. The objectives of the paper are to obtain answers to the following questions:

1. Can sampling be used to determine frequencies of occurrence of written words in a given population?
2. Does the given author use approximately the same frequency of words in his various publications?
3. Can patterns be established for a writer by which he may be identified?

In answer to the first question, it was discovered that a sample of at least 10,000 words is necessary for the inspection of the more frequently occurring words such as *the, of, and, on, etc.* Secondly, the data indicate that an author (Joyce) does not use the same frequency of words in his two books (*Ulysses* and *Finnegan's Wake*). Thirdly, Mr. Taylor found it possible to set up patterns for different authors. These patterns indicate quite clearly that two different authors (Joyce and Emerson in this case) vary distinctly in their use of words. Mr. Taylor emphasized that this is only a pilot study and that there is great need for much more counting and investigating of the English language, in order to establish all-inclusive norms.

4. *The Burkill integral as a unifying concept in undergraduate mathematics*, by Dr. W. S. Snyder, University of Tennessee.

The theory of the Burkill integral can be utilized to give a unified treatment of Rieman integration in one or more dimensions, of line or surface integrals, of arc length, continuous functions of bounded variation, and the Jordan theory of measure. The following two theorems are the essential tools which lead to all the results usually studied under the above headings in a course in Advanced Calculus. Theorem I: If f is a continuous function of intervals on the line and increases

by subdivision then its Burkill integral exists but may have the value $+\infty$. Theorem II: If f is a function of intervals (n-dimensional) which increases by subdivision and if a constant K exists such that for any interval I, $|f(I)| \leqq K|I|$, then the Burkill integral of f exists.

For precise definitions of the concepts involved, see Saks, *Theory of the Integral*, 2nd Ed., Chap. V, §3.

5. *The old and new methods in the theory of elasticity*, by Professor Tadeusz Leser, University of Kentucky.

After introducing the basic concepts and assumptions of the theory of elasticity, the author presented the tensor notation as compared with the scalar notation and also discussed the increasing use of conformal mapping for problems of twisting and bending.

6. *Inversion with respect to a cubic of the syzygetic pencil*, by Miss Elsie T. Church, University of Kentucky.

With base curve as a cubic of the pencil of syzygetic cubics,

$$F = x_1^3 + x_2^3 + x_3^3 - 3\lambda x_1 x_2 x_3 = 0,$$

an inversion analogous to quadric inversion was defined and the equations of the corresponding cubic transformation were derived. These equations are $x_1 : x_2 : x_3 = f_1 : f_2 : f_3$, where

$$f_1 = \lambda x_1'^2 x_2' - x_1' x_3'^2, \qquad f_2 = \lambda x_1' x_2'^2 - x_2' x_3'^2, \qquad f_3 = x_1'^3 + x_2'^3 - 2\lambda x_1' x_2' x_3'.$$

The curves $f_1 = 0$, $f_2 = 0$, $f_3 = 0$, intersect in seven common points and thus the transformation sets up a (1, 2) correspondence. Attention was called to the interesting results obtained with these seven points when λ is assigned the values 1, ω, ω^2, respectively.

7. *The knight's move on an infinite chess board*, by Professor A. W. Goodman, University of Kentucky.

A sequence of moves of the knight is called a grand tour if the knight occupies each square once and only once. Kurschak has proved that a grand tour is possible on an infinite chess board, *Acta Szeged*, vol. 4, pp. 12–13. Let $[r, s]$ denote the move of a generalized knight, where $[2, 1]$ is the ordinary knight move. A necessary condition for a grand tour is obviously $(r, s) = 1$ and $r + s$ odd. It is not known whether these conditions are sufficient.

AUGHTUM S. HOWARD, *Secretary*

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The tenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Manhattan College in New York City, New York, on Saturday, April 7, 1951. Dr. I. A. Dodes, the High School Vice-Chairman, presided at the morning session; Brother Bernard Alfred, Chairman of the Section, presided at the business meeting which followed the morning session, and he and Professor James Singer, Collegiate Vice-Chairman, presided at the afternoon session.

One hundred and twelve persons attended the meeting, including the following sixty-four members of the Association:

Brother Bernard Alfred, A. A. Bennett, Samuel Borofsky, C. B. Boyer, A. B. Brown, J. H. Bushey, Jewell H. Bushey, A. J. Carlan, Charles Clos, L. W. Cohen, T. F. Cope, W. H. H. Cowles, P. H. Daus, I. A. Dodes, J. N. Eastham, W. H. Fagerstrom, William Feller, Edward Fleisher, R. M. Foster, Leona Freeman, Harriet Griffin, George Grossman, G. C. Helme, W. M. Hirsch,

J. H. Hlavaty, T. R. Humphreys, L. C. Hutchinson, H. S. Kieval, J. J. Kinsella, R. J. Kohlmeyer, David Kotler, H. C. Kranzer, Helen Kutman, Martin Maltenfort, D. May H. Maria, Audrey Michaels, Martin Milgram, F. H. Miller, A. J. Mortola, W. R. Murray, D. S. Nathan, M. A. Nordgaard, C. J. Oberist, Eugene Odin, Walter Prenowitz, Hymen Rensin, Moses Richardson, John Riordan, Rose Roll, H. D. Ruderman, J. P. Russell, Charles Salkind, Arthur Schach, Abraham Schwartz, Aaron Shapiro, E. I. Shapiro, L. G. Sigler, James Singer, C. G. Solky, Mildred M. Sullivan, R. L. Swain, P. M. Treuenfels, Alan Wayne, M. E. White.

The following officers were elected for the coming year: Chairman, James Singer, Brooklyn College; Collegiate Vice-Chairman, L. F. Ollmann, Hofstra College; High School Vice-Chairman, E. I. Shapiro, Abraham Lincoln High School; Secretary, H. S. Kieval, Brooklyn College; Treasurer, Aaron Shapiro, Midwood High School. A resolution was passed which provided that the by-laws of the Section be interpreted to include Junior and Community Colleges in the naming of a representative from each collegiate department of mathematics to the Executive Committee of the Section. A resolution, that a second meeting of the Section be held in the fall of each year to be devoted to short papers, was referred to the Executive Committee for further study. The eleventh annual meeting will be held in the Spring of 1952.

Brother Bonaventure Thomas, President of Manhattan College, welcomed the people at the meeting, and then the following papers were presented.

1. *A theory of distribution*, by Professor Claude Chevalley, Columbia University, introduced by the Secretary.

The main object of Schwarz's theory of distributions is to put on a rigorous and uniform mathematical footing certain types of computational methods which are in frequent use by the physicists, like, for instance, the Heaviside Calculus.

One of the most frequent sources of difficulty in trying to make simple and general statements in analysis is the fact that a given function, even if it is continuous, does not necessarily have a derivative. The distributions are a class of mathematical objects wider than the class of functions, and inside which the operation of derivation is possible without any restriction. Thus, in particular, a continuous function always has derivatives of all orders; these derivatives are of course not functions in general, but distributions.

In order to define the notion of distribution, one introduces first the set D of indefinitely differentiable functions which are zero outside bounded sets (the bounded set depending on the function). This set is a vector space: the sum of two functions in D is in D, and the product by a constant of a function in D is in D. The distributions are then defined to be the linear functionals on D which are continuous relatively to a suitable notion of convergence for functions in D. A function $f(x)$ is identified with the functional which assigns to every g in D the number $\int_{-\infty}^{\infty} f(x)g(x)dx$. The derivative of a distribution S assigns to every function g in D the number $-S(dg/dx)$.

2. *Changing philosophy and content in tenth year mathematics*, by Dr. J. H. Hlavaty, Bronx High School of Science.

In the prolonged attempt of American secondary schools to adapt the curriculum to fit the ever-broadening educational aims, the continuous transformations in the philosophy and content of instruction in plane geometry has been a particularly instructive sidelight. A study of the recommendations of authoritative national bodies in the field of general education and in the field of mathematical instruction and a study of recent representative texts in plane geometry indicate

the following major attempts to reform instruction in demonstrative geometry: (1) Improving the methods of instruction; (2) Reformulating the aims of instruction; (3) Teaching geometry with a stress on practical applications; (4) Teaching geometry to develop the ability to think clearly; (5) The integrated mathematics movement; (6) The arithmetization of geometry; (7) Teaching geometry as an example of an empirical science.

However, it has proved impossible to rationalize the instruction in traditional demonstrative geometry either from the point of view of general education or from that of mathematical instruction. This failure is traceable to the fact that the reformers have in all cases retained traditional geometry or its method as the core of the mathematical instruction of the tenth year. It is suggested that there is a need for a course in general mathematics through the tenth year. In the tenth year the course should stress: (1) Methods of thinking (induction, deduction, statistical inference, "clear thinking"); (2) The uses of mathematics as a means of extending our understanding of the world (elementary statistics, trigonometry, spherical geometry); (3) The beginnings of these methods that lead to analytic geometry and the calculus.

3. *Changes in secondary school mathematics from 1900 to 1950*, by Mr. Joseph Orleans, George Washington High School, introduced by the Secretary.

At the beginning of the twentieth century the syllabi in mathematics in the American High Schools were determined largely by the requirements for entering the colleges. In 1902, a committee of the American Mathematical Society made recommendations for the content of the elementary algebra, the higher algebra, plane and solid geometry, and trigonometry. This report was described by David Eugene Smith as 'rather inclusive through its lack of precision.' It was to serve fairly well as starting point for reform.

In 1900 the College Entrance Examinations Board had come into being and gradually began through its examinations to exert its influence on syllabi.

In 1908 the work of the International Commission on the Teaching of Mathematics brought to the attention of teachers that other countries were ahead of us in the modernization of syllabi. In 1916, the National Committee on Mathematical Requirements began (the report appeared in 1923) with suggestions for the elimination of non-essentials, the retention of those things which should best meet the needs of pupils, and the introduction of such modern material as would strengthen the work without making it unreasonably difficult. In 1922, the College Entrance Examinations Board issued revised requirements which represented the combined judgment of the colleges and the secondary schools, and which set a new standard to eliminate much of the work for which no reasonable justification could be found. In the meantime the junior high school was developing with the introduction of informal geometry and the beginning of algebra earlier than the ninth year.

In 1930, one heard again the complaint that courses in secondary mathematics were still far from satisfactory. Recommendations were made for curriculum construction based on difficulties which pupils experience in the study of mathematics, and thus to eliminate the difficulties: (1) the introduction of arithmetic into the high school, (2) the teaching of algebra of a practical character in earlier years, (3) the stress on intuitional geometry.

Then came World War II with a wild scramble everywhere to stress the importance of mathematics for everybody with special courses—war courses—and refresher work in arithmetic, in basic mathematics, and related mathematics. In 1945, there was published the Report of the Commission on Post War Plans with the recommendations that the high school must come to grips with its dual responsibility, (1) to provide sound mathematical training for our future leaders in science, mathematics, and other learned fields, and (2) to insure mathematical competence for the ordinary affairs of life for all citizens as a part of a general education.

The questions that arise are, (1) What is the place of the able students in the present situation?, (2) What do schools do and what can they do to provide for these students?, (3) What is the effect upon their development and their preparation for work in higher institutions of learning?

4. *On chance fluctuations*, by Professor William Feller, Princeton University.

The speaker discussed fluctuations in waiting lines, queues, etc. He emphasized that the fluctuations are much larger than generally expected, and that they easily produce spurious effects resembling trends.

5. *Physical concepts in freshman mathematics*, by Professor A. A. Bennett, Brown University.

Texts and teachers currently use many terms dependent upon the human body, the solar system or clocks, in defining at freshman level, supposedly mathematical notions. On the other hand they often insist that only numbers (with their relations and operations) enter into equations. This view rejects the possible role of vectors in equations, and accepts scalars only when they are dimensionless or have been referred to arbitrarily specified units. It ignores all such rigorous studies as abstract group theory, topology, etc. But other views are possible. One may deal with "denominate numbers" either with or without reference to prior chosen units. There seems often great gain in so doing.

H. S. KIEVAL, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1–2, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.

ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.

INDIANA, Indiana University, Bloomington, May 3, 1952.

IOWA, Coe College, Cedar Rapids, April 18–19, 1952.

KANSAS, Bethany College, Lindsborg, March 29, 1952.

KENTUCKY, University of Kentucky, Lexington, April 19, 1952.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Virginia Military Institute, Lexington, April 26, 1952.

METROPOLITAN NEW YORK, Hofstra College, Hempstead, New York, April, 1952.

MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.

MINNESOTA, College of St. Catherine, St. Paul, May 10, 1952.

MISSOURI, Lindenwood College, St. Charles, May 2, 1952.

NEBRASKA, University of Nebraska, Lincoln, May 3, 1952.

NORTHERN CALIFORNIA

OHIO, Ohio State University, Columbus, April 19, 1952.

OKLAHOMA

PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.

PHILADELPHIA

ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May 23–24, 1952.

SOUTHEASTERN, Georgia Institute of Technology and Agnes Scott College, Atlanta, March 21–22, 1952.

SOUTHERN CALIFORNIA, Occidental College, Los Angeles, March 8, 1952.

SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.

TEXAS, East Texas State Teachers College, Commerce, April 25–26, 1952.

UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May 10, 1952.

WISCONSIN, Milwaukee, May 10, 1952.

Timely McGRAW-HILL Books

INTERMEDIATE ALGEBRA

By PAUL K. REES, Louisiana State University, and FRED W. SPARKS, Texas Technological College. 328 pages, $3.50

An exceptionally clear and teachable presentation of intermediate algebra which brings the student from a limited background in algebra through simultaneous quadratic equations; ratio, proportion, and variation; logarithms; etc. Topics of particular difficulty are considered at length and in detail. 2500 problems, of progressive difficulty, are a particular feature of the text.

PLANE TRIGONOMETRY

By E. RICHARD HEINEMAN, Texas Technological College. 167 pages (with tables), $3.00

Designed especially for students with average mathematical background, this text seeks to establish in them the habit of logical and independent thinking. Memory work is reduced to a minimum, and all unnecessary formulas and concepts have been omitted. Offers 1274 carefully graded problems. An alternate edition is also available which contains 1420 problems nearly all of which differ from those appearing in the regular edition.

DIFFERENTIAL EQUATIONS

By WALTER LEIGHTON, Washington University. *International Series in Pure and Applied Mathematics.* Ready in May

A *unified* treatment, unusually clear and rigorous, this text covers the material customarily given in a first course—with considerable attention devoted to underlying theory. The author has concentrated on making the *fundamental existence theorem* a central idea and one which becomes part of the student's experience.

ADVANCED ENGINEERING MATHEMATICS

By C. R. WYLIE, JR., University of Utah. 640 pages, $7.50

Provides an introduction to those fields of advanced mathematics which are currently of engineering significance. Covers such topics as ordinary and partial differential equations, Fourier series and the Fourier integral, vector analysis, numerical solution of equations and systems of equations, finite differences, least squares, etc. Relationships of various topics are emphasized.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, INC.

330 West 42nd Street **New York 36, N.Y.**

ADVANCED

By Wilfred University

This new book is intended to provide sufficient material for a course in advanced calculus up to a year or more in length. It is hoped that the great variety of topics covered will also make this work useful as a reference book.

It is assumed that the student has had courses in algebra, analytic geometry, and calculus. The first chapter in ADVANCED CALCULUS provides a concise review of these subjects; it can therefore be used to determine whether the student is prepared for the work ahead.

While it is commonly stated that advanced calculus must be regarded as a collection of unrelated topics, the author is firmly convinced that this is not the case, that there is both a natural order for studying the various parts of the subject and a simple logical structure which ties these parts together. One of the aims of this book is to demonstrate the truth of this conviction. In doing so, it is hoped that a contribution is being made to the "teachability" of the subject.

While the chapters are laid out according to a very definite plan, a certain flexibility does remain. Thus a number of topics of lesser importance can be omitted if desired. The individual chapters are to some extent self-contained, repetition being preferred to cross-referencing whenever feasible.

Perhaps the chief unifying device in the book is that of vectors, which occur in some form in practically every part of the theory. By the frequent use of vectors, one is able to relate the mathematical concepts to familiar geometrical and physical notions.

While detailed engineering and physical applications are considered in separate sections and in problems, the physical point of view is strongly maintained throughout. Indeed, it has been considered more important to make the mathematical theorems physically meaningful and intuitively evident on that basis than to supply all the details of a rigorous mathematical proof. In keeping with this approach, simple numerical examples of important concepts are given and numerical methods receive more than their usual stress. It is felt that if the student can grasp the directional derivative in terms of *differences,* and the line integral in terms of *sums,* then the passage to the limit is relatively simple.

Because of the emphasis on the physical point of view and the absence of complete mathematical proofs, it might be felt that this book would be of use only to the engineer and physicist and not to the mathematician. However, it is believed that this treatment of the subject can be of great benefit to the mathematician. Once he has obtained a strong intuitive grasp of the subject, a very small amount of ε and δ work in the modern spirit will enable him to fill in the gaps in the theory. The contact with the physical interpretations should also prove of benefit in bridging the gap between mathematicians and physicists so evident at the present time.

Many exercises and problems are furnished, with answers, on which the student can try out his understanding. Some of the less important topics are themselves treated in problems, with "hints" for solution.

ADDISON-WESLEY PRESS, INC

CALCULUS

Kaplan,
of Michigan

ADVANCED CALCULUS WILL BE PUBLISHED IN APRIL!

Professor Franz E. Hohn of the University of Illinois used the preliminary edition of Kaplan's ADVANCED CALCULUS as a text for his course, and he read in detail the entire final manuscript. Here is his estimate of the book.

"Kaplan's ADVANCED CALCULUS is outstanding from the point of view of exposition. It does not attempt to cover quite as wide a range of formal topics as do some of its predecessors. On the other hand, it contains unusually full and clear explanations of the material treated in it, and it contains an extraordinarily large number of illustrative examples, many of which are especially cogent.

"The explanations are executed with uncommon care as to matters of precise statement and as to pointing up fundamental issues. This is a welcome emphasis entirely in accord with current trends in mathematical teaching.

"The material of this book is so arranged as to possess unusual smoothness and impact, and it is selected with excellent judgment. Indeed, there is no other advanced calculus text whose contents are better geared to the requirements of workers in the applied fields, but at the same time, the student of pure mathematics will obtain here a firmer grounding in principles of analysis than he would from any other book on this level. In fact, the book strikes such a neat balance between rigor and formal technique that it is, in my opinion, *the most valuable book for teaching advanced calculus that is presently available.*

"The need for this kind of book, combining emphasis on analytical principles with sound technique, has so widely been felt that I predict that Kaplan's book will rapidly become a leader in the field."

FRANZ E. HOHN
University of Illinois

ADVANCED CALCULUS

Cloth bound—6″ x 9″—Approx. 750 pages—Beautifully illustrated—$8.50

Table of Contents

Introduction. Review of algebra, analytic geometry, and calculus.
Chapter 1. Vectors
Chapter 2. Differential calculus of functions of several variables
Chapter 3. Vector differential calculus
Chapter 4. Integral calculus of functions of several variables
Chapter 5. Vector integral calculus
Chapter 6. Infinite series
Chapter 7. Fourier series and orthogonal functions
Chapter 8. Ordinary differential equations
Chapter 9. Functions of a complex variable
Chapter 10. Partial differential equations

An *important* new book in the Addison-Wesley Mathematics Series.

SEND FOR COPIES ON APPROVAL

CAMBRIDGE 42, MASSACHUSETTS

Recent Texts of Interest

Longley-Smith-Wilson

Analytic Geometry and Calculus

A new full-year course covering plane and solid analytic geometry and the differential and integral calculus (including a chapter on differential equations). Early presentation of the elementary technique of integration.

Rosenbach-Whitman:

Intermediate Algebra for Colleges

Elementary principles are here explained with the utmost clarity for students with a limited background in algebra. Reviews fundamentals, and covers topics usually completed in two years of high school—through progressions and logarithms.

GINN and Company

Home Office: Boston Sales Offices: New York 11 Chicago 16
Atlanta 3 Dallas 1 Columbus 16 San Francisco 3 Toronto 5

Lloyd M. SMAIL'S

CALCULUS

grows in popularity as teachers discover its teaching qualities

AMONG THE MANY INSTITUTIONS NOW USING IT FOR THE *SECOND* YEAR:

Colgate University
Dartmouth College
Indiana University
Iowa State College of Agriculture and Mechanic Arts
Lehigh University
Purdue University
Southern Methodist University
United States Military Academy
University of British Columbia
University of Maryland
University of Michigan
University of Minnesota
University of New Mexico
University of Pennsylvania
University of Utah
University of Wisconsin
Western Reserve University
Wright Division, Chicago City Junior College

235 pp. $4.50

Appleton-Century-Crofts
35 West 32nd Street, New York 1, N. Y.

COLLEGE MATHEMATICS TEXTS

By WILLIAM L. HART

Mathematics of Investment, Third Edition

Presents a standard course in annuities certain, with an introduction to life annuities and life insurance . . . Emphasizes problems under the *simple case* for annuities certain, with the general case segregated in a final chapter . . . Very complete tables, particularly for small interest rates. Text, $3.25. With tables, $4.25. Bound with tables and Hart's *Essentials of College Algebra,* $5.00. Tables separately, $2.00

Introduction to the Mathematics of Business

The minimum essentials presented on the level of college students who have had only one year of high school algebra . . . May be used for a one-semester course (or less) for freshmen with more advanced algebraic preparation. $3.25

Intermediate Algebra for Colleges

Presents a collegiate substitute for third semester high school algebra . . . A frankly elementary treatment of the fundamentals of algebra in language suitable to the maturity of college students . . . Does not aim to overlap typically collegiate algebra. $3.00

By TOMLINSON FORT

Calculus

Mathematically mature but not difficult . . . designed for regular college calculus courses, providing also a thorough grounding in calculus for engineering students . . . Includes all that is given in the usual course and more, so that a suitable selection may be made . . . The first chapter is on infinite series—basic to the calculus study. $4.75

D. C. HEATH AND COMPANY

Sales Offices: **New York, Chicago, San Francisco, Atlanta, Dallas** ***Home Office:*** **Boston**

INTERMEDIATE ALGEBRA FOR COLLEGES

By **Earle B. Miller,** ***Illinois College***

A clear and carefully organized textbook for students who have had only one year of algebra in high school. The author's experience with the difficulties which beset these students has resulted in the following important features: full text explanations, an emphasis on algebraic techniques, an early introduction of the fundamental concept of function and of graphical representation, plus numerous notes, formal proofs, a helpful treatment of logarithms, and a generous number of carefully graded exercises.

361 pages, Illustrated

COLLEGE ALGEBRA

By **Earle B. Miller,** ***Illinois College; and***
Robert M. Thrall, ***University of Michigan***

For the first-year college student who wants a thorough grounding in the subject. The book opens with a review of topics from the student's earlier work. The exposition avoids the complexity of the too advanced text and the sterility of the oversimplified presentation. The subject matter follows traditional lines except in places where modern trends suggest additions which increase utility and simplify theory.

493 pages, 36 Illustrations

ANALYTIC GEOMETRY

By **Alfred L. Nelson, Karl W. Folley,** ***and***
William M. Borgman—*all of Wayne University*

A textbook recommended for use in those courses planned as a preparation for the calculus rather than a study of geometry. This makes it of maximum value to the future student of the calculus, the basic sciences, and engineering. Special attention has been given to developing a thorough understanding of the coordinate system. The book contains brief tables of trigonometric, exponential, and logarithmic functions.

215 pages, 110 Illustrations

Examination Copies Available

THE RONALD PRESS COMPANY
15 East 26th Street New York 10, N.Y.

Ready in April—

CALCULUS

By* Atherton Hall Sprague, *Amherst College

This textbook constitutes a logically complete course in the calculus. In addition, it contains a chapter on polar coordinates and an extensive one on solid analytic geometry. The treatment is rigorous, but analytic proofs which might cause difficulty are accompanied by detailed explanation. There are the usual applications to geometry and physics, with appropriate definitions bridging the gap between non-calculus concepts and those requiring the calculus. An abundance of carefully graded problems and illustrative examples are presented throughout the book. Includes answers to all problems.

Approx. 575 pages

Ready in March—

THE NATURE OF NUMBER

An Approach to Basic Ideas of Modern Mathematics

By* Roy Dubisch, *Fresno State College

This new volume is written for those seeking to gain an overall view of modern mathematics and an insight into the nature of its theory as a subject in which ideas have the dominant role. Going back to the fundamental ideas involved in counting and arithmetic, the discussion leads the reader to discern reasons and motivations by developing one representative part of modern mathematics known as the theory of linear algebras. A work of special interest to those preparing to teach mathematics or concerned with it in connection with scientific or other pursuits.

Approx. 160 pages

THE RONALD PRESS COMPANY

15 East 26th Street New York 10, N.Y.

Featuring the Use of VECTOR METHODS
Where Proofs Would Be Simplified

ANALYTIC GEOMETRY

By W. Kelso Morrill, Professor of Mathematics, The Johns Hopkins University

This is a new introductory textbook, designed for use in a one or two semester course. An important feature is the introduction of vector methods, extremely useful to students in the study of the calculus and other sciences; but this technique is used only where the author felt proofs would thus be simplified. Review exercises, with answers given for every other problem. Included are recommended successful chapter combinations for course adaptation.

400 pages, Illustrated, $3.50

For a ONE or TWO Semester Course

MATHEMATICS OF FINANCE

By Floyd S. Harper, Head, Department of Actuarial Science, Drake University

Suitable for engineering and business administration students with one idea introduced at a time in conveniently arranged lessons. Over 500 well graded exercises. Especially useful to students planning to major in Actuarial Science. 327 pages, $3.50.

Send for EXAMINATION COPIES

INTERNATIONAL TEXTBOOK COMPANY
DEPT. AMM1, SCRANTON 9, PENNSYLVANIA

Now available:

PROFESSIONAL OPPORTUNITIES IN MATHEMATICS

The report of a Committee of the

Mathematical Association of America,

reprinted from the January 1951 AMERICAN MATHEMATICAL MONTHLY

24 pages, paper covers

25¢ for single copies; 10¢ each for orders of ten or more

Send orders to: HARRY M. GEHMAN, Secretary-Treasurer
Mathematical Association of America
University of Buffalo
Buffalo 14, New York

Enthusiastically received . . .

ALGEBRA FOR COMMERCE AND LIBERAL ARTS

By A. K. BETTINGER and W. A. DWYER. Teachers who have examined this text are particularly impressed with the clear and simple exposition and the general arrangement of the material. Well-graded illustrative examples, drill exercises and word problems, and review exercises assure a high degree of mastery by students of both adequate and inadequate backgrounds. *225 pages, $3.00. Answers available.*

Currently being used at Northwestern, North Dakota, Creighton, Hofstra, Detroit, Drexel Institute, Mary Holmes Junior College, St. Louis Preparatory Seminary, and many other schools.

Fourth Revised Edition

THE NOMOGRAM

By H. J. ALLCOCK and J. REGINALD JONES. Revised by J. G. L. MICHEL. An effective, well-written text that combines general theory with practical instructions for the construction and use of all classes of computation charts having scientific or industrial applications. Numerous illustrations and examples. *237 pages, $3.75*

An invaluable book

REGULAR POLYTOPES

By H. S. M. COXETER. A highly readable study of aspects of the five regular solids with numerous excellent drawings. *321 pages, $10.00*

"Professor Coxeter . . . has presented us with what is, and will probably be for many years, the only organized treatment of the subject."—*American Mathematical Monthly*

You are cordially invited to send for examination copies.

2 W. 45 St. **PITMAN** **New York 36**

Math texts by M. Wiles Keller, Purdue University

BASIC ALGEBRA

SPRING PUBLICATION

An intermediate algebra text which offers (1) careful and thorough introduction of each topic (2) step-by-step illustration of rules (3) sample problems to cover all important variations and (4) a large number of exercises and problems for drill.

BASIC MATHEMATICS

(with J. H. Zant, *Oklahoma A & M College*)

Designed to strengthen and review students in arithmetic, algebra, and trigonometry, *Basic Mathematics* has a unique testing program which shows exactly where each student's weaknesses lie.

COLLEGE ALGEBRA

A text with a highly successful approach based on (1) extensive drill on fundamental operations (2) topics in the order of difficulty and (3) a careful pattern of review and repetitive drill.

Houghton Mifflin Company

Boston New York Chicago Dallas San Francisco

from Macmillan . .

• Calculus

John F. Randolph, *University of Rochester*

Maintaining a sound balance between formal problem material and a mathematically rigorous presentation, Professor Randolph's new text offers a flexible, well-proportioned, modern treatment of the theory and applications of calculus. There is enough routine knowledge given to provide for the needs of the average student, and the very good student is challenged by the many proofs and more advanced aspects of the subject which are found in the appendix. The problems are numerous, practical, and well-illustrated. *Ready in March*

• Elementary Differential Equations

Earl D. Rainville, *University of Michigan*

This text, containing all the material of the author's successful *Short Course in Differential Equations,* provides a complete introduction to elementary differential equations in a full year's course for students who have had standard calculus. It offers both a sound training in good techniques for obtaining solutions and an explanation of the basic theory behind the techniques. An exceptionally fine presentation of series solutions provides a thorough discussion of the powers series methods and an extensive introduction to Fourier series and the solution of boundary value problems in partial differential equations. The organization of the text is flexible and permits freedom in choice of topics for the instructor. *Ready in April*

The Macmillan Company

60 Fifth Ave., New York, N. Y.

Just Published . . .

a new book to facilitate your teaching and speed the student's learning

BUSINESS MATHEMATICS

by WALTER F. CASSIDY and C. CARL ROBUSTO, Professors of Business Mathematics, St. John's University

OUTSTANDING FEATURES:

- A streamlined and highly readable presentation makes this practically a self-teaching text. Concepts and definitions are stated in simple and easy-to-understand terms. The essential material of Business Mathematics is condensed to an amount readily assimilated by the student.

- After discussing a point, the text immediately follows the explanation with an illustrative example worked out in detail. The completeness of student understanding makes long, classroom interpretations unnecessary.

- Mathematical rigor has been avoided in the belief that it is neither desirable nor necessary on this particular level. Material proceeds from simple concepts in arithmetic to the more complex ones in life insurance.

- Requiring only a year of high-school algebra as a prerequisite, this text can be adapted to a variety of courses—Business Mathematics, Mathematics of Finance, Mathematics of Investment. Ideal as a supplementary text for courses in accounting, finance and taxation.

419 pages | 5½" x 8½" | Published 1952

Send For Your Copy Today



GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 59 NUMBER 4

CONTENTS

Mappings of Vector Spaces and the Theory of Matrices . . . . . W. G. LEAVITT 219

Continuous Functions without Derivatives . . . . K. A. BUSH 222

Sums of Powers of Numbers Having a Given Exponent Modulo a Prime . . . . . RAYMOND MOLLER 226

Maxima and Minima under Restraint . . . . . C. G. PHIPPS 230

Mathematical Notes . . . . . ARTHUR PORGES, S. S. WALTERS, LEONARD CARLITZ 236

Classroom Notes. . . . . J. P. BALLANTINE AND A. R. JERBERT, C. M. FULTON, M. J. NORRIS, SAMUEL BOURNE 242

Elementary Problems and Solutions 249

Advanced Problems and Solutions 254

Recent Publications 263

Clubs and Allied Activities 267

News and Notices 270

Mathematical Association of America

New Members 279

Report of the Treasurer for the Year 1951 281

April Meeting of the Kansas Section 283

May Meeting of the Illinois Section 285

May Meeting of the Upper New York State Section 287

APRIL 1952

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL
PATRICK DU VAL
HOWARD EVES
F. A. FICKEN
J. W. GIVENS
L. M. KELLY
H. D. LARSEN
IVAN NIVEN
EDITH R. SCHNECKENBURGER
GEORGE SPRINGER
E. P. STARKE
G. B. THOMAS
E. P. VANCE

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, SAUNDERS MACLANE, University of Chicago
Honorary President, W. D. CAIRNS, Oberlin College
First Vice-President, F. L. GRIFFIN, Reed College
Second Vice-President, JEWELL H. BUSHEY, Hunter College
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo
Editor, C. B. ALLENDOERFER, University of Washington

Additional Members of the Board of Governors: E. B. ALLEN, C. F. BARR, T. A. BICKERSTAFF, J. C. BRIXEY, R. H. CAMERON, G. R. CLEMENTS, T. F. COPE, H. S. M. COXETER, W. M. DAVIS, J. M. EARL, G. M. EWING, L. R. FORD, TOMLINSON FORT, J. S. FRAME, A. E. GAULT, R. E. GILMAN, E. H. HANSON, M. R. HESTENES, AUGHTUM S. HOWARD, C. G. JAEGER, B. W. JONES, M. S. KNEBELMAN, R. E. LANGER, D. H. LEHMER, C. C. MACDUFFEE, W. E. MILNE, H. P. PETTIT, J. C. POLLEY, G. E. RAYNOR, E. B. ROESSLER, R. G. SANGER, F. H. STEEN, EARL WALDEN, MARIE J. WEISS, F. B. WILEY

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

MAPPINGS OF VECTOR SPACES AND THE THEORY OF MATRICES

W. G. LEAVITT, University of Nebraska

1. Introduction. The concept of a matrix is best introduced, it is now generally agreed, by way of the theory of finite-dimensional vector spaces (sometimes called linear spaces). Such a space may, if one prefers, be defined abstractly [1; p. 167]. One may, however, define a vector space as a set of n-tuples (from some field). This entails no loss of generality, while having the advantage of greater familiarity. The operations in the space (addition and scalar multiplication) may then be defined precisely as if the elements of the n-tuple were just the rectangular components of a vector as ordinarily conceived, say, in physical problems (the sum of two vectors is obtained by adding their respective components, and so on). Again, one may think of the space as a set of single-rowed matrices, for here also the two operations are defined in exactly the same manner.

In any vector space there may exist subsets which are closed with respect to addition and scalar multiplication, a set being closed with respect to an operation if every operation on members of the set yields another member of the set. Such a set satisfies the postulates of a vector space, and is thus a subspace of the original space. In one approach to the theory of matrices, a matrix is regarded as simply a linear operator mapping the space either onto itself or onto a proper subspace of itself. By a linear operation one means an operation T such that $T(c_1V_1+c_2V_2)=c_1T(V_1)+c_2T(V_2)$ for any V_1, V_2 members of the space. This approach is, from many points of view, the most enlightening of all the various introductions to matrix theory. It is, for example, very natural from the point of view of geometry, where the behavior of a space under linear mappings is of fundamental importance. Such mappings, appearing as linear transformations, also have many important applications to physical problems. Again, the problem, in the theory of equations, of the solution of systems of linear equations appears naturally as an inverse mapping problem. This permits, for example, the number of independent solutions of the system to be the dimension of the subspace mapped by the coefficient matrix into the zero vector. This particular approach to the theory of matrices is found efficacious, moreover, in providing proofs which are both simple and elegant for many of the theorems of the algebra of matrices.

It is intended to illustrate this simplicity here by presenting a proof by vector space methods of a theorem of Frobenius on the ranks of matric products. For this presentation it is expected that the reader has some acquaintance with the fundamental concepts of vector spaces (independence, basis, and dimension). In addition, a few of the elementary theorems on vector spaces will be

used without proof (*e.g.*, *Any independent subset of a space forms part of a basis*, and *A space has dimension greater than or equal to that of any of its subspaces*). The reader is also expected to know the definitions of the ordinary matrix operations (addition, multiplication, and scalar multiplication).

2. Linear mappings of vector spaces. The base space S will be chosen to be the set of all n-tuples from some field (the field of real numbers, if the reader prefers). This space of n-tuples is easily seen to satisfy the abstract postulates of a vector space, and in fact every finite-dimensional vector space is known to be isomorphic with some such space [1; p. 176]. Thus unless one prefers the abstract formulation, consideration may as well be limited to spaces of such n-tuples. As indicated above, a vector may be regarded as a matrix with a single row of n elements. This permits a definition of multiplication (on the right) of vectors by any matrix having n rows. A linear mapping of S (or a subspace of S) will then be defined to be the process of multiplying the vectors of the space (on the right) by a fixed n-by-n matrix A. If the reader prefers the abstract definition of the linear mapping, the matrix is simply the symbol of the operation, and the matrix, if one wishes to exhibit it at all, appears as the set of coefficients (in terms of a basis) of the images of the base vectors. The image of S under the mapping will be designated SA.

In proving the theorem of Frobenius, use is made of the two lemmas following, and since they may be new to some readers, their proofs are included. Let S_1 be a subspace of S, and let $N(A)$ be the set of all $V \in S$ for which $VA = 0$, then the intersection $N_1 = N \cap S_1$ is the set of all members of S_1 mapped into zero. If B is any space, let $d(B)$ denote the dimension of B.

LEMMA 1. *The sets N_1 and S_1A are subspaces such that $d(S_1) = d(N_1) + d(S_1A)$.*

It is easily verified that N_1 and S_1A are vector spaces. This may be checked directly, or one may cite the fact that N_1 is the intersection of subspaces N and S_1, while S_1A is the image of S_1 under a homomorphic mapping. The dimensions $d(N_1)$ and $d(S_1A)$ therefore exist. Let the set $\{V_i\}$ be a basis for N_1 and $\{W_j\}$ a basis for S_1A. Clearly there exists some set $\{U_j\}$ of inverse images (*i.e.*, such that $U_jA = W_j$). The set $\{V_i, U_j\}$ is independent, for any linear relation among its members would map into a linear relation among the members of the basis $\{W_j\}$. Thus $d(S_1A) + d(N_1) \leqq d(S_1)$.

But $VA \in S_1A$ for any $V \in S_1$, and so $VA = \sum c_jW_j$. Thus $V - \sum c_jU_j$ maps into zero, and being therefore in N_1, is expressible in terms of $\{V_i\}$. Hence V is expressible in terms of $\{V_i, U_j\}$ and so $d(S_1) - d(N_1) \leqq d(S_1A)$. This completes the proof.

For the matrix A of the mapping, the *rank* $\rho(A)$ and the so-called *nullity* $\nu(A)$ are defined as $\rho(A) = d(SA)$ and $\nu(A) = d(N(A))$. Since $d(S) = n$, the lemma implies the relation $\rho(A) + \nu(A) = n$. The rank so defined may be shown to be equivalent to other definitions of rank, such as the number of rows of the largest non-zero minor, or the number of independent rows or columns.

This lemma also yields interesting results in the theory of equations. Thus the system of equations $VA=Q$ will be solvable if and only if $Q\in SA$, and the number of independent parameters in the solution will be precisely $\nu(A)$. That is, if V maps into Q, then the solutions are formed by the residue class of V modulo $N(A)$.

Now let S_1 and S_2 be subspaces of S. The intersection $S_1\cap S_2$ and the sum S_1+S_2 (the set of all sums of vectors from S_1 and S_2) are known to be themselves vector spaces [1; p. 170, or by checking directly].

LEMMA 2. $d(S_1+S_2)=d(S_1)+d(S_2)-d(S_1\cap S_2)$.*

Since $S_1\cap S_2$ is a subspace of both S_1 and S_2, its basis is a part of a basis for either. If $\{U_i\}$ is a basis for $S_1\cap S_2$, then bases for S_1 and S_2 may be chosen respectively as $\{U_i, V_j\}$ and $\{U_i, W_k\}$. Clearly every vector in S_1+S_2 is expressible in terms of the set $\{U_i, V_j, W_k\}$, and this set contains $d(S_1)+d(S_2)-d(S_1\cap S_2)$ vectors. Further, this set is independent, for otherwise there is either a linear relation among some or all of the members of a basis ($\{U_i, V_j\}$ or $\{U_i, W_k\}$), or there exists some linear combination W of members of $\{W_k\}$ which is expressible in terms of $\{U_i, V_j\}$. But then $W\in S_1\cap S_2$ and so again there is a relation among the members of the basis $\{U_i, W_k\}$. This completes the proof.

The above lemmas make possible an easy proof of the theorem of Frobenius. Let A, B, and C be n-by-n matrices; then

THEOREM 1. $\nu(B)+\nu(ABC)\leqq\nu(AB)+\nu(BC)$.†

Consider the mapping: S onto SA. If $V\in N(AB)$, then clearly the mapped element VA satisfies $(VA)B=0$. Thus since $N(AB)A$ is the image of $N(AB)$, it follows that $N(AB)A=SA\cap N(B)$. Then by Lemma 1, $\nu(AB)=\nu(A)+d(SA\cap N(B))$, and hence by Lemma 2, $\nu(AB)=\nu(A)+d(SA)+\nu(B)-d(SA+N(B))$. But it was noted above that $\nu(A)+d(SA)=n$, and thus

$$\nu(AB)-\nu(B)=n-d(SA+N(B)). \tag{1}$$

Substituting BC for B gives

$$\nu(ABC)-\nu(BC)=n-d(SA+N(BC)). \tag{2}$$

But clearly every solution of $VB=0$ is a solution of $VBC=0$, so that $N(B)$ is a subspace of $N(BC)$. Thus $SA+N(B)$ cannot have a dimension greater than that of $SA+N(BC)$. Relations (1) and (2) thus yield $\nu(ABC)-\nu(BC)\leqq\nu(AB)-\nu(B)$, which is the desired result.

---

* See [1; p. 179].

† See [2; p. 11]. The essential ideas of the proofs here given are to be found in that of MacDuffee.

From this theorem follow, as corollaries, two very useful formulas (Sylvester's law of nullity):

COROLLARY 1. $\nu(AB) \geqq \max(\nu(A), \nu(B))$.

Proof: In Theorem 1, first let $A=0$, and then let $C=0$.

COROLLARY 2. $\nu(AB) \leqq \nu(A)+\nu(B)$.

Proof: In theorem 1, let $B=I$.

References

1. Garrett Birkhoff and Saunders MacLane, A Survey of Modern Algebra, The Macmillan Co., New York, 1948.

2. C. C. MacDuffee, The Theory of Matrices, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 2, Springer, Berlin, 1933.

CONTINUOUS FUNCTIONS WITHOUT DERIVATIVES

K. A. BUSH, State University of New York

In this note we give an elementary example of a class of continuous functions which fail to possess a derivative. The known examples of this phenomenon all involve limiting processes in one form or another and are therefore somewhat complicated.

We shall restrict the discussion to the unit interval so that any number may be written in the form

$$x = .x_1x_2x_3\cdots$$

where each x_i is one of the numbers 0, 1, 2, $\cdots$, $b-1$ where b is an integer used as the base. Once chosen, b is to be kept fixed. We further demand $b>2$.

We say a run begins on x_i if $x_{i-1}\neq x_i$. Conventionally we agree that a run always begins on x_1. We now form a function $f[x]$ by setting

$$f[x] = .u_1u_2u_3\cdots$$

where u_i is the number of runs reduced modulo 2 that have occurred in the first i places of the b-adic expansion of x. The functional values are therefore represented in base 2.

THEOREM 1. *$f[x]$ is a continuous function.*

Proof. Suppose $2^{-p}<\epsilon$ where ϵ has been fixed in advance. Given any point

$$x^0 = .x_1^0x_2^0x_3^0\cdots,$$

we choose a neighboring point x such that

$$|x - x^0| < b^{-p}. \tag{1}$$

This can occur in two ways. We may have

$$x_i = \overset{0}{x_i}, \qquad i = 1, 2, \cdots, p.$$

[In this case we suppose that the remaining places in the expansion of the difference (1) are not all filled with the value $b-1$.] Since each u_i depends only on its predecessors, we have

$$u_i = \overset{0}{u_i}, \qquad i = 1, 2, \cdots, m \text{ where } m \geqq p$$

so that

$$|f[x] - f[x^0]| < \epsilon. \tag{2}$$

We also have the possibility

$$\begin{aligned} x_i &= \overset{0}{x_i}, && i = 1, 2, \cdots, k-1 \\ x_k &= \overset{0}{x_k} + 1 \\ x_i &= 0, \quad \overset{0}{x_i} = b - 1, && i = k+1, k+2, \cdots, p. \end{aligned}$$

[In certain cases it may be necessary to have the last of these relationships hold for $i=p+1$. A third possibility, $x_k=x_k^0-1$, is in all respects the dual of the case we discuss.]

If in particular $k=1$, we must have

$$\begin{aligned} u_1 &= \overset{0}{u_1} = 1 \\ u_2 &= \overset{0}{u_2} = 0, \end{aligned}$$

since a run must begin on x_2 and on x_2^0,

$$u_i = \overset{0}{u_i} = 0, \qquad 2 < i \leqq p$$

since no run begins in the next p places. Consequently

$$|f[x] - f[x^0]| < \epsilon.$$

For the general case we use three initial dots to indicate that the symbols in the first $k-1$ places agree. Suppose we have

$$\left.\begin{matrix} \cdots \overset{0}{x_k} + 1 & 0 & 0 & \cdots \\ \cdots \overset{0}{x_k} & b-1 & b-1 & \cdots \end{matrix}\right\} \text{ repetitions holding to the } p\text{th place,}$$

for the respective expansions of x and x^0. If x_{k-1}^0 differs from x_k^0 and from x_k^0+1, runs begin in the kth place for both x and x^0 so that (2) holds. But if x_{k-1}^0 agrees with one of these two values, a run will begin on the kth place of one expansion

and not on the other so that we can represent the two functional values by

$$\left.\begin{matrix}\cdots 1\ 0\ 0\cdots\\ \cdots 0\ 1\ 1\cdots\end{matrix}\right\}\ \text{repetitions holding to the } p\text{th place,}$$

where one of these values will be associated with x^0, although this is not necessarily the first value, and the other with x. These values satisfy (2), completing the proof.

THEOREM 2. *$f[x]$ has no derivative.*

Proof. Let us set $x=x_i^0$ except when $i=p$. Then

$$|x - x^0| = kb^{-p}, \qquad 0 < k \leqq b-1.$$

We secure five different cases tabulated below which depend upon the relationships existing between x_{p-1}^0, x_p^0, and x_{p+1}^0. These entries appear in the second line of each pair. At the right we indicate the corresponding entries in the dyadic system of the corresponding function values on the assumption that $u_{p-1}=0$. If $u_{p-1}=1$, then the zeros and ones are to be interchanged, but this will not affect the essential argument in any way.

Case I.

	u_p	u_{p+1}
$x_{p-1}^0 \neq x_p \neq x_{p+1}^0$	$\cdots 1$	$0\cdots$
$x_{p-1}^0 = x_p = x_{p+1}^0$	$\cdots 0$	$0\cdots$

Case II.

$x_{p-1}^0 \neq x_p = x_{p+1}^0$	$\cdots 1$	$1\cdots$
$x_{p-1}^0 = x_p \neq x_{p+1}^0$	$\cdots 0$	$1\cdots$

Case III.

$x_{p-1}^0 = x_p \neq x_{p+1}^0$	$\cdots 0$	$1\cdots$
$x_{p-1}^0 \neq x_p = x_{p+1}^0$	$\cdots 1$	$1\cdots$

Case IV.

$x_{p-1}^0 = x_{p+1}^0$		
$x_{p-1}^0 = x_p = x_{p+1}^0$	$\cdots 0$	$0\cdots$
$x_{p-1}^0 \neq x_p \neq x_{p+1}^0$	$\cdots 1$	$0\cdots$

Case IVa.

$x_{p-1}^0 \neq x_{p+1}^0$		
$x_{p-1}^0 = x_p \neq x_{p+1}^0$	$\cdots 0$	1
$x_{p-1}^0 \neq x_p \neq x_{p+1}^0$	$\cdots 1$	0

In the first four cases, the function values differ only in the pth place. We therefore have

$$|f[x] - f[x^0]| = 2^{-p}.$$

For these cases we therefore have

$$\frac{2^{-p}}{kb^{-p}} = \left|\frac{f[x] - f[x^0]}{x - x^0}\right|.$$

If x^0 has the property that an infinite number of values of p can be found such that one of the four cases first listed holds, then we can always choose a neighboring point x such that the differential quotient can be made arbitrarily large in absolute value by choosing p sufficiently large. If Case IVa holds when $i \geqq p_0$, then a run must begin on each digit x, $i \geqq p_0$ so that for $p \geqq p_0$ we have

$$f[x^0] = \cdots 1\ 0\ 1\ 0 \cdots$$
$$f[x] = \cdots 0\ 1\ 0\ 1 \cdots$$

in accordance with our choice of x. Hence the absolute value of their difference is easily computed to be $[4/3]2^{-[p+1]}$. Again the absolute value of the differential quotient can be made arbitrarily large by choosing p sufficiently large so that no derivative can exist.

If $b=2$, the function reduces to a straight line, but the non-trivial cases possess an interesting intrinsic property embodied in the following theorem.

THEOREM 3. *If* $1/2<t<1$ *and if* t *is not representable as a finite sum of elements of the form* 2^{-n}, *then* $f[x]-t=0$ *has a non-denumerable number of solutions.*

Proof. Consider the decimal expansion of t in base 2. The restriction on t implies that t contains an infinite number of runs. The expansion therefore contains an infinite number of 1's preceded by 0. It is possible to generate 1 in the pth place preceded by 0 by $b-1$ different choices of x_p. Hence there are $[b-1]^a$ such numbers where a is the cardinal number of the integers. This theorem throws considerable light on the non-existence of the derivative.

SUMS OF POWERS OF NUMBERS HAVING A GIVEN EXPONENT MODULO A PRIME

RAYMOND MOLLER, Catholic University of America

In his *Disquisitiones Arithmeticae*, C. F. Gauss [3] showed that the sum of the primitive roots of a prime p is congruent modulo p to $(-1)^k$ if $p-1$ is a product of k distinct primes, and to zero if $p-1$ contains a square factor; or, as we may now state it more concisely, to $\mu(p-1)$. This theorem was later generalized by Stern [5], who proved that the sum of the numbers belonging to any divisor d of $p-1$, where p is a prime, is congruent modulo p to $\mu(d)$. A further generalization by Forsyth [2] established congruence theorems for the sum of the nth powers of the primitive roots of a prime, but the statement of his results was so complicated by the enumeration of various special cases that Dickson, in his *History* [1], summarized only a small part of them and dismissed the rest as "not so simple." The purpose of this paper is to obtain one simple congruence relationship which will express the sum of the nth powers of the numbers belonging to any divisor d of $p-1$ in terms of the Möbius function and the Euler ϕ-function.

If g is one number belonging to d modulo p, then the set of all numbers belonging to d is

$$g^j \qquad [0 < j \leqq d, (j, d) = 1]. \tag{1}$$

When the members of set (1) are raised to the power n, each member of the new set will be a number belonging to d/d', where $d' = (n, d)$. However, since there are only $\phi(d/d')$ numbers belonging to d/d', the $\phi(d)$ members of the new set will not be distinct modulo p, except in the trivial case where $d' = 1$ and the new set is identical with (1). The members of the new set are all of the numbers

$$g^{nj} \qquad [0 < j \leqq d, (j, d) = 1]. \tag{2}$$

A number g^{nx}, with no restriction on x, will be congruent modulo p to any given member g^{nj_1} of set (2) if and only if

$$nx \equiv nj_1 \pmod{d},$$

which in turn is true if and only if

$$x \equiv j_1 \pmod{d/d'}.$$

There is exactly one solution for x in the range $0 < x \leqq d/d'$, and d' solutions in the range $0 < x \leqq d$; however, not all of these values of x are possible values of j [corresponding to members of (2)], because of the restriction $(j, d) = 1$. To determine how many of these solutions are values of j, we must determine how many members of the set

$$j_1 + dt/d' \qquad (t = 0, 1, 2, \cdots, d' - 1), \tag{3}$$

consisting of all solutions of the congruence, are prime to d. Since all of the members of (3) are prime to d/d', any member will be prime to d if and only if it is prime to d'.

Let $d'=\prod p_i^{\alpha_i}\prod q_i^{\beta_i}$, where the p_i and q_i are primes such that $(p_i, d/d')=1$ and $(q_i, d/d')=q_i$. The members of set (3) which are prime to d' are those which are not divisible by any p_i, since none of them is divisible by any q_i. For the congruence $j_1+dt/d'\equiv 0 \pmod{q_i}$ has no solution inasmuch as $(q_i, d/d')=q_i$ does not divide j_1, which is prime to d and *a fortiori* to d/d'. Now corresponding to each p_i there is one t in the range $0\leqq t<p_i$ such that $j_1+dt/d'\equiv 0 \pmod{p_i}$, and d'/p_i such values in the range $0\leqq t<d'$. The values of t for which the above congruence is satisfied modulo p_i and those for which it is satisfied modulo p_j are not necessarily distinct. However, for each duplication the congruence must be satisfied modulo p_ip_j, and so the number of such duplications is one in the range $0\leqq t<p_ip_j$ and d'/p_ip_j in the range $0\leqq t<d'$. In like manner, for each t which satisfies the congruence modulo p_i, modulo p_j, $\cdots$, and modulo p_s, the congruence must be satisfied modulo $p_ip_j\cdots p_s$, so that there is just one such t in the range $0\leqq t<p_ip_j\cdots p_s$ and $d'/p_ip_j\cdots p_s$ such values of t in the range $0\leqq t<d'$.

Consequently, the number N of members of set (3) which are prime to d', being the number of values of j_1+dt/d' $(t=1, 2, \cdots, d'-1)$ possessing none of the properties of divisibility by a p_i, is equal (by a well-known combinatorial theorem [4]) to

$$d' - \sum d'/p_i + \sum d'/p_{i_1}p_{i_2} - \cdots + (-1)^k \sum d'/p_{i_1}p_{i_2}\cdots p_{i_k},$$

where the summations are taken over all combinations of prime divisors of d' such that $(p_i, d/d')=1$. This may be written

$$N = d' \prod_{p_i} (1 - 1/p_i).$$

If we now denote by r_i those prime factors of d which do not divide d', then the p_i, q_i, and r_i are all of the prime divisors of d, and the q_i and r_i are all of the prime divisors of d/d'. Thus $\phi(d)=d\prod(1-1/p_i)(1-1/q_i)(1-1/r_i)$ and $\phi(d/d')=(d/d')\prod(1-1/q_i)(1-1/r_i)$, so that

$$\phi(d)/\phi(d/d') = d' \prod (1 - 1/p_i) = N.$$

Since the expression for N is independent of j_1, we see that all of the members of (2) fall into subsets each containing the same number of members congruent to each other modulo p; or, in other words, any number which does appear in (2) is repeated exactly $\phi(d)/\phi(d/d')$ times. Now since the $\phi(d)$ numbers in (2) belong to d/d', only $\phi(d/d')$ of them can be distinct, so that in order to account for the total of $\phi(d)$ numbers, every number belonging to d/d' must appear. Therefore the set of numbers obtained by raising to the nth power all of the numbers belonging to d modulo p consists of all of the numbers belonging to d/d' modulo p, each repeated $\phi(d)/\phi(d/d')$ times. But since the sum of

the numbers belonging to d/d' is congruent modulo p to $\mu(d/d')$, the sum of the nth powers of the numbers belonging to d must be congruent modulo p to

$$\frac{\phi(d)}{\phi(d/d')}\mu(d/d').$$

We have thus proved the following theorem:

THEOREM I: *If the numbers g_d are all of the numbers belonging to d modulo p, p being any prime and d any divisor of $p-1$, then for any n*

$$\sum g_d^n \equiv \frac{\phi(d)}{\phi[d/(n, d)]}\mu[d/(n, d)] \pmod{p},$$

where ϕ is the Euler ϕ-function and μ is the Möbius function.

We shall now consider the function

$$F(x, n) = \sum_{d \mid x} \frac{\phi(d)}{\phi[d/(n, d)]}\mu[d/(n, d)]$$

which by Theorem I is congruent modulo p to the sum of the nth powers of all numbers belonging modulo p to any of the divisors of x, where p is any prime such that x divides $p-1$. (There are of course infinitely many such primes for any given x.) If $(d, d')=1$, we have

$$\left\{\frac{\phi(d)}{\phi[d/(n, d)]}\mu[d/(n, d)]\right\}\left\{\frac{\phi(d')}{\phi[d'/(n, d')]}\mu[d'/(n, d')]\right\}$$

$$= \frac{\phi(dd')}{\phi[dd'/(n, d)(n, d')]}\mu[dd'/(n, d)(n, d')] = \frac{\phi(dd')}{\phi[dd'/(n, dd')]}\mu[dd'/(n, dd')]$$

so that $F(x, n)$ has the multiplicative property [4]

$$F(x, n)F(x', n) = F(xx', n) \quad \text{when} \quad (x, x') = 1.$$

We proceed to evaluate $F(q^\alpha, n)$, where q is a prime. If $(n, q^\alpha)=q^\beta(0 \leqq \beta < \alpha)$, then

$$F(q^\alpha, n) = \sum_{d \mid q^\beta}\{\phi(d)\} + \frac{\phi(q^{\beta+1})}{\phi(q)}\mu(q) = q^\beta - q^\beta = 0.$$

If $(n, q^\alpha)=q^\alpha$,

$$F(q^\alpha, n) = \sum_{d \mid q^\alpha}\phi(d) = q^\alpha.$$

Using the multiplicative property of $F(x, n)$, we conclude that if n is not a multiple of x, $F(x, n)=0$, for there would be at least one $q_i^{\alpha_i}$ dividing x such that $(n, q_i^{\alpha_i})=q_i^{\beta_i}$, $(0 \leqq \beta_i < \alpha_i)$, and consequently at least one zero factor in

$\prod_i F(q_i^{\alpha_i}, n) = F(x, n)$. On the other hand, if n is a multiple of x, we see that $F(x, n) = \prod_i q_i^{\alpha_i} = x$. Thus we have proved

THEOREM II: *The sum of the nth powers of all numbers belonging modulo p to any of the divisors of x, where p is any prime such that x divides p−1, is congruent modulo p to x or to zero, according as n is or is not a multiple of x.*

As a special case, if we take $x = p-1$, we have the well-known theorem that the sum of the nth powers of all the numbers from 1 to $p-1$ inclusive is congruent modulo p to $p-1$ or to zero according as n is or is not a multiple of $p-1$; for every number from 1 to $p-1$ belongs modulo p to one of the divisors of $p-1$.

References

1. Dickson, L. E., History of the Theory of Numbers, Washington, 1919, Volume I, page 193.
2. Forsyth, A. R., Primitive roots of prime numbers and their residues, Messenger of Mathematics, Volume XIII (1883–4), pp. 180–185.
3. Gauss, C. F., Disquisitiones Arithmeticae, arts. 80-81.
4. Hardy, G. H., and Wright, E. M., An Introduction to the Theory of Numbers, Oxford, 1938, pp. 232, 235.
5. Stern, M. A., Bemerkungen über hohere Arithmetik, Journal für Mathematik, Volume VI (1830), pp. 147–153.

Additional Remarks by H. S. Zuckerman, University of Washington. It is interesting to note that it is possible to prove Theorem **II** first and then obtain Theorem **I** from it. We let $f(d, n) = \sum g_d^n$ and have $F(x, n) = \sum_{d|x} f(d, n)$ and $x \mid (p-1)$. Now $u^x \equiv 1 \pmod p$ if and only if u_0 belongs to the exponent d modulo p and $d \mid x$ so we have $F(x, n) \equiv \sum u^n \pmod p$ where the sum is taken over all u (mod p) such that $u^x \equiv 1 \pmod p$. If u_0 belongs to the exponent x modulo p then $(u_0 u)^x \equiv 1 \pmod p$, $(u_0, p) = 1$ and hence the set of $u_0 u$ is congruent to the set of u modulo p, in some order. Then $\sum u^n \equiv \sum (u_0 u)^n \equiv u_0^n \sum u^n \pmod p$ and $\sum u^n \equiv 0 \pmod p$ unless $u_0^n \equiv 1 \pmod p$, that is $x \mid n$. We then find

$$\text{(A)} \qquad F(x, n) \equiv \begin{cases} 0 \pmod p & \text{if } x \nmid n, \\ x \pmod p & \text{if } x \mid n, \end{cases}$$

and this is Theorem **II**.

From the Möbius inversion formula we find $f(x, n) = \sum_{d|x} F(d, n)\mu(x/d)$ and, using (A), $f(x, n) \equiv \sum d\mu(x/d) \pmod p$ where the sum extends over all d that divide both x and n. We can write $x/(x, n) = a$, $x = aa'b$, where $(aa', b) = 1$ and every prime dividing a' also divides a. Then $(x, n) = a'b$ and we have

$$\sum_{d|(x,n)} d\mu\left(\frac{x}{d}\right) = \sum_{d|a'b} d\mu\left(\frac{aa'b}{d}\right) = \sum_{\delta|a'b} \frac{a'b}{\delta}\mu(a\delta)$$

$$= \sum_{\delta|b} \frac{a'b}{\delta}\mu(a)\mu(\delta) = a'\mu(a)\phi(b)$$

since $\mu(a\delta) = 0$ unless $(a, \delta) = 1$. In order to express this in terms of x and n we

insert a factor $\phi(a)$ and find

$$a'\mu(a)\phi(b) = \frac{a'\phi(a)\phi(b)}{\phi(a)}\mu(a) = \frac{\phi(aa')\phi(b)}{\phi(a)}\mu(a) = \frac{\phi(aa'b)}{\phi(a)}\mu(a)$$
$$= \frac{\phi(x)}{\phi\left(\frac{x}{(x, n)}\right)}\mu\left(\frac{x}{(x, n)}\right).$$

Now we have

$$f(x, n) \equiv \frac{\phi(x)}{\phi\left(\frac{x}{(x, n)}\right)}\mu\left(\frac{x}{(x, n)}\right) \pmod{p}$$

which is Theorem I.

MAXIMA AND MINIMA UNDER RESTRAINT*

C. G. PHIPPS, University of Florida

1. Introduction. The problem of finding the maximum or minimum of a function whose several variables are required to satisfy one or more auxiliary equations is treated extensively in the literature and in standard texts. In general, the working rules are given without an adequate explanation of why they work. The methods of solution usually fall into one of three categories.

First, the auxiliary equations may be used to eliminate an equal number of dependent variables from the given function. In attempting to use this method, the elimination may prove to be difficult or even impossible in finite form. However, since the method reduces the problem to one without restraint, it need not concern us further.

Second, the differentials of all functions may be taken. If the differentials of the dependent variables are then eliminated, enough equations are formed to solve for all the quantities involved. This is the method used here. It has many variations.

Third, Lagrange multipliers may be used. The rules for their use are essentially short-cuts in the solution by the second method above. In addition, the use of these multipliers formalizes the problem and lends a kind of symmetry to the solution which the other methods do not possess.

* Adapted from an address as retiring Chairman of the Southeastern Section of the Mathematical Association of America delivered at Vanderbilt University, March 16, 1951.

The purpose of the present article is (1) to effect an algebraic solution by the second method, (2) to show its connection with the use of Lagrange multipliers, and (3) to give an analytic and geometric interpretation often overlooked. No special cases are considered.

All functions involved are assumed to be continuous in a given region R and to have continuous first and second partial derivatives with respect to each of the variables. Additional restrictions upon these functions are stated in the appropriate places.

2. The general problem. Let $f(x_1, x_2, \cdots, x_n)$ be a function defined in the region R and whose n arguments are restrained to satisfy the m auxiliary equations

$$g^k(x_1, x_2, \cdots, x_n) = a_k, \qquad k = 1, 2, \cdots, m < n. \tag{2.1}$$

If subscripts on the functions are used to indicate partial differentiation with respect to the correspondingly numbered variable, it is assumed that the matrix

$$(g_i^k), \qquad k = 1, 2, \cdots, m; \; i = 1, 2, \cdots, n,$$

is non-singular in R.

A *stationary point* of the function f is defined as a point at which its differential is zero; *i.e.*,

$$df = \sum_{i=1}^{n} f_i dx_i = 0, \tag{2.2}$$

where the differentials of the x_i are subject to the m equations

$$dg^k = \sum_{i=1}^{n} g_i^k dx_i = 0, \qquad k = 1, 2, \cdots, m. \tag{2.3}$$

Thus the stationary points of f may include, in addition to the points at which it attains its maxima and minima, some points at which it attains neither.

Since the matrix above is non-singular, it is possible to number the variables so that the determinant $|g_1^1, g_2^2, \cdots, g_m^m|$ does not vanish, then to take the variables $x_1, x_2, \cdots, x_m$ as dependent, and to eliminate the differentials of these variables from (2.2) by a choice of multipliers $\lambda_1, \cdots, \lambda_m$ such that

$$f_j - \sum_{k=1}^{m} \lambda_k g_j^k = 0, \qquad j = 1, 2, \cdots, m. \tag{2.4}$$

With this choice the equation in (2.2) is transformed into

$$\sum_{i=m+1}^{n} \left(f_i - \sum_{k=1}^{m} \lambda_k g_i^k \right) dx_i = 0.$$

Since this is an identity in the differentials appearing here, the coefficient of each differential is zero. Hence it is possible to extend (2.4) to all variables and write

$$f_i - \sum_{k=1}^{m} \lambda_k g_i^k = 0, \qquad i = 1, 2, \cdots, n. \tag{2.5}$$

These n equations together with the m equations in (2.1) serve to determine the coordinates of the stationary points and the values of all the multipliers. Since any other choice of dependent variables would lead to the same set of equations as in (2.5), the set of multipliers at each stationary point is unique but will in general be different for different stationary points.

3. The second order conditions. If the stationary point is a minimum, it is necessary that the second differential of f be positive or zero there; it is sufficient to insure a minimum if the differential is positive. The only tests which are convenient to apply are those based on this sufficiency.

The second differential of f is given by

$$d^2f = \sum_{i=1}^{n}\sum_{j=1}^{n} f_{ij}dx_i dx_j + \sum_{k=1}^{m} f_k d^2x_k > 0. \tag{3.1}$$

The second differentials of the g-functions are similar except that they must vanish; *i.e.*,

$$d^2g^k = 0, \qquad k = 1, 2, \cdots, m. \tag{3.2}$$

The left member of (3.1) can be made into a quadratic form in the differentials of all the variables by adding appropriate multiples of the second differentials of the g-functions. Thus

$$d^2f - \sum_{k=1}^{m} \lambda_k d^2g^k = \sum_{i=1}^{n}\sum_{j=1}^{n}\left(f_{ij} - \sum_{k=1}^{m} \lambda_k g_{ij}^k\right) dx_i dx_j > 0, \tag{3.3}$$

where the λ_k are the same multipliers as those defined by the equations in (2.4). The conditions under which the left side of (3.3) is a positive definite quadratic form subject to the m linear restraints in (2.3) are easily written down.*

Let

$$b_{ij} = f_{ij} - \sum_{k=1}^{m} \lambda_k g_{ij}^k, \qquad i, j = 1, 2, \cdots, n;$$

* See H. B. Mann, Quadratic forms with linear restraints, this MONTHLY, vol. 50, pp. 430–433. His conditions for a negative definite form under restraint should read $(-1)^i h_i > 0$ instead of $(-1)^{p+i} h_i > 0$.

$$h_t = \begin{vmatrix} 0 & \cdots & 0 & g_1^1 & \cdots & g_t^1 \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ 0 & \cdots & 0 & g_1^m & \cdots & g_t^m \\ g_1^1 & \cdots & g_1^m & b_{11} & \cdots & b_{1t} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ g_t^1 & \cdots & g_t^m & b_{t1} & \cdots & b_{tt} \end{vmatrix}, \qquad t = m, \cdots, n.$$

The expression in (3.3) will be positive definite if and only if h_t has the sign of $(-1)^m$ for all the specified values of t. These conditions for a minimum are only sufficient.

If the stationary point is a maximum, the sense of the inequality in (3.3) is reversed. The conditions upon h_t for this case are easily derived from the conditions for a positive definite form.

This negative definite form can be changed into one which is positive definite by multiplying by -1. At the same time let each equation in (2.3) also be multiplied by -1. Let k_t be the determinant associated with this form and corresponding to h_t of the previous form. Since k_t and h_t are determinants of order $m+t$, we have that

$$h_t = (-1)^{m+t}k_t = (-1)^{2m+t}\left| k_t \right| = (-1)^t \left| k_t \right|.$$

Therefore, for a negative definite quadratic form under restraint, h_t has the sign of $(-1)^t$ for the values of t indicated.

If the signs of h_t do not follow either pattern, the point is neither a maximum nor a minimum. If, however, some of the h_t are zero but otherwise conform to the pattern of signs, the stationary point cannot be classified by this test.

4. An interpretation. Since every set of equations to be solved contains the equations of (2.1), the a-constants of these equations are parameters of the solution. Hence, the coordinates of each stationary point, the value of the multipliers there, and the stationary value of f are all functions of these parameters.

Let c be the stationary value of f, then

$$c = f(x_1, x_2, \cdots, x_n) = F(a_1, a_2, \cdots, a_m). \tag{4.1}$$

If now each equation in (2.1) is differentiated implicitly with respect to any one a_j, and if each derived equation is multiplied by the corresponding λ_k and collectively subtracted from the equation similarly derived from (4.1), the result rearranges to read

$$\frac{\partial c}{\partial a_j} = \lambda_j, \qquad j = 1, 2, \cdots, m. \tag{4.2}$$

The results in (4.2) furnish an analytic interpretation of the multipliers used in eliminating the dependent differentials from (2.2). Mathematicians have effectively prevented themselves from reaching this interpretation by always setting the g-functions equal to zero. Mathematical economists have of necessity fared better although, under the influence of the mathematicians, they have always transposed their constants to the left side of the equation before applying this method.

This interpretation has given the economists two important concepts. If f is the utility function of an individual, and if the first equation of (2.1) is the budget equation of the individual in which a_1 is his income, then λ_1 is the marginal utility of income. Again, if f is the cost of producing a certain amount of some good, and if $g^1 = a_1$ is the production equation, then the marginal cost of production in producing a_1 units of this good is λ_1. For a more detailed discussion of these concepts and a fuller explanation of the terminology the reader is referred to the last two items of the bibliography.

5. A second interpretation. Define the function

$$u = f(x_1, \cdots, x_n) - \sum_{k=1}^{m} \lambda_k g^k(x_1, \cdots, x_n) \tag{5.1}$$

as a point function in n-dimensional space. Consider this function at a stationary point of f as determined above with the λ_k taken as the multipliers associated with this point. If ds is the differential of arc along any arbitrary curve through this point, it follows from (2.5) that

$$\frac{du}{ds} = \sum_{i=1}^{n} \left(f_i - \sum_{k=1}^{m} \lambda_k g_i^k \right) \frac{dx_i}{ds} = 0; \tag{5.2}$$

i.e., the derivative of u is zero in every direction from this point.

In turn, the result in (5.2) implies that, at a stationary point of f, its directional derivative is a linear combination of the directional derivatives of the g-functions taken in the same direction; also, the coefficients of this linear combination are the multipliers associated with this point. This interpretation is interesting when visualized in three-space (using vector analysis) with either one or two restraining equations.

6. Lagrange multipliers. Lagrange first proposed the use of the function u about 1797.* From this fact these multipliers take their name. Like most others since, he gave no explanation beyond the statement that the rules he gave worked.

---

* See the work cited here or the various reprints of his *Theorie des fonctions analytiques.*

These rules may now be formulated as a direct consequence of the facts indicated in (5.2). They are:

(1) form the function u as defined in (5.1) above;
(2) treat the m multipliers *as if* they were constant;
(3) treat the n variables *as if* they were independent;
(4) set the n derivatives of u equal to zero *as if* the function were unrestricted.

By following these rules, one is lead immediately to the equations of (2.5) which together with those in (2.1) are sufficient in number to determine all unknowns. In addition, the second differential of u yields the quadratic form in (3.3).

The function in (5.1) may be written in a slightly more general form as

$$v = f - \sum_{k=1}^{m} \lambda_k(g^k - a_k). \tag{6.1}$$

This function may then be treated, up to a certain point, *as if* the n x_i and the m λ_k were all independent variables. The first derivatives of v set equal to zero yield all the necessary equations. Also, one may form the determinants h_i from the second derivatives *as if* the function were without restriction. With this, the analogy ceases, for the determinants of lower orders are not relevant, and the signs of the others, for sufficiency, must be those given in section 3.

Bibliography

1. Lagrange, J. L., Oeuvres de Lagrange, Fourth Edition, Paris, 1881, vol. IX, pp. 290–295.
2. Courant, Richard, Differential and Integral Calculus, New York, 1937, vol. II, pp. 188–199.
3. Chaundy, Theodore, The Differential Calculus, Oxford, 1935, pp. 256–282.
4. Bliss, G. A., Lectures on the Calculus of Variations, Chicago, 1946, pp. 210–213.
5. Mann, H. B., Quadratic forms with linear restraints, this MONTHLY, vol. 50, pp. 430–433.
6. Hicks, J. R., Value and Capital, Oxford, 1946, Mathematical Appendix, pp. 301–337.
7. Samuelson, P. A., Foundations of Economic Analysis, Cambridge, Mass., 1948.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

A CONTINUED FRACTION CIPHER

ARTHUR PORGES, Los Angeles, California

The elementary properties of continued fractions form the basis of a cipher with unusual characteristics, the most interesting being that a message can be enciphered in blocks containing an arbitrary number of letters, different for each block, yet every such letter-group can be represented uniquely in cipher by a single pair of integers.

Let each letter of the alphabet be assigned a unique positive integer, $\alpha_i \neq 1$, $(i = 1, 2, \cdots, 26)$. A word ρ, or any consecutive letter sequence, after being expressed numerically as a series of integers, α_i, spaced (or separated by dashes), may be enciphered in blocks of any number Δ of letters, $(\Delta \neq 1)$, as follows.

Let the α_i's be consecutive partial quotients of the continued fraction

$$\beta = \alpha_a + \cfrac{1}{\alpha_b + \cfrac{1}{\alpha_c + \cdots}}.$$

The expression of β as an ordinary fraction gives the unique equivalent: $\beta = \phi/\delta$. Then ρ is written in cipher as the integer-pair $\phi - \delta$, or simply ϕ (space) δ.

Obviously, to decipher the sequence $\phi_1\ \delta_1\ \phi_2\ \delta_2 \cdots \phi_r\ \delta_r$, each ϕ_k/δ_k is expanded into a continued fraction; the partial quotients will be the α_i's of the "clear" letter-groups.

Especially interesting is the fact that a message of any length can be enciphered as a single (presumably large) pair of integers.

As an illustration of the method, consider the cipher sequence

566 43 4038 191 181 12 1289 91 289 129.

The quotient 566/43, expanded into a continued fraction, gives the partial quotients, 13, 6, 7. Assuming the simple choice of α_i's given by A = 2, B = 3, etc., these quotients represent the letters L E F. Similarly, 4038/191, with partial quotients 21, 7, 13, 2, gives the four letters, T F L A. Continuing, we easily obtain the remaining letter blocks N K, M E N, A C E D, all of which reads LEFT FLANK MENACED.

ON ASCOLI'S THEOREM

S. S. WALTERS, University of California at Los Angeles

Preliminary to proving Ascoli's Theorem in real variables, one proves the following theorem:

If $\{f_n(x)\}$ is an equicontinuous family for x in a fixed compact set C in n-space such that $\lim_{n\to\infty} f_n(x)$ exists on a set dense in C, then $\lim_{n\to\infty} f_n(x)$ exists for all x in C.

It is then quite natural to inquire if there exists a family of continuous functions which converges on a dense set and also diverges on a dense subset of C. It is the purpose of this paper to present a very elementary example of *a sequence of functions $\{f_n(x)\}$ (which are continuous and uniformly bounded on the interval $0 \leqq x \leqq 1$) which converges on a countable dense subset of $0 \leqq x \leqq 1$ and diverges elsewhere.*

For each fixed non-negative integer n, let

$$f_n(x) = \begin{cases} 2^{n+1}\left(x - \dfrac{j}{2^n}\right) & \text{for } \dfrac{j}{2^n} \leqq x \leqq \dfrac{2j+1}{2^{n+1}}, \\ -2^{n+1}\left(x - \dfrac{j+1}{2^n}\right) & \text{for } \dfrac{2j+1}{2^{n+1}} \leqq x \leqq \dfrac{j+1}{2^n}, \end{cases}$$

where $j = 0, 1, \cdots, 2^n - 1$. That is, $f_n(x)$ is that function whose graph is the polygon obtained by joining successively with straight lines the points in the plane

$$(0, 0), \left(\frac{1}{2^{n+1}}, 1\right), \left(\frac{1}{2^n}, 0\right), \cdots, \left(\frac{j}{2^n}, 0\right), \left(\frac{2j+1}{2^{n+1}}, 1\right),$$
$$\left(\frac{j+1}{2^n}, 0\right), \cdots, (1, 0).$$

It is then evident, from the graphs of $f_{n+1}(x)$ and $f_n(x)$, that

$$f_{n+1}(x) = 2f_n(x)$$

for x in the leftmost and rightmost quarters of the interval

$$E_x\left[\frac{j}{2^n} \leqq x \leqq \frac{j+1}{2^n}\right],$$

while

$$f_{n+1}(x) = 2(1 - f_n(x))$$

in the two middle quarters of this interval. Thus, for any x in the unit interval and each positive integer n, either

$$f_{n+1}(x) = 2f_n(x) \quad \text{or} \quad f_{n+1}(x) = 2(1 - f_n(x)).$$

In other words, either

$$f_{n+1}(x) - f_n(x) = f_n(x) \quad \text{or} \quad f_{n+1}(x) - f_n(x) = 2 - 3f_n(x).$$

If $\lim_{n\to\infty} f_n(x)$ exists, then clearly

$$\lim_{n\to\infty} [f_{n+1}(x) - f_n(x)] = 0.$$

Thus, either there is only a finite number of n's such that

$$f_{n+1}(x) - f_n(x) = f_n(x),$$

or there is only a finite number of n's such that

$$f_{n+1}(x) - f_n(x) = 2 - 3f_n(x);$$

for otherwise we should have the absurdity

$$0 = \lim_{n\to\infty} f_n(x) = 2 - 3 \lim_{n\to\infty} f_n(x) = 2.$$

Hence, either there exists an integer N such that $n \geqq N$ implies $f_{n+1}(x) = 2(1 - f_n(x))$, or there is an integer N such that $n \geqq N$ implies $f_{n+1}(x) = 2f_n(x)$. In the former case we see that $n \geqq N$ implies

$$f_{n+1}(x) - \tfrac{2}{3} = -2(f_n(x) - \tfrac{2}{3}),$$

whence $n \geqq N$ implies

$$f_n(x) - \tfrac{2}{3} = (-2)^{n-N}(f_N(x) - \tfrac{2}{3}).$$

Since $\lim_{n\to\infty} f_n(x)$ exists, then $f_N(x) = 2/3$. In the latter case we see that $n \geqq N$ implies

$$f_n(x) = 2^{n-N} f_N(x).$$

From the existence of $\lim_{n\to\infty} f_n(x)$, we see that $f_N(x) = 0$. We have then shown that if $\lim_{n\to\infty} f_n(x)$ exists, either $f_n(x) = 2/3$ for some n or $f_n(x) = 0$ for some n, or, equivalently (as is seen from the definition of f_n), either $x = j/2^n$ for some n and $j = 0, 1, \cdots, 2^n$ or $x = j/2^n \pm 1/3 \cdot 2^n$ for some n and $j = 0, 1, \cdots, 2^n$. Conversely: (a) if $x = j/2^n$ for some j and n, then $x = 2j/2^{n+1}$, whence $f_{n+1}(x) = f_n(x) = 0$ for all larger n, and thus $\lim_{n\to\infty} f_n(x) = 0$; (b) if $x = j/2^n \pm 1/3 \cdot 2^n$, then $f_{n+1}(x) = 2(1 - f_n(x))$ and hence $f_{n+1}(x) = 2/3$ for all larger n, whence $\lim_{n\to\infty} f_n(x) = 2/3$.

We have now shown that $\lim_{n\to\infty} f_n(x)$ exists on the countable dense set of all points of the form $j/2^n \pm k/3 \cdot 2^n$, $n = 0, 1, \cdots, j = 0, 1, \cdots, 2^n$, $k = 0, 1$, and $\lim_{n\to\infty} f_n(x)$ does not exist elsewhere in the unit interval.

NOTE ON A PAPER OF SHANKS

LEONARD CARLITZ, Duke University

In his paper [4] Shanks proved the following formula

$$\binom{x}{i}^k = \sum_{s=1}^{ik-i+1} A_{ks}^{i} \binom{x+s-1}{ik}; \tag{1}$$

he showed that the A_{ks}^{i} are positive integers and that $A_{ks}^{i} = A_{k,ik-i-s+2}^{s}$, and also gave a determinantal formula for the A_{ks}^{i}.

In this note we shall make several remarks concerning (1). In particular we shall generalize formulas (3) and (4) below.

Consider first the case $i=1$; then (1) becomes

$$x^k = \sum_{s=1}^{k} A_{ks} \binom{x+s-1}{k}, \tag{2}$$

where $A_{ks} = A_{ks}^{1}$. This formula is due to Worpitzky [6] but the numbers A_{ks} had been introduced in a different manner by Euler [2; 487–491], who gave the explicit expression

$$A_{ks} = \sum_{r=0}^{s} (-1)^r \binom{k+1}{r} (s-r)^k \qquad (k \geqq 1). \tag{3}$$

The recursion formula

$$A_{k+1,s} = (k+2-s)A_{k,s-1} + sA_{ks} \tag{4}$$

is due to Worpitzky. An interesting formula of a different kind is quoted by Vandiver [5; 506]. Let

$$R_k = R_k(x) = \sum_{s=1}^{k} A_{ks} x^{s-1},$$

and put $H_k = R_k/(x-1)^k$, then

$$(H+1)^k = xH_k \qquad (k \geqq 1), \tag{5}$$

where the left member is to be expanded by the binomial theorem and superscripts degraded to subscripts.

Before returning to (1) we derive a formula which follows readily from some results due to Nielsen [3; 26–30]. Nielsen shows that

$$(-1)^{m+n} x^n = \sum_{s=0}^{m+1} \binom{x+s-1}{m} B_s^{m,n} \qquad (m \geqq n),$$

where

$$B_s^{m,n} = \sum_{r=0}^{s} (-1)^r \binom{m+1}{r} (s-r)^n.$$

Let $f(x) = \sum_{n=0}^{m} a_n x^n$ be an arbitrary polynomial of degree $\leqq m$. Then

$$\begin{aligned} f(x) &= \sum_{n=0}^{m} (-1)^{m-n} a_n \sum_{s=0}^{m+1} \binom{x+s-1}{m} B_s^{m,n} \\ &= \sum_{s=0}^{m+1} \binom{x+s-1}{m} \sum_{r=0}^{s} (-1)^r \binom{m+1}{r} \sum_{n=0}^{m} (-1)^{m-n} a_n (s-r)^n \\ &= (-1)^m \sum_{s=0}^{m+1} \binom{x+s-1}{m} \sum_{r=0}^{s} (-1)^r \binom{m+1}{r} f(r-s). \end{aligned}$$

Now

$$\sum_{r=0}^{m+1} (-1)^r \binom{m+1}{r} f(r-m-1) = 0,$$

since it is essentially the $(m+1)$th difference of a polynomial of degree $\leqq m$. We have therefore the following result. *Let $f(x)$ be an arbitrary polynomial of degree $\leqq m$; then we have*

$$f(x) = (-1)^m \sum_{s=0}^{m} C_s \binom{x+s-1}{m}, \tag{6}$$

where the coefficients C_s are determined by

$$C_s = \sum_{r=0}^{s} (-1)^r \binom{m+1}{r} f(r-s). \tag{7}$$

Note that when $f(0) = 0$, the term corresponding to $s = 0$ in (6) drops out.

If in (6) we take $f(x) = \binom{x}{i}^k$, $m \geqq ik$, then the right member of (7) reduces to

$$(-1)^{ki} \sum_{r=0}^{s} (-1)^r \binom{m+1}{r} \binom{s-r+i-1}{i}^k; \tag{8}$$

in particular, for $m = ik$ we get

$$A_{ks}^{i} = \sum_{r=0}^{s} (-1)^r \binom{ik+1}{r} \binom{s-r+i-1}{i}^k \qquad (i, k \geqq 1). \tag{9}$$

For $i = 1$, (9) clearly reduces to (3).

In order to extend (4) to the general case we require the following identity:

$$(x)_i = \sum_{j=0}^{i} \alpha_j (x+s+j-1)_j (x+s-\lambda-1)_{i-j}, \tag{10}$$

where $(x)_i = x(x-1)\cdots(x-i+1)$, $(x)_0 = 1$, and

$$\alpha_j = \binom{i}{j} \frac{(\lambda + 1 - s)_j (s + i - 1)_{i-j}}{(\lambda + i)_i}. \tag{11}$$

This formula is readily proved by an induction with respect to i; λ is arbitrary, but for the application we take $\lambda = ik$. Then multiplying both sides of (1) by $\binom{x}{i}$ and using (10) we get

$$A^i_{k+1,s} = \frac{1}{i!} \sum_{j=0}^{i} \binom{i}{j} (ik - s + j + 1)_j (s + i - j - 1)_{i-j} A^i_{k,s-j}. \tag{12}$$

For $i = 1$ this reduces to (4).

If we define the operator E by means of $Ef(s) = f(s-1)$ and agree that E operates only on s, we can put (12) in the form

$$A^i_{k+1,s} = \frac{1}{i!} \Omega_i A^i_{ks},$$

where

$$\Omega_i = \sum_{j=0}^{i} \binom{i}{j} (ik - s + j + 1)_j (s + i - j - 1)_{i-j} E^j. \tag{13}$$

Then Ω_i can be factored as follows:

$$\Omega_i = (\mu E + s)(\mu E + s + 1)\cdots(\mu E + s + i - 1), \tag{14}$$

where $\mu = ik - s + 2$. It is easy to prove (14) by induction if we put it in slightly stronger form, namely, let

$$\Omega'_i = \sum_{j=0}^{i} \binom{i}{j} (\lambda - s + j + 1)_j (s + i - j - 1)_{i-j} E^j. \tag{15}$$

Then (14) holds for Ω'_i where now $\mu = \lambda - s + 2$. Indeed it is only necessary to substitute from (15) in $\Omega'_1 (E + s + i)$ and verify that the result reduces to Ω'_{i+1}.

We remark that the linear factors in (14) permute.

For formulas similar to (12) and (14) see [1].

References

1. L. Carlitz, On arrays of numbers, Amer. J. Math., vol. 54, 1932, pp. 739–752.
2. L. Euler, Institutiones calculi differentialis, Petrograd, 1755.
3. N. Nielsen, Traité élémentaire des nombres de Bernoulli, Paris, 1923.
4. E. B. Shanks, Iterated sums of powers of the binomial coefficients, this MONTHLY, vol. 58, 1951, pp. 404–407.
5. H. S. Vandiver, An arithmetical theory of the Bernoulli numbers, Trans. Amer. Math. Soc., vol. 51, 1942, pp. 502–531.
6. J. Worpitzky, Studien über die Bernoullischen und Eulerschen Zahlen, Journal für die reine und angewandte Mathematik, vol. 94, 1883, pp. 202–232.

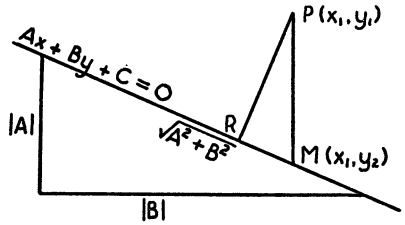
CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

DISTANCE FROM A LINE, OR PLANE, TO A POINT*

J. P. BALLANTINE and A. R. JERBERT, University of Washington



To find the perpendicular distance from the line $Ax+By+C=0$, to the point $P(x_1, y_1)$, we begin by computing the y-distance,

$$MP = y_1 - y_2 = y_1 - (-Ax_1 - C)/B = (Ax_1 + By_1 + C)/B, \tag{1}$$

where y_2 is the y-value obtained by substituting $x=x_1$ in the equation of the line. Since the y and x intercepts of the latter are in the ratio $(-C/B)/(-C/A) = A/B$, the intercept triangle has sides proportional to $|A|$, $|B|$, $\sqrt{A^2+B^2}$. MRP is evidently a similar triangle so that,

$$RP/MP = |B|/\sqrt{A^2+B^2},$$

whence,

$$\begin{aligned} d = RP &= \frac{|B|}{\sqrt{A^2+B^2}} \cdot MP \\ &= \frac{|B|(Ax_1+By_1+C)}{B\sqrt{A^2+B^2}}, && \text{by equation (1),} \\ &= \frac{Ax_1+By_1+C|}{(\text{signum } B)\sqrt{A^2+B^2}} && (\text{signum } B = |B|/B). \end{aligned} \tag{2}$$

Any $K \neq 0$, which is multiplied into the coefficients A, B, C of the equation of the line evidently "divides out" of the factors in the right member of equation (2). These factors and their product d are therefore invariant under such a multiplication. The second factor MP, yields $+$ and $-$ values for points above and below the line, respectively, and this remains true for d since the first factor, $|B|\sqrt{A^2+B^2}$, is always positive.

* This paper is based on Professor J. P. Ballantine's treatment of the distance formula on page 235 of his book Essentials of Engineering Mathematics, published by Prentice Hall, 1938.

In three dimensions the x, y, and z distances from the plane $A_1x+A_2y+A_3z+A_4=0$, to the point $P(x_1, y_1, z_1)$ are given by the expressions,

$$(A_1x_1 + A_2y_1 + A_3z_1 + A_4)/A_i, \qquad i = 1, 2, 3,$$

respectively. To obtain the perpendicular distance we multiply by the cosine of the acute angle between the chosen axis and the normal to the plane, *i.e.* by $|A_i|/\sqrt{A_1^2+A_2^2+A_3^2}$. Thus, as before,

$$d = (A_1x_1 + A_2y_1 + A_3z_1 + A_4)/(\text{signum } A_i)\sqrt{A_1^2 + A_2^2 + A_3^2}.$$

In this manner d is $+$ or $-$ according as the point P is right or left, front or back, above or below, the plane, respectively.

A SIMPLE PROOF OF THE BINOMIAL THEOREM

C. M. FULTON, University of California, Davis

As most mathematics teachers know from experience, our freshman students find the usual formal proof of the binomial theorem hard to understand. In this paper we present a different proof whose simplicity, we feel, will speak for itself.

The tool for our proof will be the easily proven identity

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1}). \tag{1}$$

We use it first to write

$$(1 + x)^n - (1 + y)^n = (x - y)[(1 + x)^{n-1} + (1 + x)^{n-2}(1 + y) + \cdots + (1 + y)^{n-1}]. \tag{2}$$

Now, the students will grant the validity of the expansion

$$(1 + x)^n = 1 + {}_nC_1x + {}_nC_2x^2 + \cdots + {}_nC_nx^n \tag{3}$$

with coefficients ${}_nC_r$ to be determined, n being a positive integer. We can express the left side of (2) with the aid of (3) and apply (1) to the differences of like powers. Thus,

$$(1 + x)^n - (1 + y)^n = (x - y)[{}_nC_1 + {}_nC_2(x + y) + {}_nC_3(x^2 + xy + y^2) + \cdots + {}_nC_n(x^{n-1} + x^{n-2}y + \cdots + y^{n-1})]. \tag{4}$$

We now equate the right sides of (2) and (4) canceling the common factor $x-y$. In the resulting identity it is still permissible to make $x=y$. Hence,

$$n(1 + x)^{n-1} = {}_nC_1 + 2{}_nC_2x + 3{}_nC_3x^2 + \cdots + n{}_nC_nx^{n-1}. \tag{5}$$

On the other hand, because of (3), we also have

$$n(1 + x)^{n-1} = n(1 + {}_{n-1}C_1x + {}_{n-1}C_2x^2 + \cdots + {}_{n-1}C_{n-1}x^{n-1}). \tag{6}$$

On comparing the right sides of (5) and (6) we see that ${}_nC_1=n$. Then necessarily ${}_{n-1}C_1=n-1$. Since the coefficients of x are equal we find next ${}_nC_2=n(n-1)/2$.

Therefore, ${}_{n-1}C_2=(n-1)(n-2)/2$. This allows us to determine ${}_nC_3$, *etc.* In general,

$$ {}_nC_r = \frac{n}{r}\,{}_{n-1}C_{r-1}, \quad {}_nC_0 = 1 \qquad (r = 1, 2, \cdots, n). $$

Finally, from these recurrence formulas, we may easily prove, by induction, that

$$ {}_nC_r = \frac{n!}{r!(n-r)!} \qquad (r = 1, 2, \cdots, n). $$

INTEGRABILITY OF CONTINUOUS FUNCTIONS

M. J. Norris, College of St. Thomas

In many current calculus texts the proof that a continuous function is integrable is unsatisfactory; for it is assumed without mention, or at best stated to be true with proof deferred to advanced courses, that a function continuous on a closed interval is uniformly continuous there. Now we do not wish to advocate teaching uniform continuity to sophomores, but we would like to suggest another possible remedy in a different type of proof.

The method of this note can be used with any definition such that integrability of a function can be proved by showing that the function is bounded and that upper and lower integrals, satisfying (d) and (e) below, are equal. However, it seems particularly well suited to the definition of integrability suggested herein, which we feel is somewhat simpler than those involving generalized limit processes.

Given the closed interval $[a, b]$ we consider partitions, $P(x_0, x_1, x_2, \cdots, x_n)$, of $[a, b]$ into the n subintervals $[x_{i-1}, x_i]$, $(1 \leqq i \leqq n)$, where $a=x_0<x_1<x_2 \cdots <x_n=b$. We say that the partition P_2 is a refinement of the partition P_1 if every endpoint used in P_1 is also used in P_2. If now f is a bounded function on $[a, b]$, and P is a partition of $[a, b]$, we let $U(P)$ and $L(P)$ be respectively the sums $\sum_{i=1}^{n} M_i(x_i-x_{i-1})$ and $\sum_{i=1}^{n} m_i(x_i-x_{i-1})$, where M_i is the least upper bound of f on $[x_{i-1}, x_i]$ and m_i is the greatest lower bound of f there. It is easy to prove the following results:

(a) $L(P) \leqq U(P)$.

(b) If P_2 is a refinement of P_1, then $L(P_2) \geqq L(P_1)$ and $U(P_2) \leqq U(P_1)$.

(c) $L(P_1) \leqq U(P_2)$ for any two partitions P_1 and P_2.

We obtain (c) by taking a common refinement of P_1 and P_2 and then using (a) and (b).

It is now natural to consider the greatest lower bound, $\overline{\int_a^b} f dx$, of the $U(P)$'s and the least upper bound, $\underline{\int_a^b} f dx$, of the $L(P)$'s. Plainly we have:

(d) If M and m are the least upper bound and greatest lower bound, respectively, of f on $[a, b]$, then $m(b-a) \leqq \underline{\int_a^b} f dx \leqq \overline{\int_a^b} f\, dx \leqq M(b-a)$.

(e) If $a<c<b$, then $\underline{\int_a^c} f dx + \underline{\int_c^b} f dx = \underline{\int_a^b} f dx$ and $\overline{\int_a^c} f dx + \overline{\int_c^b} f dx = \overline{\int_a^b} f dx$.

We have (e) as a consequence of the facts that the values for $\underline{\int_a^b} f dx$ and $\overline{\int_a^b} f dx$

are completely determined by partitions which are refinements of $P(a, c, b)$ and that any partition of $[a, c]$ can be combined with any partition of $[c, b]$ to yield a partition of $[a, b]$.

Finally we say that f is integrable on $[a, b]$ if $\underline{\int}_a^b f dx = \overline{\int}_a^b f dx$. We are now ready to prove the theorem on integrability of continuous functions.

THEOREM. *If f is continuous on the closed interval $[a, b]$, then f is integrable on $[a, b]$.*

Proof. As a continuous function on a closed interval f is bounded on $[a, b]$. Thus we can define the function $Z(x)$ by

$$Z(x) = \overline{\int}_a^x f dx - \underline{\int}_a^x f dx \qquad (a < x \leqq b).$$

For $b - x \geqq h > 0$ we have

$$Z(x + h) - Z(x) = \overline{\int}_x^{x+h} f dx - \underline{\int}_x^{x+h} f dx.$$

Thus, if $M(h)$ and $m(h)$ are the maximum and minimum, respectively, of f on $[x, x+h]$, we have

$$0 \leqq Z(x + h) - Z(x) \leqq M(h)h - m(h)h,$$

$$0 \leqq \frac{Z(x + h) - Z(x)}{h} \leqq M(h) - m(h).$$

Now $\lim_{h \to 0} M(h)$ and $\lim_{h \to 0} m(h)$ both exist and are equal to $f(x)$, so $\lim_{h \to 0} [Z(x+h) - Z(x)]/h$ exists and is zero. A similar argument is available if $a - x < h < 0$, so $Z'(x)$ exists and is zero for $a < x < b$ and $Z(x)$ is continuous for $a < x \leqq b$. Thus $Z(x)$ is constant for $a < x \leqq b$; and as $\lim_{x \to a} Z(x)$ is clearly zero, we must have $Z(x)$ identically zero for $a < x \leqq b$. In particular, $\underline{\int}_a^b f dx = \overline{\int}_a^b f dx$.

Incidentally, the proof above is valid, after a few minor changes in wording, for functions which have right and left limits at every inner point of $[a, b]$, a right limit at a, and a left limit at b.

COORDINATE GEOMETRY FROM THE VECTOR POINT OF VIEW

SAMUEL BOURNE, University of Connecticut

Introduction. From a mathematical point of view, the vector approach to coordinate geometry is superior to the conventional slope approach, in that concepts do not depend on the dimensionality of the space, and proofs are simplified. From a practical point of view, its superiority rests in the early introduction of vectors into the student's mathematical education, thereby enabling him to utilize the algebra of vectors in his study of the physical sciences. The above opinions are supported by years of teaching this subject at the Johns

Hopkins University, under Professor F. D. Murnaghan. We give a few examples to illustrate them.

Vectors. In defining a vector, we are cognizant of the mathematical immaturity of the student and we constantly appeal to his geometric experience. If $P_1=(x_1, y_1)$ and $P_2=(x_2, y_2)$, we define the direction numbers of $\overrightarrow{P_1P_2}$ to be $\Delta x=x_2-x_1$ and $\Delta y=y_2-y_1$, and we write $\overrightarrow{P_1P_2}=(\Delta x, \Delta y)$. Conversely, each ordered pair of numbers determines directed line segments having these numbers as direction numbers. Hence, we may define a *vector as the collection of directed line segments having equal direction numbers.* This definition is adequate at this early date. Each directed line segment is called a representative line segment of the vector and the length of such a line segment is defined to be the magnitude of the vector. We denote by $\vec{v}(P_1\rightarrow P_2)$ a vector, a representative segment of which is $\overrightarrow{P_1P_2}$. If $\overrightarrow{P_1P_2}=(\Delta x, \Delta y)$ and $r=+\sqrt{(\Delta x)^2+(\Delta y)^2}\neq 0$, then we define the direction cosines (l, m) of P_1P_2 to be $l=\Delta x/r=\cos\theta$ and $m=\Delta y/r=\sin\theta$, where θ is an angle from the positive direction of the x-axis to the direction $\overrightarrow{P_1P_2}$. Obviously, $l^2+m^2=\cos^2\theta+\sin^2\theta=1$.

Angle. We let θ be an angle from the vector $\vec{v}_1$ to the vector $\vec{v}_2$; then we have

THEOREM 1. *If the direction cosines of* $\vec{v}_1$ *are* (l_1, m_1) *and if the direction cosines of* $\vec{v}_2$ *are* (l_2, m_2), *then* $\cos\theta=l_1l_2+m_1m_2$.

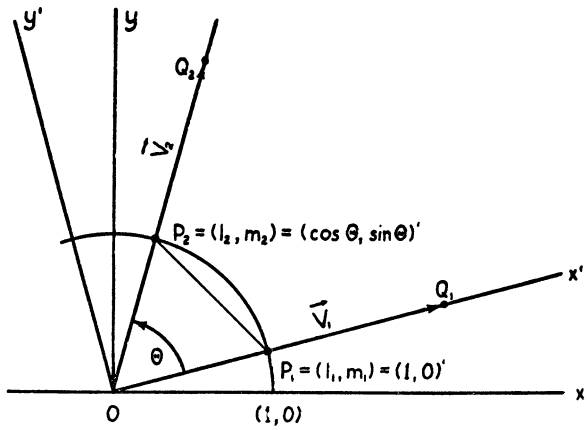


FIG. 1

We let $\overrightarrow{OQ_1}$ and $\overrightarrow{OQ_2}$ be representative segments of the vectors $\vec{v}_1$ and $\vec{v}_2$, respectively, and P_1 and P_2 the points in which the unit circle, at the origin, intersects these segments. Then, $P_1=(l_1, m_1)$ and $P_2=(l_2, m_2)$ relative to xy-axes, and $P_1=(1, 0)$ and $P_2=(\cos\theta, \sin\theta)$, relative to $x'y'$-axes. We have

$$|\overrightarrow{P_1P_2}|^2=(l_2-l_1)^2+(m_2-m_1)^2=2-2(l_1l_2+m_1m_2),$$

$$|\overrightarrow{P_1P_2}|^2 = (\cos\theta - 1)^2 + \sin^2\theta = 2 - 2\cos\theta;$$

therefore

$$\cos\theta = l_1l_2 + m_1m_2.$$

Since

$$l_1 = \frac{\Delta_1 x}{r_1}, \quad l_2 = \frac{\Delta_2 x}{r_2}, \quad m_1 = \frac{\Delta_1 y}{r_1} \quad \text{and} \quad m_2 = \frac{\Delta_2 y}{r_2}, \quad r_1r_2 \neq 0,$$

we have that

$$\cos\theta = \frac{(\Delta_1 x)(\Delta_2 x) + (\Delta_1 y)(\Delta_2 y)}{r_1r_2}.$$

This equality motivates us to define *the symmetric product* $\overrightarrow{v_1} \cdot \overrightarrow{v_2}$ *of the vector* $\overrightarrow{v_1} = (x_1, y_1)$ *by the vector* $\overrightarrow{v_2} = (x_2, y_2)$ *to be* $x_1x_2 + y_1y_2$. We prove, with ease, the well-known properties of this product with their corresponding geometric interpretations.

Straight line. We use this symmetric product to prove

THEOREM 2. *An equation of a straight line is an equation of the form* $ax+by+c=0$. *Conversely, the graph of the equation* $ax+by+c=0$ *is a straight line.*

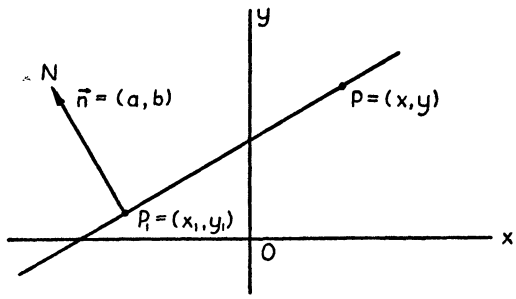


FIG. 2

We let $P_1 = (x_1, y_1)$ be a fixed point on the given line, $\overrightarrow{n} = (a, b)$ a vector normal to this line, and $P = (x, y)$ any point on this line. Then

$$\overrightarrow{v}(P_1 \to N) \cdot \overrightarrow{v}(P_1 \to P) = 0, \qquad P \neq P_1;$$

therefore

$$a(x - x_1) + b(y - y_1) = 0, \quad \text{for all} \quad P.$$

Therefore

$$ax + by + c = 0, \quad \text{where} \quad c = -ax_1 - by_1.$$

The proof of the converse follows by a retracing of steps.

Distance from a line to a point. Again, by a judicious employment of the symmetric product, we may prove

THEOREM 3. *The distance d from the line $ax+by+c=0$ to the point $P_1=(x_1, y_1)$ is given by*

$$\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}.$$

Proof:

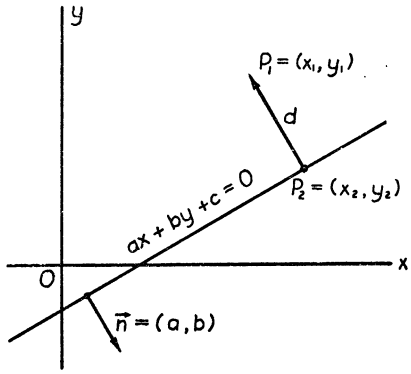


FIG. 3

We let $\vec{n}=(a, b)$ be a normal to the line $ax+by+c=0$ and $P_2=(x_2, y_2)$ the foot of the perpendicular from $P_1=(x_1, y_1)$ to this line. Then $\vec{n}$ and $\overrightarrow{P_1P_2}$ are parallel, and if θ is an angle from $\vec{n}$ to $\overrightarrow{P_1P_2}$ we have:

$$(a, b)\cdot(x_1-x_2, y_1-y_2)=\sqrt{a^2+b^2}\,d\cos\theta;$$

therefore

$$a(x_1-x_2)+b(y_1-y_2)=\pm d\sqrt{a^2+b^2},$$

and

$$d=\frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}.$$

Conclusion. The above few examples, in the plane, illustrate adequately the effectiveness of this approach. We may point out that no trigonometry need be presupposed. The definition of the vector product of two space vectors is motivated by the necessity to find direction numbers of the line of intersection of two planes. The laws of trigonometry are merely a by-product of the relationships of the algebra of plane vectors, while the laws of spherical trigonometry spring from the corresponding relationships of the algebra of space vectors.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1011. *Proposed by D. D. Wall, Los Angeles, California*

If traffic signals are "set for 30," at what other constant speeds may one travel? Assume signals evenly spaced, all change at the same time, and that they alternate in color spacewise as well as timewise.

E 1012. *Proposed by Oystein Ore, Yale University*

In a multiplication like

$$\begin{array}{r} 11 \\ \times 11 \\ \hline 11 \\ 11 \\ \hline 121 \end{array}$$

there occur only the two digits 1 and 2. Find all multiplications involving exactly two digits.

E 1013. *Proposed by P. A. Piza, San Juan, Puerto Rico*

Show that for $a=0, 1, 2, \cdots, 10$ and for any positive integral n except 1 and 3, the expression

$$(10^n - 1)(909a + 1)$$

yields palindromic numbers.

E 1014. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a triangle having sides a, b, c and area S, if the ratio

$$(a^2 + b^2 + c^2)/4S$$

is a whole number $m>1$, then the sides a, b, c cannot all be integral.

E 1015. *Proposed by C. A. Shook and Albert Wilansky, Lehigh University*

Evaluate

$$\int \frac{dx}{x[J_n(x)]^2},$$

where J_n is the usual Bessel function.

SOLUTIONS

A Tetrahedron and Some Spheres

E 980 [1951, 564]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a sphere through the vertices A, B, C of a tetrahedron $ABCD$ cuts the edges DA, DB, DC in A', B', C', then the distances of its center from those of the spheres $ABCD$ and $A'B'C'D$ are equal, respectively, to the radii of the spheres $A'B'C'D$ and $ABCD$

Solution by S. T. Thompson, Plattsburg, N. Y. The planes ABC and $A'B'C'$ are antiparallel with respect to the trihedral angle D. It follows that DO is perpendicular to $A'B'C'$ and DO' to ABC. But, if P is the center of the given sphere, PO' is perpendicular to $A'B'C'$ and PO to ABC. Thus $DOPO'$ is a parallelogram and the theorem is established. The analogous theorem for the plane is also true.

Also solved by Leon Bankoff and Joseph Langr.

Geometric Progression with Two Ratios

E 981 [1951, 564]. *Proposed by David Mandelbaum, Hillside, N. J.*

Derive a formula for the sum of the first n terms of a progression in which the first term is a, each even placed term is obtained from its preceding term by multiplying by the constant u, and each odd placed term (after the first) is obtained from its preceding term by multiplying by the constant v.

Solution by Ferrel Atkins, Bowling Green State University. The first n terms of the given series will be made up of $[(n+1)/2]$ terms of the series

$$a + a(uv) + a(uv)^2 + \cdots$$

and $[n/2]$ terms of the series

$$au + au(uv) + au(uv)^2 + \cdots,$$

where $[x]$ denotes the greatest integer not greater than x. Thus

$$S_n = a\left\{\frac{1-(uv)^{[(n+1)/2]}}{1-uv}\right\} + au\left\{\frac{1-(uv)^{[n/2]}}{1-uv}\right\}.$$

If $|uv|<1$, then

$$S_\infty = a(1+u)/(1-uv).$$

Also solved by A. N. Aheart, P. M. Anselone, Leon Bankoff, J. H. Braun, Fred Discepoli, F. F. Dorsey, D. H. Friedman, C. V. Fronabarger, B. K. Gold, Boyd Henry, Vern Hoggatt, J. M. Kingston, Sam Kravitz, Sidney Kravitz,

L. H. Lange, H. R. Leifer, Kovina Milosevich, R. Muguercia, Prasert Na Nagara, C. S. Ogilvy, Margaret Olmsted, C. F. Pinzka, Azriel Rosenfeld, C. W. Trigg, R. Z. Vause, Jr., G. W. Walker, E. J. Zirkel, and the proposer.

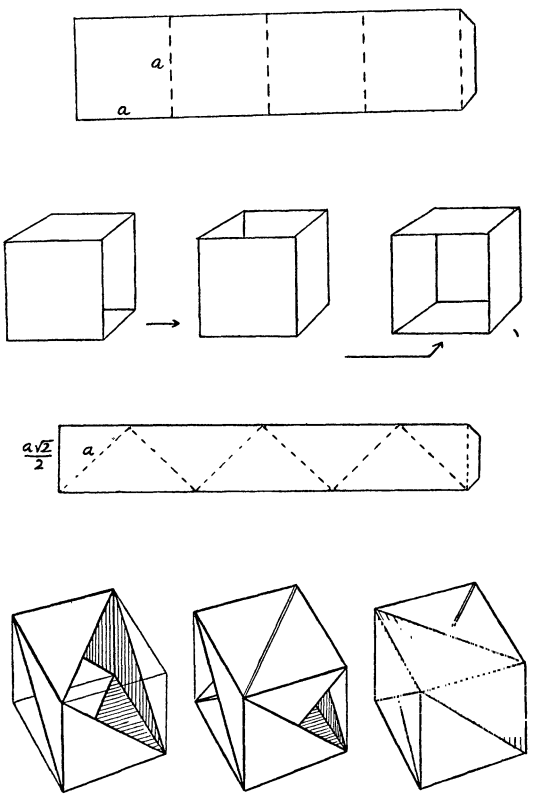
Milosevich pointed out that this sum was evaluated by L. Conte, *Rendite a termini variabili in progressione arithmetica o geometrica a due ragioni*, Giornale di Matematica Finanziaria, vol. VII, No. 1–2, 1949, pp. 27–46.

Folding Cylinders into a Cube

E 982 [1951, 564]. *Proposed by C. W. Trigg, Los Angeles City College*

Show that there are two different sets of three congruent right cylindrical surfaces which may be folded and assembled into a cube with no open edges.

Solution by Leon Bankoff, Los Angeles, California. The following diagrams are self-explanatory.



Also solved by the proposer.

A Geometric Inequality

E 983 [1951, 564]. *Proposed by A. W. Goodman, University of Kentucky*

Let M and N be two points one unit apart. With M and N as centers and with unit radii draw arcs ANB and AMB. Let Q be any point on arc AMB and P_1 and P_2 any points on arc ANB such that N is the midpoint of arc P_1P_2. Show that

$$QP_1 + QP_2 \leqq 2 \leqq (QP_1)^2 + (QP_2)^2.$$

Solution by C. W. Trigg, Los Angeles City College. Let O be the midpoint of chord P_1P_2. The locus of Q' such that $Q'P_1+Q'P_2=2$ is an ellipse with semi-minor axis OM (since $P_1M+P_2M=2$), and major axis lying along P_1P_2, centered at O and equal to 2. Now O lies between M and N, so the arc AMB lies within the ellipse and is tangent to it at M. Hence $QP_1+QP_2 \leqq Q'P_1+Q'P_2=2$.

The locus of Q'' such that $(Q''P_1)^2+(Q''P_2)^2=2$ is a circle with its center at O, and radius OM (since $(P_1M)^2+(P_2M)^2=2$). Now $OM<NM$, so the circle $O(M)$ lies wholly within the circle $N(M)$ and is tangent to $N(M)$ at M. Hence $2=(Q''P_1)^2+(Q''P_2)^2 \leqq (QP_1)^2+(QP_2)^2$. Hence the proposition.

Also solved by Ferrel Atkins, Leon Bankoff, J. H. Braun, Fred Discepoli, C. V. Fronabarger, W. J. Klimczak, Sam Kravitz, F. M. Morgan, Prasert Na Nagara, Margaret Olmsted, and R. P. Tapscott.

An Iterated Function

E 984 [1951, 564]. *Proposed by Joseph Rosenbaum, Hartford, Conn.*

(a) Find $f(x)$ when $f[f(x)]=x^2-2$.

(b) More generally, find $f_1(x)$ when $f_n(x)=x^2-2$, where $f_n(x)$ is defined by the relation $f_{r+1}(x)=f_1[f_r(x)]$.

I. *Solution by J. L. Botsford and R. E. Wild, University of Idaho.* For the general case take

$$f_1(x) = 2 \cos (2^{1/n} \arccos x).$$

Then

$$f_r(x) = 2 \cos (2^{r/n} \arccos x),$$

and

$$f_n(x) = 2 \cos (2 \arccos x) = x^2 - 1.$$

II. *Solution by the Proposer.* For the general case take

$$f_1(x) = [(x + \sqrt{x^2 - 4})/2]^{2^{1/n}} + [(x - \sqrt{x^2 - 4})/2]^{2^{1/n}}.$$

The desired relation may be conveniently verified by making the substitution $x=r+1/r$.

A Problem in Partitioning

E 985 [1951, 564]. *Proposed by Leo Moser, Texas Technological College*

What is the smallest integer N such that if the integers 1, 2, 3, $\cdots$, N are distributed into three classes in any manner whatsoever at least one of the classes will contain a solution of $a+b=c$?

I. *Solution by G. W. Walker, Buffalo, N. Y.* The partitioning

1, 2, 4, 8, 11, 22

3, 5, 6, 7, 19, 21, 23

9, 10, 12, 13, 14, 15, 16, 17, 18, 20

shows that $N>23$. We might also have either 16 or 17 (but not both) in the first row instead of in the third. There are just these three partitionings of the first 23 positive integers into three rows without a case of $a+b=c$ in some row. In none of these cases can we add 24 to any row. Hence $N=24$.

If N_s is the smallest integer that cannot by any arrangement be accommodated in s rows, then $N_4=67$ and $N_5=197$. It can be shown that

$$2N_s < N_{s+1} < 3N_s.$$

Also solved similarly by J. H. Braun, who claimed that

$$N_s \geqq 3(3^s + 2s - 1)/4.$$

II. *Remark by the Proposer.* In the above solution it is assumed, in the relation $a+b=c$, that a, b, c are distinct. Without this limitation we may show that $N\leqq 16$. For, if the numbers 1, 2, $\cdots$, 16 are put into three classes A, B, C, then at least one of these, say A, will contain at least six numbers, say $a_1<a_2<a_3<a_4<a_5<a_6$. Now consider the five numbers $b_1=a_6-a_5$, $b_2=a_6-a_4$, $b_3=a_6-a_3$, $b_4=a_6-a_2$, $b_5=a_6-a_1$. We may assume that none of these is in A, for otherwise A would contain a solution to $a+b=c$. We may therefore suppose that at least three of these are in B or C. Suppose $b_1'<b_2'<b_3'$ are three b's in B. Next consider $c_1=b_3'-b_2'$, $c_2=b_3'-b_1'$. If c_1 or c_2 is in A or B, then this class would contain a solution to $a+b=c$. Hence we may assume that c_1 and c_2 are in C. Finally, consider c_2-c_1. If this number is in A, B, or C, then that class would contain a solution to $a+b=c$.

On the other hand, the partitioning

1, 4, 10, 13

2, 3, 11, 12

5, 6, 7, 8, 9

shows that $N\geqq 14$.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4483. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Find the sum of

$$1+\frac{1}{2}\,\frac{1}{3^2}+\frac{1\cdot 3}{2\cdot 4}\,\frac{1}{5^2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\,\frac{1}{7^2}+\cdots.$$

4484. *Proposed by Murray Gerstenhaber and R. S. Varga, Harvard University*

Given a real number $y>0$. Let $g_n=h_n/n$ be the rational number which best approximates y, where h_n is a positive integer and $n=1, 2, 3, \cdots$. Let

$$\lambda_n=\begin{cases}+ & \text{if } \ h_n/n-y\geqq 0,\\ - & \text{if } \ h_n/n-y<0.\end{cases}$$

Show that λ_n is periodic, *i.e.* there exists a positive integer k such that λ_{mk} is of one sign for all positive integers m, if and only if y is a positive rational number.

4485. *Proposed by Simon T. Kao, Catholic University of America, Washington, D. C.*

Through the orthocenter H of a triangle ABC draw any pair of perpendicular lines l_1 and l_2, and let A_1, A_2; B_1, B_2; C_1, C_2 be the respective points of intersection with the three sides BC, CA, AB. Show that the three points P, Q, R which divide the three segments A_1A_2, B_1B_2, C_1C_2 in the same ratio r, lie on a line. This is a generalization of the Droz-Farny theorem.

4486. *Proposed by D. J. Newman, Harvard University*

Let $f(x)$ be any function continuous on the positive axis. Prove that there exists an entire function which is always larger than this $f(x)$ along the positive axis. (That is, prove that there are arbitrarily large entire functions.)

4487. *Proposed by Paul Erdös, University College, London, England*

Let

$$\sum_{k=1}^{\infty} d_k=\infty, \qquad d_k>0, \qquad D_n=\sum_{k=1}^{n} d_k;$$

$$\sum_{k=1}^{\infty} a_k < \infty, \qquad a_k > 0.$$

Further assume that $a_k/d_k \geqq a_{k+1}/d_{k+1}$. Prove

$$\lim_{k\to\infty} \frac{a_k D_k}{d_k} = 0.$$

SOLUTIONS

Sets Having a Divisor Property

4358 [1949, 480]. *Proposed by Paul Erdös, University College, London, England*

Let $a_1 < a_2 < \cdots$ be a sequence of integers with the property that there does not exist an infinite subsequence of the a's in which no one divides another. Prove that the products

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_k^{\alpha_k}, \qquad 0 \leqq \alpha_i$$

have the same property.

If the a's are chosen to be the first k primes, then we obtain a well-known result due to Dickson.

Solution by Paul Erdös and Richard Rado, University College, and King's College, London. Let S be a set of positive integers. We say that S has the *divisor property* if every infinite subset of S contains two distinct numbers one of which divides the other. The present problem may be restated as the

THEOREM. *If S possesses the divisor property, and if S' is the set of all numbers of the form $a_1 a_2 \cdots a_n$, where $n>0$, $a_\nu \in S$, then S' possesses the divisor property.*

The letters μ, ν, ρ, k, m denote positive integers, S is a fixed set of positive integers greater than 1, and the letter a denotes elements of S. By S^* we denote the set of all numbers of the form $a_1 a_2 \cdots a_n$, where $n \geqq 0$. Thus $1 \in S^*$. The letters A, B, C, D, E, F denote elements of S^*. We assume that S has the divisor property and proceed to show that S^* has the divisor property.

LEMMA.† *If $a_\nu \in S$ for all ν, then there are indices ν_ρ such that $\nu_1 < \nu_2 < \cdots$, $a_{\nu_1} | a_{\nu_2} | \cdots$.*‡

Proof. Let M be the set of all μ such that $a_\mu > a_\nu$ if $\mu < \nu$. Then M is finite, since otherwise we could find $\mu_\rho \in M$ such that $\mu_1 < \mu_2 < \cdots$. Then $a_{\mu_1} > a_{\mu_2} > \cdots$, which is impossible. Now we can choose an index μ_1 greater than all elements of M and, successively, μ_2, μ_3, $\cdots$ such that $\mu_1 < \mu_2 < \cdots$, $a_{\mu_1} \leqq a_{\mu_2} \leqq \cdots$. Put $M_1 = \{\mu_1, \mu_2, \cdots\}$. Let M_2 be the set of all $\mu \in M_1$ such that a_μ is

† This is problem 4330 [1950, 493–494].

‡ The symbol $x|y|z|\cdots$ means $x|y$, $y|z$, $\cdots$.

not a divisor of a_ν if $\mu<\nu$, $\nu\in M_1$. Then M_2 is finite. For, otherwise, since S has the divisor property, we could find distinct indices μ, $\nu\in M_2$ such that $a_\mu \mid a_\nu$. Then, by definition of M_2, $\nu<\mu$ and therefore, by definition of M_1, $a_\nu \leqq a_\mu$, $a_\nu = a_\mu$, $a_\nu \mid a_\mu$, which contradicts $\nu\in M_2$. Now we can choose $\nu_1\in M_1$, ν_1 greater than all elements of M_2, and, successively, ν_2, ν_3, $\cdots \in M_1$ such that $\nu_1<\nu_2 < \cdots$, $a_{\nu_1} \mid a_{\nu_2} \mid \cdots$, and the lemma is proved.

Proof of the theorem. Let $A_\nu\in S^*$ for all ν. Our aim is to find distinct indices μ, ν such that $A_\mu \mid A_\nu$. We define, inductively, numbers n_k, B_k, C_k as follows:

(i) We put $n_0=1$, $B_0=1$, $C_0=A_1$. We note that it is possible to choose m_ρ, D_ρ, E_ρ such that $n_0<m_1<m_2< \cdots$,

$$A_{m_\rho} = D_\rho E_\rho, \qquad B_0 \mid D_1 \mid D_2 \mid D_3 \mid \cdots,$$

a possible choice being $m_\rho=\rho+1$, $D_\rho=1$, $E_\rho=A_{\rho+1}$.

(ii) Suppose that we have already defined, for some k, numbers n_{k-1}, B_{k-1}, C_{k-1} in such a way that it is possible to choose numbers m'_ρ, D'_ρ, E'_ρ such that $n_{k-1}<m'_1<m'_2< \cdots$, $A_{m'_\rho}=D'_\rho E'_\rho$, $B_{k-1} \mid D'_1 \mid D'_2 \mid \cdots$. Then we choose m'_ρ, D'_ρ, E'_ρ in such a way that E'_1 is least possible, and we put $n_k=m'_1$, $B_k=D'_1$, $C_k=E'_1$.

The numbers n_k, B_k, C_k have now been defined for all k, and they satisfy the following conditions:

$$n_1 < n_2 < n_3 < \cdots, \qquad A_{n_\rho} = B_\rho C_\rho, \qquad B_1 \mid B_2 \mid \cdots.$$

Moreover, for every k, the number C_k has the following *minimum property*. Whenever $n_{k-1}<m''_1<m''_2< \cdots$, $A_{m''_\rho}=D''_\rho E''_\rho$, and $B_{k-1} \mid D''_1 \mid D''_2 \mid \cdots$, then $C_k \leqq E''_1$.

(iii) We now distinguish two cases.

Case 1. Suppose that there is a k such that $C_k=1$. Then $A_{n_k}=B_k \mid B_{k+1} \mid A_{n_{k+1}}$, and the desired result holds for $\mu=n_k$, $\nu=n_{k+1}$.

Case 2. Suppose that $C_k>1$ for every k. Then there are numbers a_k, F_k such that $C_k=a_kF_k$. By the lemma, there are indices ν_ρ such that $\nu_1<\nu_2< \cdots$, $a_{\nu_1} \mid a_{\nu_2} \mid \cdots$. Then

$$A_{n_{\nu_\rho}} = B_{\nu_\rho} C_{\nu_\rho} = (B_{\nu_\rho} a_{\nu_\rho}) F_{\nu_\rho}, \qquad n_{\nu_1-1} < n_{\nu_1} < n_{\nu_2} < \cdots,$$
$$B_{\nu_1-1} \mid B_{\nu_1} a_{\nu_1} \mid B_{\nu_2} a_{\nu_2} \mid \cdots, \qquad F_{\nu_1} = C_{\nu_1}/a_{\nu_1} < C_{\nu_1},$$

which contradicts the minimum property of C_{ν_1}. This completes the proof of the theorem.

Note. The following more general theorem can be proved. Let $\{a_\alpha\}$ be a partially ordered set with the property that every infinite subset $\{a_{\alpha_k}\}$ contains a finite subset b_1, b_2, $\cdots$, b_l such that every a_{α_k} is no less than at least one of the b's. A set with this property may be called partially well ordered. (For a well ordered set $l=1$.)

Consider now the set of all vectors of finite length

$$(a_{\alpha_1}, a_{\alpha_2}, \cdots, a_{\alpha_n})$$

formed from the a's. We define

$$(a_{\alpha_1}, a_{\alpha_2}, \cdots, a_{\alpha_r}) \leqq (a_{\beta_1}, \cdots, a_{\beta_l})$$

if there exists a one to one mapping of the a_α's into the a_β's so that each a_α is less than or equal to the corresponding a_β. Our theorem now states that these vectors form a partially well ordered set. The theorem of problem 4358 is the special case in which the partial order in the set of integers is defined by $a \leqq b$ if $a \mid b$.

G. Higman and B. H. Neumann have found, independently of each other and of the present writers, the following still more general theorem: The set of vectors of finite length remains partially well ordered if we define

$$(a_{\alpha_1}, \cdots, a_{\alpha_r}) \leqq (a_{\beta_1}, \cdots, a_{\beta_l})$$

if there exists a mapping $f(x)$ of the integers $1, 2, \cdots, r$ into the integers $1, 2, \cdots, l$ so that $f(x)$ is monotone increasing and

$$a_{\alpha_i} \leqq a_{\beta_{f(i)}}.$$

Subdivisions of an Interval

4411 [1950, 564]. *Proposed by L. C. Hsu, National Tsing-Hua University, Peiping, China*

Show that if α is a real number >0, then for any given $\epsilon>0$ and any integer $k>0$ there can be found a number $\delta=\delta(\epsilon)>0$ and an integer $N \geqq k$ such that, if

$$\Delta x_1 + \Delta x_2 + \cdots + \Delta x_N = \Delta x_1' + \Delta x_2' + \cdots + \Delta x_N' = \alpha,$$
$$S_k = \sum \Delta x_{\nu_1} \cdots \Delta x_{\nu_k}, \qquad S_k' = \sum \Delta x_{\nu_1}' \cdots \Delta x_{\nu_k}',$$

then

$$|S_k - S_k'| < \epsilon$$

whenever $0<\Delta x_i<\delta$, $0<\Delta x_i'<\delta$, where S_k and S_k' are extended over all possible terms with distinct subscripts $\nu_1, \cdots, \nu_k$.

Solution by Morgan Ward, Pasadena, California. With the Proposer's notation we shall prove that if K is any assigned positive integer, for a suitable $N \geqq K$ and a suitable $\delta>0$,

$$|S_k - S_k'| < \epsilon \qquad k = 1, 2, \cdots, K, \tag{1}$$

provided that the Δx, $\Delta x'$ are in absolute value less than δ.

Choose N for the present merely larger than K, and let

$$\sigma_r = \sum \Delta x_i^r; \qquad \sigma_r' = \sum \Delta x_i'^r, \qquad (r = 1, 2, \cdots, k)$$

be the Newton power sums of the Δx; $\Delta x'$. Then S_k; S_k' are the elementary symmetric functions of the Δx; $\Delta x'$. Hence

$$S_1 = \sigma_1 = S_1' = \sigma_1' = \alpha, \tag{2}$$

and in general by Newton's formulas for $2 \leqq k \leqq K$,

$$\begin{aligned} S_k &= \frac{1}{k!}\{\sigma_1^k + \sum C_{(r)}\sigma_1^{r_1}\sigma_2^{r_2}\cdots\sigma_k^{r_k}\} \\ S_k' &= \frac{1}{k!}\{\sigma_1'^k + \sum C_{(r)}\sigma_1'^{r_1}\sigma_2'^{r_2}\cdots\sigma_k'^{r_k}\}. \end{aligned} \tag{3}$$

Here the $C_{(r)}$ are integers, and the sum is extended over all $r_i \geqq 0$ such that $r_1+2r_2+3r_3+\cdots+kr_k=k$. Furthermore, $r_1<k$ for every term in the summation, so that

$$r_2 + r_3 + \cdots + r_k \geqq 1. \tag{4}$$

Assign any positive ϵ less than 1, and let

$$M = \max \frac{\alpha^k}{k!} \sum |C_{(r)}|, \qquad k = 2, 3, \cdots, K.$$

Now choose the positive integer N so that both $N \geqq K$ and

$$2\alpha M/N < \epsilon. \tag{5}$$

Let $\delta = \alpha/N$. Then

$$N\delta = \alpha, \qquad 0 < \delta < 1. \tag{6}$$

Now if all the $|\Delta x|$, $|\Delta x'|$ are less than δ,

$$|\sigma_r|, \ |\sigma_r'| < N\delta^r = \alpha\delta^{r-1}, \qquad r = 2, 3, \cdots, K. \tag{7}$$

Hence formulas (3), (2) and (1) give the inequality

$$|S_k - S_k'| < \frac{2}{k!} \sum |C_{(r)}| \, \alpha^k \delta^{r_2+2r_3+\cdots+(k-1)r_k} < 2M\delta = \frac{2\alpha M}{N} < \epsilon$$

by (4), (6) and (5). Since (1) is trivial if $k=1$, the proof is complete.

The proof is readily modified to show that if we subject the S_k to L fixed constraints:

$$S_k = S_k' = \alpha_k, \qquad k = 1, 2, \cdots, L,$$

then the inequalities (1) hold for any fixed K, provided merely that $\alpha_1=\alpha$ is positive.

Also solved by O. E. Stanaitis and the Proposer.

Probability, Disjunct Events

4412 [1950, 635]. *Proposed by Jacques Dutka, Rutgers University*

Of the freshmen admitted to a certain college last fall it was found that:

74% were seventeen years of age or older,
89% weighed 125 pounds or more,
78% measured $5\frac{1}{2}$ feet tall or more.

Let p be the probability that a freshman chosen at random had all three characteristics. Determine bounds for the value of p, and generalize the problem.

Solution by C. F. Pinzka, Xavier University, Cincinnati. Consider the generalization in which a fraction f_i of the students have the property P_i $(i=1, 2, \cdots, n)$. Since the fraction not having the property P_i is $(1-f_i)$, the minimum fraction having all the properties is $1-\sum(1-f_i)$ or 0, whichever is greater, and which we denote by

$$\max\left[1-\sum(1-f_i), 0\right] = \max\left[\sum f_i - (n-1), 0\right].$$

The maximum fraction having all the properties is the minimum of the f_i, which we denote by $\min(f_i)$. Thus

$$\max\left[\sum f_i - (n-1), 0\right] \leqq p \leqq \min(f_i).$$

For the numerical example given, $0.41 \leqq p \leqq 0.74$.

This analysis implies that the properties are logically disjunct. That is, it is possible for a student to have any given subset of the properties and lack the remaining properties.

Also solved by Ferrel Atkins, N. J. Fine, G. P. Henderson, Charles McCracken, Jr., O. Dale Smith, S. L. Thompson, and the Proposer.

Editorial Note. Smith finds the problem in Birkhoff and MacLane, *Modern Algebra*, pp. 324–325. Azriel Rosenfeld refers to the solution to Knot X (#1) in Lewis Carroll, *A Tangled Tale* (Complete works of Lewis Carroll, Modern Library, p. 1071.)

Number of Integers of Special Form

4413 [1950, 635]. *Proposed by Paul Erdös, University College, London, England*

Let $a_1=2\cdot 3$, $a_2=3\cdot 5$, $a_3=5\cdot 7$, $\cdots$, $a_k=p_k\cdot p_{k+1}$, $\cdots$, where p_k is the kth prime. Denote by $f(x)$ the number of integers $\leqq n$ composed entirely of the a's (*i.e.*, the integers of the form $\prod a_i^{\alpha_i}$, $0\leqq\alpha_i$). Prove that

$$f(n) = cn^{1/2} + o(n^{1/2}), \tag{1}$$

where $\frac{1}{2}<c<1$.

Solution by the Proposer. It will turn out that

$$c = \prod_{i=1}^{\infty} \frac{1-p_i^{-1}}{1-(p_ip_{i+1})^{-1/2}}. \tag{2}$$

(It is easy to see that the product for c converges. In fact it remains convergent

if $p_1<p_2<\cdots$ is replaced by any increasing sequence of real numbers.)

Let k be a large integer. Denote by $1=A_1(k)<A_2(k)<\cdots$ the integers composed of $a_1, a_2, \cdots, a_k$; and by $B_1(k)<B_2(k)<\cdots$ the integers composed of $a_{k+1}, a_{k+2}, \cdots$. We shall put A_i, B_i for $A_i(k)$, $B_i(k)$. Denote by $g_k(m)$ the number of B's which do not exceed m. Clearly

$$f(n) = \sum_{i=1}^{\infty} g_k(n/A_i). \tag{3}$$

Denote by $h_j(n)$ the number of integers $\leqq n$ composed of $p_{j+1}^2, p_{j+2}^2, \cdots$; *i.e.*, $h_j(n)$ equals the number of squares $k^2\leqq n$ for which p_i does not divide k, $1\leqq i\leqq j$. We have

$$h_{j+1}(m) \leqq g_j(m) \leqq h_j(m); \tag{4}$$

the first inequality of (4) follows by letting p_k^2 correspond to a_k, $k=j+1, j+2, \cdots$; the second inequality follows by letting a_k correspond to p_{k+1}^2. From (4), with $j=0$, we have $\frac{1}{2}n^{1/2}<f(n)<n^{1/2}$.

From (3) and (4) we have

$$\sum_{i=1}^{\infty} h_{k+1}(n/A_i) \leqq f(n) \leqq \sum_{i=1}^{\infty} h_k(n/A_i). \tag{5}$$

By the sieve of Eratosthenes we have

$$n^{1/2}\prod_{i=1}^{k}(1-1/p_i) - 2^k < h_k(n) < n^{1/2}\prod_{i=1}^{k}(1-1/p_i) + 2^k. \tag{6}$$

Also trivially,

$$h_k(m) \leqq m^{1/2}. \tag{7}$$

Let t be large but fixed (*i.e.*, independent of n). From (5) and (7),

$$\sum_{i=1}^{t} h_{k+1}(n/A_i) < f(n) < \sum_{i=1}^{t} h_k(n/A_i) + n^{1/2}\sum_{i=t+1}^{\infty} 1/A_i^{1/2}.$$

Using (6) we obtain

$$\begin{aligned} n^{1/2}\prod_{i=1}^{k+1}(1-1/p_i)\sum_{i=1}^{t} 1/A_i^{1/2} - 2^{k+1}t < f(n) \\ < n^{1/2}\prod_{i=1}^{k}(1-1/p_i)\sum_{i=1}^{t} 1/A_i^{1/2} + 2^k t + n^{1/2}\sum_{i=t+1}^{\infty} 1/A_i^{1/2}. \end{aligned} \tag{8}$$

Now $\sum 1/A_i^{1/2}$ clearly converges. Furthermore

$$\lim_{k\to\infty}\prod_{i=1}^{k}(1-1/p_i)\sum_{i=1}^{\infty} 1/A_i^{1/2}$$

exists. In fact

$$\sum_{i=1}^{\infty} \frac{1}{A_i^{1/2}} = \prod_{i=1}^{k} \left[1 + \frac{1}{(p_i p_{i+1})^{1/2}} + \frac{1}{p_i p_{i+1}} + \frac{1}{(p_i p_{i+1})^{3/2}} + \cdots \right]$$

$$= \prod_{i=1}^{k} \frac{1}{1 - (p_i p_{i+1})^{-1/2}}.$$

Thus with c, given by (2), choosing t and k large enough, we have

$$\left| f(n) - cn^{1/2} \right| < \epsilon n^{1/2}, \qquad n > n_0,$$

which is the desired relation (1).

Circumscribed Steiner Ellipse of a Triangle

4414 [1950, 635]. *Proposed by Lucien Droussent, Clermont Ferrand, France*

The trilinear polar, with respect to a triangle T, of any point P on the circumscribed Steiner ellipse of T is tangent to the conjugate conic of T that has P for center. (The circumscribed Steiner ellipse of a triangle ABC is the circumscribed ellipse whose tangents at A, B, C are parallel to the opposite sides BC, CA, AB.)

Solution by the Proposer. It is convenient to use barycentric coördinates so that the centroid G of the fundamental triangle T is (1, 1, 1), the line at infinity is $\alpha+\beta+\gamma=0$, and the circumscribed Steiner ellipse has the equation $\beta\gamma+\gamma\alpha+\alpha\beta=0$ and has G for its center. If α_0, β_0, γ_0 are the coördinates of the point P, then

$$\frac{1}{\alpha_0} + \frac{1}{\beta_0} + \frac{1}{\gamma_0} = 0. \tag{1}$$

The trilinear polar of P has for its equation

$$\frac{\alpha}{\alpha_0} + \frac{\beta}{\beta_0} + \frac{\gamma}{\gamma_0} = 0 \tag{2}$$

and hence evidently passes through G. The tangential equation of the conjugate conic of T with center P is

$$\alpha_0 u^2 + \beta_0 v^2 + \gamma_0 w^2 = 0. \tag{3}$$

The proof is completed by observing that, by virtue of (1), the coördinates $(1/\alpha_0, 1/\beta_0, 1/\gamma_0)$ of the line (2) satisfy (3).

Integers Having Common Factors with Members of a Sequence

4416 [1950, 691]. *Proposed by D. J. Newman, Harvard University*

Let $a_1, a_2, \cdots$ be a sequence of integers such that $\sum_{n=1}^{\infty} 1/a_n$ diverges.

Show that almost all integers have a factor in common with some a_n.

Solution by Samuel Skolnik, Los Angeles City College. Let $\{r_i\}$ be the set of all prime factors of a_n $(n=1, 2, \cdots)$. For any N we have

$$\prod_{r_i \leqq a_N} \left(\frac{1}{1-1/r_i}\right) = \prod_{r_i \leqq a_N} \left(1+\frac{1}{r_i}+\frac{1}{r_i^2}+\cdots\right) \leqq \sum_{a_N} \frac{1}{a_n},$$

where the last sum is taken over all a_n none of whose prime factors exceeds a_N. This condition is satisfied by all $a_n \leqq a_N$ so that

$$\prod_{r_i \leqq a_N} \left(\frac{1}{1-1/r_i}\right) \leqq \sum_{n=1}^{N} \frac{1}{a_n},$$

and since $a_N \to \infty$ implies $N \to \infty$, and since $\sum_{n+1}^{\infty} 1/a_n$ diverges, we conclude that $\prod_i (1-1/r_i)^{-1}$ diverges; hence $\prod_i (1-r_i)$ diverges to zero.

Let $r_1, r_2, \cdots, r_2$ be the first k primes, in the order of magnitude, of the set $\{r_i\}$. The number of integers not greater than n, and not divisible by any of the primes $r_1, \cdots, r_k$ is

$$f(n, k) = [n] - \sum_i [n/r_i] + \sum_{i,j} [n/r_i r_j] - \cdots, \tag{1}$$

where $i, j, \cdots$ are unequal and run from 1 to k, and where $[x]$ indicates the integral part of x. The total number of brackets in (1) is $1+k+k(k-1)/2 + \cdots = 2^k$, and since $0 \leqq x-[x]<1$ we have

$$f(n, k) \leqq n - \sum_i n/r_i + \sum_{i,j} n/r_i r_j - \cdots + 2^k = n \prod_{r_i \leqq r_k} (1 - 1/r_i) + 2^k.$$

Let $f(n)$ denote the number of integers less than n, which are not divisible by any of the primes r_i. Then

$$f(n) \leqq f(n, k) \leqq n \prod_{r_i \leqq r_k} (1 - 1/r_i) + 2^k.$$

But $\prod_i (1-1/r_i)$ diverges to zero and hence, for arbitrary ϵ and for k sufficiently large we have $\prod_{r_i \leqq r_k} (1-1/r_i) < \frac{1}{2}\epsilon$, and for n sufficiently large, $2^k/n < \frac{1}{2}\epsilon$, whence $f(n) < \epsilon n$ and

$$\lim_{n \to \infty} \frac{f(n)}{n} = 0$$

which establishes the theorem.

It was assumed throughout the foregoing that the members of the sequence $\{a_n\}$ are distinct. The theorem is evidently false if, *e.g.*, $a_n = 2$, $(n=1, 2, \cdots)$. The notation and techniques employed are those of Hardy and Wright, *An Introduction to the Theory of Numbers*, pp. 233, 338, 341.

Solved also by the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Secondary Mathematics, A Functional Approach for Teachers. By H. F. Fehr, Boston, D. C. Heath and Company, 1951. 11+431 pages. $4.25.

This book has been written for teachers and prospective teachers of secondary school mathematics. It is a mathematics book and not a methods text. Rather the author presents, in a form suitable for use as a textbook, topics in mathematics which have a clearly determined relationship to the content of secondary school mathematics and the study of which will broaden the teacher's knowledge of the mathematics he will teach. The book can be recommended also for the professional library of secondary school teachers in service as a book which will renew and increase their interest in more advanced mathematics and insight into topics in many cases previously studied.

It is intended that, as a text, this book be used to supplement existing courses in teacher education programs. The materials in the book have been used by the author over a period of twelve years in classes for seniors and graduate students at Montclair State Teachers College and Teachers College, Columbia University. In the preface the author states that his book has, in part, the same purpose for secondary school mathematics teachers in the United States that Felix Klein's three volumes on *Elementary Mathematics From An Advanced View-Point* had for teachers in Germany. The author also expresses his desire to continue the pioneer work in America of his former teacher and friend, John A. Swenson, who was an early leader in this country in the teaching of advanced mathematics, with special reference to the implications and relationships of the content to the topics of secondary school mathematics.

One indication of the wide choice of topics presented is given by such chapter headings as Congruence in Geometry, Elementary Curve Tracing, The Binomial Series and the Exponential Function, the Derivative, Denumerability and Non-denumerability, Complex Exponents and Roots, Three-Dimensional Geometry, Hyperbolic Functions, and Geometric Constructions with Compasses and Straightedge. One of the sources of strength of this book is the selection of topics, all important for the teacher and yet varied enough to allow choices for classroom use and professional study. With so many topics considered, no one can be given extensive consideration. This is true even of geometric constructions, a topic which would seem to lend itself especially well to this type of treatment. However, this plan provides further evidence that *Secondary Mathematics* is not intended to replace other existing courses for teachers.

The style of writing is clear and direct. Examples are included of the various concepts or theorems developed. Practice exercises are sufficiently numerous

for advanced undergraduate or first year graduate students. There is a good list of references with each chapter. Answers are given for odd-numbered exercises. Many of the references to the secondary school come in the various chapters under the heading of Discussion. Illustrative of these would be:

"Compare the Euclidean solution of a locus theorem as presented here with that given in current high school textbooks. To what extent can the high school treatment be improved to include necessary and sufficient concepts? Should the analytic geometry method of locus be introduced in high school geometry? If so, to what extent?"

"The foregoing treatment of complex exponents is too difficult for high school presentation. At what point in collegiate mathematics should it first be presented? How can complex exponents be made meaningful to a senior high school mathematics class?"

As further illustration of the professionalized treatment of the material attention might be called to the list in the chapter on the derivative of problems on maxima and minima suitable for high school students.

College staff members who are interested in improving their offerings for teachers, particularly from the point of view of helping students in their classes appreciate the value for them, as secondary school teachers, of having a knowledge and understanding of mathematics beyond the calculus, should examine this book carefully. If all teachers of secondary mathematics had studied and understood the content of this book within college courses or as a part of their professional reading, surely more effective work would be done in our classrooms.

J. R. MAYOR
University of Wisconsin

The Algebra of Vectors and Matrices. By T. L. Wade. Cambridge, Mass., Addison-Wesley, 1951. ix+189 pages. $4.50.

According to its preface, this book is intended as "an elementary exposition of the *algebra of vectors and matrices*, that exposition being articulated with the basic concepts of modern algebra in the broad sense, to wit, group, integral domain, field, ring, basis, dimension and isomorphism." Also, the student is assumed to have progressed no further than analytic geometry and to know the elements of determinant theory.

The book may be recommended as an apparently usable one in a field which suffers from a paucity of texts, but not without a number of rather serious reservations.

First of all, one may question whether a student with only the stated minimal background is prepared for the book's immediate plunge into abstract group theory, and also whether he knows, without previous discussion, the meaning of such terms as, for example (p. 18), "one-to-one correspondence."

Secondly, one may cite a large number of slips of various types. For example, even and odd permutations are defined (p. 6), but the essential fact that a permutation cannot be both even and odd is not proved, nor is it even stated;

in defining a ring (p. 12), the right distributive law has been omitted; often (*e.g.*, p. 19) constant k lacks a necessary restriction: "$k \neq 0$." To this list may be added certain assertions which might perhaps just as well be deleted from the text. For example: (p. 48) "One should draw the distinction between 'perceptual' or visualizable spaces and 'conceptual' or logically conceived spaces. . . . Ordinary three space is usually thought of as a perceptual space, although a configuration in it of moderate complexity may not be altogether perceptual"; (p. 47) " . . . by a space of two dimensions is meant any set of objects which may be put in a one-to-one correspondence with the totality of vectors of two dimensions"; (p. 49) " . . . if a set of vectors are linearly dependent, their corresponding scalar coordinates are linearly dependent."

I. H. Rose
University of Massachusetts

An Introduction to Probability Theory and Its Applications. By William Feller. New York, John Wiley and Sons, Inc., 1950. xii+419 pages. $6.00.

The discussions and definitions of mathematical probability found in college algebra books reduce the theory to a pure *a priori* system and thereby detract from the real purpose of probability theory. Too often the problems and examples are either trivial or so detached from application as to appear ridiculous. The history of mathematics, as historians of science have shown, displays a fascinating interplay between theory and application. Certainly it is reasonable to expect that learning will be facilitated if some similar pattern is followed in the presentation of material to the neophyte.

Dr. Feller in this new text has such a novel view and presentation. As it should be with all basic concepts, he introduces the concept of probability with great caution by using intuitive examples before proceeding to a more rigorous and abstract definition. The use of sample space, in this volume discrete sample space, as the basis for definition is intuitively more pleasant and to this reader more elegant. Not only does this approach avoid the falseness of the usual ratio definition but it lends itself more readily to transition to statistical applications.

The symbolism, $Pr(A)$, for the probability of an event A is totally differentiated from the usual permutation symbol, which Dr. Feller carefully avoids. The symbol, $(n)_r$, is used to mean the permutation of n things r at a time, although the concept of permutations is not directly associated with the symbol in the text. Specifically the author states, "The number of different possible samples of size r from a population of n elements is n^r if the sampling is with replacement, and $(n)_r$ if it is without replacement." This may be hedging, but the approach seems to be more gratifying to this reader. Similarly the usual combination idea is introduced through partitions, the symbol used is $\binom{n}{r}$ for the number of different groups of size $r \leqq n$ that can be selected from n elements. This treatment leads easily to multinomial coefficients.

The two chapters on combination of events and conditional probabilities

are as fine as ever have been written. The discussions of applications in these chapters are reason enough for merit. Two other features of these chapters that should not escape mention are the handling of the Laplace "law of succession" and the unique approach to conditional probability from definition thus avoiding a rote usage of Bayes' Rule.

Much of the material presented is of course standard; binomial and Poisson distributions, the normal distribution and the law of large numbers. However, the theory of recurrent events has been specially designed for this volume and leads directly to Markov Chains.

The various applications to occupancy problems, runs, trunking problems, waiting time, matching and guessing, *et cetera*, throughout the book should give the volume appeal to a diverse audience.

The book is not without error but, as might be expected, these are relatively rare. The definitions of event, page 13, and sample space, page 10, seem to be equivalent yet apparently not intended to be so, unless we conceive of the event as being the "act" that brings into being the sample space. If this be the intention of the author he has gone to great lengths to achieve it. Without being picayune it might also be pointed out that the concept of asymptotic equality, first introduced in Stirling's Formula, is vague. Still further a juxtaposition of definition and usage occurs concerning the concept of density function. The "bad" features of the book are, however, trivial by comparison and in no way detract from the readability or teachability of the work.

In general the text is more appealing than most probability texts. That it is a genuine contribution to the literature of mathematical statistics will be borne out by the tests of time and usage.

F. A. C. SÉVIER
Princeton, New Jersey

NEW BOOKS RECEIVED

Finite Deformation of an Elastic Solid. By F. D. Murnaghan. New York, John Wiley and Sons, Inc., 1951. 140 pages. $4.00.

Les Recreations Mathematiques (Parmi Les Nombres Curieux). By Victor Thébault. Paris, Gauthier-Villars, 1952, vi+297 pages. $7.50.

Introduction to Mathematical Thinking. By Friedrich Waisman. New York, Frederick Ungar Publishing Company, 1951. 12+260 pages. $4.50.

Elements of the Topology of Plane Sets of Points, Second Edition. By M. H. A. Newman. London, Cambridge University Press, 1951. vi+214 pages. 27s. 6d.

An Introduction to Applied Mathematics. By J. C. Jaeger. New York, Oxford University Press, 1952. xiii+445 pages. $7.00.

Aristotle's Syllogistic. By Jan Lukasiewicz. New York, Oxford University Press, 1951. xi+141 pages. $3.00.

Analytic Geometry. By W. K. Morrill. Scranton, International Textbook Company, 1951. 14+383 pages. $3.50.

Proceedings of a Second Symposium on Large-Scale Digital Calculating Machines. By Harvard University Computation Laboratory and the Navy Department Bureau of Ordnance. Cambridge, Harvard University Press, 1951. 38+393 pages. $8.00.

Tables of the Error Function and of Its First Twenty Derivatives (Annals 23). By the Staff of the Harvard University Computation Laboratory. Cambridge, Harvard University Press, 1952. 28+276 pages. $8.00.

Meaningful Mathematics. By H. S. Kaltenborn. New York, Prentice-Hall, Inc., 1951. xiv+397 pages. $4.75.

History of the Theories of Aether and Electricity. By Sir Edmund Whittaker. New York, The Philosophical Library, 1951. xiv+343 pages. $12.00.

Linear Transformation in n-Dimensional Vector Space. By H. L. Hamburger and M. E. Grimshaw. New York, Cambridge University Press, 1951. 10+195 pages. $4.50.

L'Enseignement des Mathematiques Generales Par Les Problemes, I. By G. Bouligand and J. Rivaud. Paris, Librairie Vuibert. 6+372+4 pages. 2000 francs.

Application of Linear Programming to the Theory of the Firm. By Robert Dorfman. Berkeley, University of California Press, 1951. ix+98 pages. $3.50.

Graphic Aids in Engineering Computation. By R. P. Hoelscher, J. N. Arnold, and S. H. Pierce. New York, McGraw-Hill Book Company, 1952. 8+197 pages. $4.50.

Trigonometry—Plane and Spherical. By L. L. Smail. New York, McGraw-Hill Book Company, 1952. xii+406 pages. $3.75.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of special features, student papers, bibliographies of program topics, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB REPORTS, 1950–51

Mathematics Club, Sacramento State College

The *Mathematics Club of Sacramento State College* was organized in January, 1951, and met monthly during the spring semester of 1951. The following topics were presented:

Logic of mathematics, by Dr. Hal D. Draper

Methods of integration: by parts and long division, by Dr. Louis C. Graue

Mathematical theory of games, by Mr. Norman Rudy

Development of number system, by Mr. W. W. Wallace, Sacramento Junior College

What is the difference between mathematics, philosophy of mathematics and philosophy? by Mr. Maxon A. Bernoff, Sacramento Junior College.

The officers elected for 1951–52 are: President, Robert Rimmele; Vice-President, George May; Secretary, John Otter; Treasurer, Mathias Hilling; Program Chairman, Robert Gustafson.

Mathematical Section of the Order of Magnitude, Pasadena City College

The *Mathematical Section of the Order of Magnitude* meets every two weeks at *Pasadena City College*. The topics presented in 1950–51 were:

Mathematical models and soap film experiments, by Mr. W. H. Glenn

Vector analysis, by Robert Phelan

Congruences of numbers, by Marvin Ridley

The seven bridges of Königsberg, by Stuart Bowen

Projective geometry, by Mr. Laing

Relationships and correspondences between slopes and their eccentricities, by Mr. Lewis

Fourier series, by Mr. Hardy

Mathematics conference report, by Marvin Ridley

Prime numbers, by Marvin Ridley

Hyperbolic trigonometric functions, by William Lindley

Higher plane curves, by Mr. Hardy

Logic in mathematics, or What is mathematics? by Dr. E. T. Bell.

Both the *Mathematical Section* and the *Astronomy Section of the Order of Magnitude* participated in Club Day and Vocational Guidance Day at the College.

Officers for the second semester were: President, Andy Markell; Vice-President, Bob Phelan; Secretary-Treasurer, Carol Blackinton.

Pi Mu Epsilon, University of Oregon

The *Oregon Alpha* chapter of *Pi Mu Epsilon* held four regular meetings including initiations and elections during 1950–51. The following papers were read:

Scales of notation, by Mr. Wesley L. Nicholson

A problem in symbolic logic, by Miss Dorothy J. Christensen.

The DeCou Prize award of $50 given annually to the outstanding upper division student in mathematics was presented to Miss Dorothy J. Christensen.

The annual joint spring picnic of the chapter and the Mathematics Department of the University of Oregon was held May 15.

Officers elected for 1951–52 are: Director, Wesley L. Nicholson; Vice-Director, Roger H. Moore; Secretary-Treasurer, Sam Saunders.

Junior Mathematics Club, University of Chicago

The *Junior Mathematics Club of the University of Chicago* meets bi-weekly, its main activities being the talks presented at these meetings and the semi-

weekly teas given for the Department of Mathematics. During 1950–51 the following talks were presented:

Notion of mathematical structure, by Donald Bratton
Lie-admissible algebras, by Louis Weiner
Structure theory of algebras and rings, by Alexander Rosenberg
Integral inequalities suggested by game theory, by Joseph Bram
The weak law of large numbers, by Dr. Murray Rosenblatt
Kolchin: Picard-Vessiot theory, by William Howard
Tate: functional equation of the zeta function, by Sanford Colbert
Method of gradients for the determination of the characteristic roots and vectors of a symmetric matrix, by Dr. William Karush
Weak topologies, by Ernest Michael
Curves of constant breadth, by William Ballard
Some integral inequalities, by Daniel Waterman
Partitions of unity and Tietze's extension theorem, by Robert Bartle.

The activities of the club were highlighted by two parties and the inauguration of the club reading room.

Officers elected for 1951–52 are: President, Fred Wright; Treasurer, Thomas King.

Kappa Mu Epsilon, Kansas State Teachers College, Emporia

The *Kansas Beta* chapter of *Kappa Mu Epsilon* at *Emporia State Teachers College* met monthly throughout the school year 1950–51. Papers presented at these meetings were:

The 200-inch telescope, by Brooks Becker
Short cuts in arithmetic, by George Crumley
Mechanical brains, by Richard Shur
Electronic computers, by Robert Klotz
Similar triangles, by Elva Libben
History of negative numbers, by Richard McAlister
History and development of Kappa Mu Epsilon, by Dr. O. J. Peterson
Galaxies and nebulae, by Raymond Eccles
Structure of the universe, by Billy Burgert
Famous calculus problems, by Don Allison
Probability, by Dale Smith
How to solve it, by A. H. Albert
Report on present occupations of former Kappa Mu Epsilon members, by Don Hanifan
Report on meeting of Kansas Association of Teachers of Mathematics, by Max Woods and Richard Gregory
Report on the national convention of Kappa Mu Epsilon at Springfield, Missouri, by the eleven delegates.

Delta-Y Club, D'Youville College

Delta-Y members and guest speakers this year discussed many of the interesting side-lights of mathematics. Each monthly meeting included one or more of the following topics:

An introduction to topology, by Miss Helen Hand
Mathematics and the business world, by Miss Alice Staebell
Some envelopes of the Lemoine line, by Miss June McArtney
Poetry and mathematics, by Helen Bertell
Magic squares, by Emily Monczynski
Mathematical wrinkles, by Shirley Hebler
Astronomy with an opera glass, by Ann Dolce
Mathematics and the imagination, by Marian Adrian
The fourth dimension, by Mary Frances McHugh
Panel discussion of the sliderule, by Mary Catherine Forde, Patricia Leising, Shirley Hebler, and Emily Monczynski.

The President elected for 1951–52 is Marilyn Repsher.

Mathematics Club, Marquette University

The following lectures were presented to the *Mathematics Club of Marquette University* during 1950–51:

Trisection of the angle, by David Rux
Squaring the circle, by Robert Wurm
Mathematical puzzles, by Dorothy Stidola
Mathematics and music, by Keith Betz
Magic squares, by Frank Michuden
Flatland, by Irving Treu
Unclear physics, by Dr. Barkow.

The President elected for 1951–52 is Philip Blank.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

PURCHASE OF THE A.M.S. LIBRARY BY THE UNIVERSITY OF GEORGIA

The American Mathematical Society has sold its Library to the University of Georgia. This library consists of between thirteen and fourteen thousand bound volumes. These added to the not inconsiderable collection already owned by the University gives the University of Georgia a very fine library for research

in mathematics. It is particularly strong in long runs of periodicals. Its collection of doctor's dissertations, not only American but foreign, is unrivaled. The University of Georgia is continuing the entire exchange list of the American Mathematical Society and expects to do all that is possible to keep the dissertation collection up to date. It is believed that a complete file of dissertations can be administered for the benefit of mathematicians everywhere. With this in mind department chairmen are urged to report the names of all persons who take the Ph.D. degree to the librarian of the University of Georgia at Athens. He will contact the author relative to a copy of the dissertation.

It is hoped that all mathematicians will feel free to utilize any item in the entire collection. It is the desire of the University of Georgia to organize its library for the benefit of all competent users, and rules for administering the mathematics collection are being framed with this in mind. It is particularly hoped that mathematicians in the Southeast will be benefited by the location of this library at Athens.

PRELIMINARY ACTUARIAL EXAMINATIONS AND PRIZE AWARDS

The Society of Actuaries has announced the winners of the prize awards for the May 1951 Preliminary Examinations. These awards were given to the nine undergraduates who ranked highest in score for Part 2 of these examinations. The first prize of $200 was awarded to F. O. Wyse of Harvard University. Additional prizes of $100 each were awarded to the following: A. R. Auer, Columbia University; Anita Bredt, University of Pennsylvania; W. E. Chapman, III, Duke University; J. P. Lazarus, Rensselaer Polytechnic Institute; Judith A. Richman, University of Pennsylvania; J. B. Ross, Harvard University; C. D. Silletto, University of Iowa; Howard Young, City College of the City of New York.

On May 21, 1952, the Preliminary Actuarial Examinations will be given to undergraduate students of mathematics who may be interested in going into the actuarial profession. These examinations consist of three parts: (1) Language Aptitude Examination, (2) General Mathematics Examination, and (3) Special Mathematics Examination. For detailed information regarding these examinations write to: The Society of Actuaries, 208 South La Salle Street, Chicago 4, Illinois.

The Society of Actuaries will award one $200 and eight $100 prizes to the nine successful undergraduates who rank highest in the General Mathematics Examination and who concurrently pass, or previously have passed, the Language Aptitude Examination.

ESSAY CONTEST OF THE INSTITUTE FOR THE UNITY OF SCIENCE

The Institute for the Unity of Science is conducting an Essay Contest during 1952. The theme of the Contest is Mathematical Logic as a Tool of Analysis. The essay should contain a detailed account of the actual accomplishments which may be credited to the techniques of modern formal logic in both applied

and theoretical science as well as in philosophy. On the other hand, considerations of the role of these techniques in researches on the foundations of pure mathematics and of formal logic are specifically excluded from the scope of the essay. But a discussion of unsuccessful or dubious applications of mathematical logic is highly pertinent. Proposed extensions of these techniques to new problems and areas, or critiques of the limitations of these techniques, should be developed in terms as concrete as possible.

A prize of $500 will be awarded for the best essay submitted on the theme, and two additional prizes of $200 each will be given for the two next best essays. For further information regarding rules of the Contest write to: Institute for the Unity of Science, American Academy of Arts and Sciences, 28 Newbury Street, Boston 16, Massachusetts.

REQUEST FOR BOOKS

The Franklin and Marshall College Library wishes to purchase the following: Watson—Complex Integration and Cauchy's Theorem; Eagle—Fourier's Theorem; Emch—Projective Geometry; Schmall—Analytic Geometry, 2nd edition; Scioberretie—mathematics book published about 1933, or later; Silberstein—Synopsis of Applicable Mathematics. Anyone who wishes to dispose of these or other advanced mathematics books and periodicals may write to Franklin and Marshall College Library, Lancaster, Pennsylvania.

SUMMER COURSES

The following institutions announced advanced courses in mathematics for the summer of 1952:

The Catholic University of America. June 30 to August 9: Professor Ramler, college geometry, synthetic projective geometry, ordinary differential equations; Professor Finan, theory of equations II; Professor Rice, advanced calculus I, calculus of observations; Mr. Clark, partial differential equations of mathematical physics.

Columbia University, Department of Mathematics. July 7 to August 15: Professor Koopman, introduction to higher algebra, Fourier series and integrals; Professor Rosenbloom, differential equations, mathematical logic; Professor Murray, probability, numerical methods for the integration of ordinary and partial differential equations; Professor Lorch, foundations of mathematics, theory of functions of a complex variable; Professor Levi, differential geometry; Professor Kolchin, ideal theory in commutative rings.

Columbia University, Teachers College. July 7 to August 15: Professor Fehr, teaching arithmetic in the elementary school, professionalized subject matter in advanced secondary mathematics, part II; Professor Rosskopf, teaching non-academic mathematics in the high school, supervision and problems of teaching secondary school mathematics; Professor Shuster, business arithmetic and mathematics, field work in mathematics; Professor Van Engen, history of mathematics, teaching algebra in secondary schools; Professor Yates, applications of plane geometry, survey of higher mathematics for teachers.

De Paul University. June 9 to August 5: Professor De Cicco, calculus of variations, differential geometry; Professor Caton, statistical sampling theory, differential equations.

Duke University. June 11 to July 19: Professor Carlitz, finite differences, thesis seminar; Professor Thomas, differential equations, projective geometry. July 22 to August 29: Professor Dressel, solid analytic geometry; Professor Gergen, mathematical statistics. August 5 to August 15: Professor W. W. Rankin, Director, Institute for Teachers of Mathematics.

Kent State University. June 16 to July 25: Professor Johnson, college geometry; Professor Olson, differential geometry; Professor Brooks, mathematical statistics. July 28 to August 29: Professor Kaiser, differential equations, probability.

Massachusetts Institute of Technology. June 9 to July 18: Professor Tucker, Princeton University, theory of games. July 21 to August 29: Professor Rademacher, University of Pennsylvania, special functions.

Michigan State College. June 23 to August 1: Professor Grove, differential geometry; Professor Hill, complex variable; Professor Powell, advanced calculus I; Professor Stewart, college geometry, theory of numbers; Mr. Coy, correlation analysis. June 22 to August 22: Professor Bell, theory of matrices and groups, higher algebra; Professor Harter, analysis of variance; Professor Herzog, Fourier series, advanced calculus II; Professor Kelly, projective geometry; Professor Nordhaus, theory of equations; Professor Stelson, differential equations; Professor Wells, vector analysis, integral equations.

State University of Iowa. June 9 to August 6: Professor Chittenden, matrices and determinants, infinite series; Professor Conkwright, differential equations; Professor Hogg, elements of statistics, statistical hypotheses; Professor Knowler, mathematics of business and industry, topics in actuarial science; Professor Smiley, introduction to advanced mathematics, Galois theory of fields; Professor Wylie, astronomy.

University of Buffalo. June 30 to August 9: Professor Gehman, synthetic projective geometry; Professor Montague, introduction to topics in higher algebra, history of mathematics; Professor Schneckenburger, differential equations, theory of sets.

University of California at Los Angeles. June 23 to August 2: Professor Van der Corput, University of Amsterdam and Institute for Numerical Analysis, asymptotic expansions; Professor Kaplansky, University of Chicago, higher algebra; Professor Arens, complex variables.

University of Chicago. June 23 to August 30: Professor Spanier, introduction to the theory of groups and rings, convexity; Professor Chern, fundamental concepts of geometry, differential geometry II; Professor Stone, theory of sets; Professor Barnard, tensor analysis; Professor Halmos, theory of functions of real variables, topological algebra II; Professor Segal, Lebesgue and Stieltjes integrals, mathematical foundations of quantum mechanics; Professor Schilling, representation of finite groups.

University of Colorado. June 16 to August 26: Professor Scherk, University of Saskatchewan, differential geometry; Professor Edrei, differential equations, functions of a real variable; Professor Jones and Professor Kendall, modern geometry; Professor Snively, vector analysis; Professor Stahl, Fourier series and boundary value problems. July 24 to August 26: Professor Brown, University of Tennessee, teaching of secondary mathematics, mathematics workshop in curriculum problems.

University of Kentucky. June 16 to August 9: Professor Cowling, tensor analysis; Professor Goodman, vector analysis; Professor Pence, college geometry; Professor South, statistics seminar; Professor Ward, theory of groups.

University of Michigan. June 23 to August 15: Professor Anning, college geometry; Professor Brauer, higher algebra, structure of rings; Professor Carver, theory of statistics; Professor Churchill, heat conduction and diffusion; Professor Coburn, operational mathematics, vector analysis; Professor Craig, analytic theory of frequency functions; Professor Dwyer, multivariate analysis, probability; Professor Hay, theoretical mechanics, tensor analysis; Professor E. H. C. Hildebrandt, history of geometry, teaching of algebra; Professor Kaplan, functions of complex variable and applications, partial differential equations; Dr. Leisenring, synthetic projective geometry; Professor Nesbitt, intermediate mathematics of life insurance; Professor Piranian, infinite series and summability; Professor Rainich, quadratic numbers and quadratic forms, potential theory; Professor Rainville, intermediate differential equations; Professor Samelson, theory of functions of a complex variable, differential geometry; Professor Tornheim, theory of matrices; Professor Young, foundations of mathematics, unified topology.

University of Minnesota, Department of Mathematics. June 17 to July 26: Professor Gelbaum, advanced calculus I, Fourier series and orthogonal functions; Professor Loud, theory of numbers, calculus of finite differences. July 29 to August 30: Professor Gelbaum, advanced calculus II, special functions; Professor Hatfield, advanced algebraic theory, solid analytic geometry.

University of Minnesota, Institute of Technology. June 16 to July 25: Professor Loye, differential equations, intermediate calculus; Professor Munro, vector analysis, theory of complex variables. July 28 to August 29: Professor Turrittin, advanced calculus, vector analysis with applications.

University of North Carolina. June 12 to July 16: Professor Winsor, college geometry; Professor Garner, calculus of finite differences; Professor Hill, elementary mathematical statistics; Professor Cameron, vector spaces and matrices; Professor Hoyle, advanced calculus I; Professor Brauer, some recent results in algebra. July 17 to August 20: Professor Linker, differential equations; Professor Lasley, analytic projective geometry; Professor Jones, elementary point set theory; Professor Whyburn, advanced calculus II.

University of Oklahoma. June 6 to August 1: Professor Bernhart, college geometry; Professor Springer, elementary differential equations; Professor LaFon, ordinary and partial differential equations; Professor Andree, theory of

numbers; Professor Brixey, theory of groups; Professor Goffman, combinatorial topology.

University of Southern California. June 23 to August 1: Professor Snapper, advanced calculus, theory of numbers; Professor Henkin, foundations of mathematics; Professor Yood, calculus of variations, advanced topics in analysis.

University of Washington. June 23 to July 23, July 24 to August 22: Professors Brownell, Haller, and Leipnik, ordinary and partial differential equations; Professor Avann, vector analysis; Professor Paulson, matrices and determinants; Professor Yagi, higher calculus; Professor Ball, introduction to modern algebra; Professor McFarlan, calculus of variations.

University of Wisconsin. June 30 to August 22: Professor Bing, topics in geometry, higher mathematics for engineers; Professor Buck, survey of the foundations of arithmetic; Professor Fullerton, advanced analytic geometry; Dr. Immel, introduction to statistics; Professor Sokolnikoff, higher mathematics for engineers, determinants and matrices; Professor Young, Schwartz' theory of currents and distributions; Professor Mann, Ohio State University, topics in mathematical statistics, topics in the theory of numbers.

University of Wyoming. June 9 to July 11: Professor Varineau, theory of equations, fundamental concepts of mathematics; Professor Schwid, ordinary differential equations, solid analytic geometry; Professor Neubauer, history of mathematics; Professor W. N. Smith, mathematical theory of probability. July 14 to August 15: Professor Barr, advanced calculus; Professor S. R. Smith, partial differential equations; Professor Steen, college geometry.

West Virginia University. June 4 to July 15: Professor Peters, differential equations, group theory; Professor Stewart, theory of probability. July 16 to August 22: Professor Cunningham, modern geometry, theory of determinants and analytic geometry of space; Professor Vehse, calculus of variations.

ANNUAL CONFERENCE FOR TEACHERS OF MATHEMATICS

The Second Annual Conference for Teachers of Mathematics will be held from June 30 to July 12 on the Los Angeles campus of the University of California, according to Clifford Bell, Head of Mathematics Extension for the University.

Purpose of the conference is to bring together teachers interested in mathematics to study common problems in the teaching of mathematics and to learn new uses of mathematics in various fields of endeavor.

The University's Department of Mathematics, Education and Mathematics Extension, and University Extension are sponsoring the conference in cooperation with the California Mathematics Council and the National Council of Teachers of Mathematics. W. W. Rankin, Director and Professor of Mathematics, Duke University, will come to Los Angeles to preside at the conference. Members of the planning committee are L. J. Adams, chairman, department of mathematics, Santa Monica City College; Dale Carpenter, academic education branch, Curriculum Division of the Los Angeles City Schools; Reuben Palm,

director, secondary education, Los Angeles County Schools and Paul White, Head, department of mathematics, University of Southern California.

Information concerning registration and living accommodations is available on request to the Department of Conferences, University of California Extension, Los Angeles 24.

PERSONAL ITEMS

Professor H. M. Gehman, Secretary-Treasurer of the Association, was the representative of the Association at the conference of organizational members of the American Council on Education which was held in Washington, D. C. on January 25–26, 1952.

Professor Einar Hille was elected to the Royal Swedish Academy of Sciences on April 11, 1951.

Professor Anne L. Lewis of the Woman's College of the University of North Carolina represented the Association at the inauguration of President C. C. Jernigan of Queens College, Charlotte, North Carolina on March 29, 1952.

Professor W. T. Martin of the Massachusetts Institute of Technology represented the Association at the inauguration of President L. W. Jones of Rutgers University.

Frank B. Jewett Postdoctoral Fellowships of the Bell Telephone Laboratories for 1952–1953 have been awarded to the following mathematicians: Mr. Murray Gerstenhaber of Harvard University, Dr. Henry Helson of Yale University, and Mr. E. T. Onat of Brown University.

Arizona State College announces the following: Mr. Joseph Kindle, formerly of the University of Cincinnati, has been appointed Special Lecturer for the year 1951–52; Professor F. C. Gentry is on leave for the year 1951–52 and is teaching at the University of New Mexico.

California Institute of Technology announces the following appointments for 1952–1953: Dr. H. A. Dye, former Bateman Fellow, has been promoted to an instructorship; Mr. F. B. Fuller of Princeton University has been appointed to an Instructorship; Mr. R. A. Dean of Ohio State University has been appointed Bateman Fellow.

Case Institute of Technology announces that Dr. R. A. Clark and Dr. F. C. Leone have been promoted to assistant professorships.

Hampton Institute reports the following: Professor R. D. Gordon of Washington College has been appointed to an associate professorship; Miss Gloria C. Ford, previously a student at the University of Pennsylvania, has been appointed to an instructorship; Miss Bobbie E. Jones, formerly a graduate fellow at Howard University, has been appointed to an instructorship; Miss Rosalind M. Eagleson and Miss Elaine H. Young are on leave of absence.

Illinois Institute of Technology announces that Mr. W. F. Darsow and Mr. Albert Soglin have been appointed to instructorships. On January 10, 1952 the Mathematics Department of the Institute opened an exhibition on Geometry in the Museum of Science and Industry, Chicago; a guidebook explaining the exhibition was published by Professor Karl Menger.

Iowa State College makes the following announcements: Assistant Professor N. P. Yeardley of Purdue University has been appointed to an assistant professorship; Mr. W. J. Schwartz, formerly an instructor at Montana State College, has been appointed to an instructorship; Graduate Assistants Karl Conrad, R. W. Klopfenstein and H. D. Mills have been promoted to instructorships.

Miami University reports that Mr. C. C. Crell and Mr. E. R. Epperson, previously graduate students at the University, have been appointed to instructorships.

Michigan State College reports the following: Associate Professor C. P. Wells has been promoted to a professorship; Assistant Professor E. A. Nordhaus has been promoted to an associate professorship; Dr. Ingram Olkin, formerly research associate at the University of North Carolina, has been appointed to an assistant professorship; Dr. J. B. Kelly, who was a member of the Institute for Advanced Study during 1950–51, has been appointed to an instructorship; Mr. K. H. Carlson, formerly graduate assistant at the University of Wisconsin, and Mr. J. G. Hocking, previously teaching fellow at the University of Michigan, have been appointed to instructorships.

Mount Holyoke College reports: Professor Marie Litzinger is on leave of absence; Assistant Professor I. H. Rose of the University of Massachusetts has been appointed to a visiting professorship.

Northwestern University announces the following: Instructor W. M. Boothby has been promoted to an assistant professorship; Dr. George Springer, previously Moore instructor at Massachusetts Institute of Technology, has been appointed to an assistant professorship; Dr. J. S. MacNerney, Mr. W. E. Jenner, and Mr. T. C. Littlejohn have been appointed to instructorships; Dr. Oswald Wyler has been appointed Lecturer; Professor R. P. Boas is on leave for the year on a Guggenheim Fellowship.

Oregon State College makes the following announcements: Assistant Professors B. H. Arnold and B. W. Brewer have been promoted to associate professorships; Professor W. E. Milne has returned to the headship of the department after a year's leave of absence during which he was consultant at the National Bureau of Standards, Institute for Numerical Analysis at Los Angeles; Dr. F. E. Ehlers has been granted leave for 1951–52 in order to accept a position with the Aerodynamics Unit of the Engineering Department of Boeing Aircraft in Seattle. Dr. H. W. Eves was the recipient of the Carter Award for 1950–51; this award is given to "the best teacher in the sciences" according to student vote.

The University of Delaware announces the appointments of Dr. J. H. Barrett to an assistant professorship and Mr. A. C. Nelson to an instructorship.

At the University of Georgia: Professor H. S. M. Coxeter of the University of Toronto gave the Colloquium Lectures entitled "Regular Honeycombs in Elliptic Space" on January 8–11, 1952; he also lectured under the auspices of Pi Mu Epsilon on "The Golden Section."

University of Hawaii reports: Assistant Professor Christopher Gregory has been promoted to an associate professorship; Dr. Paolo Comba, formerly a student at the California Institute of Technology, has been appointed to an assistant professorship.

University of Massachusetts announces the appointments of Mr. Bernard Bussel and Mr. A. A. Kheiralla to instructorships and of Mr. Sidney Kissin to a graduate assistantship.

At Yale University: Assistant Professor C. E. Rickart has been promoted to an associate professorship; Associate Professor G. P. Hochschild of the University of Illinois has been appointed Visiting Associate Professor for the year 1951–52; Dr. Henry Helson, who was a lecturer at the University of Uppsala, Sweden during the year 1950–51, has been appointed to an instructorship; Dr. John Wermer, previously a teaching fellow at Harvard, has been appointed to an instructorship; Mr. J. Y. Barry, Mr. G. R. Dubé, Mr. L. W. Green, Mr. R. B. Smith and Mr. Morris Weisfeld have been appointed Assistants-in-Instruction for the year 1951–52; Professor Nathan Jacobson is on leave of absence during the year 1951–52 and is spending the year in France on a Guggenheim Fellowship and Fulbright Award.

Yeshiva University announces: Assistant Professor Henry Lisman has been promoted to an associate professorship; Mr. Barry Ginsburg has been appointed to an assistant instructorship.

Associate Professor R. E. Bellman of Stanford University is on leave of absence and has accepted a temporary position at the Forrestal Research Center, Princeton University.

Mr. Jonas Beraru has a position as Design Engineer for the Ford Instrument Company, Long Island City, New York.

Mr. R. W. Bird has accepted a position as Statistical Accountant with the Pacific Indemnity Company, Los Angeles, California.

Dr. A. W. Boldyreff, formerly a staff member of Sandia Corporation, Albuquerque, New Mexico, is now a research specialist at North American Aviation, Downey, California.

Mr. T. C. Kimball, mathematics master at the Lawrenceville School, died on January 8, 1952. He had been a member of the Association for twenty-five years.

Reverend G. A. O'Donnell of Boston College died on January 1, 1952. He had been a member of the Association for twenty-seven years.

Professor E. W. Schreiber, head of the Mathematics Department, Western Illinois State Teachers College, died on December 17, 1951. He had been a member of the Association for thirty-one years.

Dr. P. F. Neményi of the Naval Research Laboratory, Washington, D. C. died on March 1, 1952.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following ninety-six persons have been elected to membership by the Board of Governors on applications duly certified.

A. G. ANDERSON, Ph.D. (Michigan) Instr., Oberlin College, Ohio

R. R. ARCHER, Student, Massachusetts Institute of Technology, Cambridge, Mass.

W. A. BECK, B.S. (Case) Instr., Purdue University, Lafayette, Ind.

J. E. BOSSHART, M.A. (Northwestern) Lecturer, Western Reserve University, Cleveland, Ohio

JENEVA J. BREWER, M.A. (Wichita) Instr., University of Wichita, Kans.

A. T. BUTSON, M.A. (Michigan S. C.) Grad. Assistant, Michigan State College, East Lansing, Mich.

J. F. CANU, M.Sc. (Chicago) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

F. D. CHAMBERLAIN, B.A. (Western Washington) Capt., U. S. Air Force, Hamilton, Calif.

N. A. CHILDRESS, M.A. (Mississippi) Instr., University of Mississippi, University, Miss.

K. P. CLANCY, Student, Fordham University, New York, N. Y.

PAOLO COMBA, Ph.D. (C.I.T.) Asst. Professor, University of Hawaii, Honolulu, Hawaii

S. D. CONTE, Ph.D. (Michigan) Asst. Professor, Wayne University, Detroit, Mich.

D. W. CROWE, M.A. (Nebraska) Teaching Fellow, University of Michigan, Ann Arbor, Mich.

REV. F. T. DALY, M.S. (St. Louis) University of Notre Dame, Ind.

C. R. DEPRIMA, Ph.D. (N.Y.U.) Head, Mathematics Branch, Office of Naval Research, Washington, D. C.

W. A. DOLID, Student, Alfred University, N. Y.

JIM DOUGLAS, JR., M.A. (Rice) Asst., Rice Institute, Houston, Tex.

LEILA A. DRAGONETTE, Ph.D. (Pennsylvania) Research Associate, University of Chicago, Ill.

M. E. DRUMMOND, JR., B.S. (East Central S. C.) Research Assistant, University of Oklahoma, Norman, Okla.

J. E. F. DUTT, B.S. (Manhattan) Grad. Student, Columbia University, New York, N. Y.

R. A. EBNER, Ed.M. (Harvard) Instr., University of Connecticut, Storrs, Conn.

J. G. ELLIOTT, M.A. (Michigan S. C.) Grad. Assistant, Michigan State College, East Lansing, Mich.

D. E. ENOS, Student, Baker University, Baldwin City, Kans.

TREVOR EVANS, M.A. (Oxford) Asst. Professor, Emory University, Ga.

J. E. FORBES, M.S. (Bradley) Instr., Bradley University, Peoria, Ill.

MRS. VIRGINIA FORBES, M.S. (Illinois) Teacher, Community Unit High School, Heyworth, Ill.

A. O. FOREHAND, JR., B.S. (Rollins) Grad. Assistant, University of Alabama, Tuscaloosa, Ala.

R. R. FOSSUM, B.S. (Idaho) Ensign, U. S. Navy, San Francisco, Calif.

BARRY GINSBURG, B.A. (Yeshiva) Asst., Yeshiva University, New York, N. Y.

HERBERT GOERTZEL, A.B. (N.Y.U.) Mathematics Assistant, Oak Ridge National Laboratory, Tenn.

SAMUEL GOLDBERG, Ph.D. (Cornell) Instr., Lehigh University, Bethlehem, Pa.

D. E. GOWAN, B.A. (South Dakota) Computer, General Electric Company, Richland, Wash.

FLORENCE N. GREVILLE, M.A. (Columbia) Analyst, Department of Defense, Washington, D. C.

L. A. GUEST, M.S. (Oklahoma A & M) Grad. Assistant, University of Oklahoma, Norman, Okla.

LAURA GUGGENBUHL, Ph.D. (Bryn Mawr) Asst. Professor, Hunter College, New York, N. Y.

R. L. GULLEY, B.A. (Roosevelt) 2121 F Street, N. W., Washington, D. C.

B. F. HANDY, JR., M.S. (Howard) Research Assistant, Massachusetts Institute of Technology, Cambridge, Mass.

J. D. HANKINS, M.S. (Missouri) Instr., University of Missouri, Columbia, Mo.

MRS. LOUISE F. HANSON, B.S.E.E. (Wisconsin) Asst. Professor, Olivet College, Mich.

ARTHUR HERSKOWITZ, Junior Accountant, Greenberg & Tucker, New York, N. Y.

R. F. HOELKER, Ph.D. (Münster) Redstone Arsenal, Huntsville, Ala.

J. E. HOFFMAN, B.S.C.E. (Oklahoma) Grad. Assistant, University of Oklahoma, Norman, Okla.

R. C. JAMES, Ph.D. (C.I.T.) Asso. Professor, Haverford College, Pa.

J. H. B. KEMPERMAN, Ph.D. (Amsterdam) Visiting Asst. Professor, Purdue University, Lafayette, Ind.

R. W. KLOPFENSTEIN, M.S. (Iowa S. C.) Instr., Iowa State College, Ames, Iowa

D. L. KUNTZ, B.A. (Knox) Illinois Bell Telephone Company, Rock Island, Ill.

L. S. LEVY, Student, Yeshiva University, New York, N. Y.

VICTOR LEWICKE, A.M. (Columbia) Mathematician, Army Map Service, Washington, D. C.

A. C. LINDBERG, M.A. (Nebraska) Asso. Professor, Dana College, Blair, Neb.

WALTER LITTMAN, Student, New York University, N. Y.

J. M. LONG, B.S. (William & Mary) Grad. Assistant, Michigan State College, East Lansing, Mich.

C. P. LOUTHAN, 78 West First Avenue, Columbus, Ohio

BECKHAM MARTIN, Ph.D. (Purdue) Physicist, Owens-Illinois Glass Company, Toledo, Ohio

SAMUEL MELAMED, B.S. (McGill) Lecturer, McGill University, Montreal, Quebec

W. E. MIENTKA, M.A. (Columbia) Instr., University of Massachusetts, Amherst, Mass.

JACK MINKER, M.S. (Wisconsin) Dynamics Engineer, Bell Aircraft Corporation, Niagara Falls, N. Y.

C. R. MITCHELL, M.E. (Rhode Island C. of Ed.) Asst. Professor, Rhode Island College of Education, Providence, R. I.

G. C. NOONAN, JR., U. S. Navy, San Diego, Calif.

E. S. NORTHAM, M.S. (Michigan) Grad. Assistant, Michigan State College, East Lansing, Mich.

ILSE L. NOVAK, Ph.D. (Radcliffe) Member, Institute for Advanced Study, Princeton, N. J.

R. D. ORR, Student, Baker University, Baldwin City, Kans.

E. N. PARKER, Ph.D. (C.I.T.) Instr., University of Utah, Salt Lake City, Utah

W. M. PEREL, M.A. (Indiana) Teaching Assistant, Indiana University, Bloomington, Ind.

L. M. RAUCH, Ph.D. (California) Research Engineer, University of Michigan, Ann Arbor, Mich.

HELENE RESCHOVSKY, Ph.D. (Vienna) Instr., University of Connecticut, Storrs, Conn.

P. B. RICHARDS, M.S.M.E. (Harvard) Analytical Engineer, Babcock & Wilcox Research Laboratory, Alliance, Ohio

DOLORES A. RICKENS, M.A. (Columbia) Asso. Professor, College of St. Elizabeth, Convent Station, N. J.

V. E. ROBERTS, B.S. (Massachusetts) Instr., University of Alaska, College, Alaska

LOUIS ROBINSON, M.A. (Syracuse) Grad. Student, Syracuse University, N. Y.

WELLINGTON ROUNDS, Student, Rutgers University, New Brunswick, N. J.

D. A. RUSSO, M.S. (N.Y.U.) Grad. Student, New York University, N. Y.

F. P. J. SANSOM, Student, Baker University, Baldwin City, Kans.

J. F. SCHELL, B.S. (Dayton) Grad. Assistant, Indiana University, Bloomington, Ind.

S. L. SCHLUNEGER, Student, University of Tulsa, Okla.

L. L. SCOTT, Ph.D. (Illinois) Asst. Professor, University of Mississippi, Miss.

MILTON SIEGEL, B.S. (C.C.N.Y.) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

T. F. SINGLETON, B.S. (St. Louis) Instr., Regis College, Denver, Colo.

F. M. SIOSON, Weather Bureau, Geophysical Observatory, Quezon City, Philippines

SISTER M. XAVERIA, Ph.D. (Michigan) Professor, Marygrove College, Monroe Campus, Mich.

SISTER RITA, M.A. (Catholic) Head, Mathematics Department, Trinity Preparatory School, Ilchester, Md.

S. T. SMITH, Student, University of Rochester, N. Y.

GEORGE SPRINGER, Ph.D. (Harvard) Asst. Professor, Northwestern University, Evanston, Ill.

DORIS S. STOCKTON, M.Sc. (Brown) Instr., University of Connecticut, Storrs, Conn.

JOSEPH TAJEN, B.S. (C.C.N.Y.) Test Engineer, Ordnance Aerophysics Laboratory, Daingerfield, Tex.

O. E. TAULBEE, M.A. (Michigan S. C.) Grad. Assistant, Michigan State College, East Lansing, Mich.

PETER TERWEY, JR., M.A. (Texas) Grad. Student and Part-time Instr., University of North Carolina, Chapel Hill, N. C.

R. P. TEUTSCH, Student, Florida Southern College, Lakeland, Fla.

W. J. THOMSEN, M.S. (Iowa) Research Associate, University of Iowa, Iowa City, Iowa

D. B. J. TOMIUK, B.Sc. (Loyola, Montreal) McGill University, Montreal, Quebec

R. H. TSOW, Student, Gonzaga University, Spokane, Wash.

D. J. WALKER, M.A. (Nebraska) Asst. Professor, University of Idaho, Moscow, Idaho

CECILIA T. WELNA, M.A. (Connecticut) Asst. Instr., University of Connecticut, Storrs, Conn.

F. G. WERNER, Student, Miami University, Oxford, Ohio

E. T. WESTER, M.A. (Oklahoma) Asst. Professor, Central State College, Edmond, Okla.

E. S. WOLK, M.Sc. (Brown) Instr., University of Connecticut, Storrs, Conn.

HENRY ZATZKIS, Ph.D. (Syracuse) Instr., University of Connecticut, Storrs, Conn.

REPORT OF THE TREASURER FOR THE YEAR 1951

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1951. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the office of the Association.

I. TOTAL FUNDS OF THE ASSOCIATION ON JANUARY 1, 1951

M & T Trust Co., Buffalo.....	$ 8,705,06	Current Fund.....	$ 8,705.06
Securities.....	59,952,08	Carus Fund.....	7,701.88
		Chace Fund.....	10,117.07
		Houck Fund.....	8,214.71
		Chauvenet Fund.....	626.05
		General Fund.....	33,292.37
	$68,657.14		$68,657.14

II. Current Fund

Balance, January 1, 1951	$ 8,705.06	Monthly	
Dues	14,742.70	Publication	$14,553.91
Initiation fees	564.00	Reprints (net)	260.68
Subscriptions	3,992.16	Editor's Office	986.28
Sale of back numbers	716.75	Secretary-Treasurer's Office	
Advertisements	3,744.00	Clerical help	6,568.56
Sale of exchange periodicals	69.25	Postage and printing	1,666.08
Interest on General Fund	1,150.76	Office expenses and furniture	646.37
Income from Hardy Fund	120.00	Bank fees	101.04
Transfer from General Fund	4,484.95	Auditing fee	50.00
		Board of Governors	1,674.59
		Meetings	686.17
		Representatives	222.42
		Subventions	988.76
		Balance, December 31, 1951	9,884.77

	III. Carus Fund	IV. Chace Fund	V. Houck Fund	VI. Chauvenet Fund
Balance, January 1, 1951	$ 7,701.88	$10,117.07	$8,214.71	$626.05
Sale of Monographs	3,869.98	—	—	—
Sale of Papyrus	—	320.00	—	—
Sale of Slaught Papers	—	338.84	—	—
Sale of Index	—	—	14.60	—
Interest	272.00	355.69	292.92	20.92
Increase in value of securities	83.58	109.29	90.01	6.43
Balance, December 31, 1951	$11,927.44	$11,240.89	$8,612.24	$653.40

VII. General Fund

Balance, January 1, 1951	$33,292.37	Transfer to Current Fund	$ 4,484.95
Increase in value of securities	353.59	Balance, December 31, 1951	29,161.01

VIII. Total Funds of the Association on December 31, 1951

Current Fund	$ 9,884.77	M & T Trust Co., Buffalo	$ 9,884.77
Carus Fund	11,927.44	Securities	61,594.98
Chace Fund	11,240.89		
Houck Fund	8,612.24		
Chauvenet Fund	653.40		
General Fund	29,161.01		
	$71,479.75		$71,479.75

APRIL MEETING OF THE KANSAS SECTION

The thirty-sixth annual meeting of the Kansas Section of the Mathematical Association of America was held at the University of Kansas, Lawrence, Kansas, on Saturday, April 7, 1951. Professor L. M. Reagan, Chairman of the Section, presided. The morning session was a joint meeting with the Kansas Association of Teachers of Mathematics.

There were one hundred twenty-nine persons registered including the following fifty-nine members:

A. H. Albert, Don Alkire, P. H. Arnold, Wealthy Babcock, Florence L. Black, R. N. Bradt, J. H. Curtiss, Lucy T. Dougherty, Paul Eberhart, J. E. Faulkner, W. C. Foreman, J. W. Forman, Abraham Franck, Albert Furman, W. H. Garrett, F. C. German, W. L. Graves, Laura Z. Greene, J. R. Hanna, P. W. Healy, N. C. Hoover, D. M. Houser, K. C. Hsu, Emma Hyde, W. C. Janes, L. E. Laird, K. E. Lake, Violet H. Larney, C. F. Lewis, Anna Marm, Margaret E. Martinson, Thirza A. Mossman, J. D. Neff, Agnes E. Nibarger, S. T. Parker, O. J. Peterson, P. S. Pretz, G. B. Price, O. M. Rasmussen, C. B. Read, C. A. Reagan, L. M. Reagan, J. D. Riley, H. Q. Rolfs, R. G. Sanger, Robert Schatten, W. R. Scott, Sister M. Helen, G. W. Smith, R. G. Smith, E. C. Stopher, E. B. Stouffer, W. T. Stratton, Wilmont Toalson, C. B. Tucker, Frances E. Walsh, E. B. Wedel, Ferna E. Wrestler, P. M. Young.

The following officers were elected for the year 1951–1952: Chairman, Anna Marm, Bethany College; Vice-Chairman, P. M. Young, Kansas State College; Secretary-Treasurer, Laura Z. Greene, Washburn University.

The following papers were presented:

1. *Some novel methods for estimating the eigenvalues of linear operators*, by Dr. J. H. Curtiss, Chief, National Applied Mathematics Laboratories of the National Bureau of Standards.

The paper was concerned with probabilistic methods of computing the eigenvalues of certain linear operators. In particular, the time-independent Schrodinger equation $\frac{1}{2}\Delta u - Vu + \lambda u = 0$ was considered where V and the unknown function u are functions of several independent variables, Δ is the Laplacian operator, and λ is a constant. The pertinent boundary conditions are that u tends to zero as any coordinate becomes infinite, the squares of each of the first partial derivatives of u are each integrable over the entire plane, and the first partial derivatives have simple discontinuities at the origin. Two categories of classical methods of calculating the eigenvalues were first mentioned. These were: (1) the methods based on variational principles and (2) the methods based on substituting a difference equation for the differential equation, which reduces the problem to a matrix eigenvalue problem. The speaker then went on to describe a variant of the latter method developed by Dr. W. R. Wasow of the National Bureau of Standards, Los Angeles, in which the matrix eigenvalues are approximated by statistics obtained from random walks (see NBS Journal of Research, January, 1951).

2. *Algebraically closed groups*, by Professor W. R. Scott, University of Kansas.

Call a group G algebraically closed if, for every finite set S of equations and inequalities, either (i) there is a solution of S in G, or (ii) there is no solution of S in any group containing G as a subgroup. It is shown that every group G is a subgroup of an algebraically closed group H.

3. *Introducing the limit concept in elementary calculus*, by Professor P. W. Healy, Southwestern College.

It was pointed out that the examples used in the standard calculus textbooks do not serve to make clear to the beginning student the distinction between the value of a function at a point $x=a$ and its limit as $x\to a$. It was suggested that an intuitive approach be employed, using such functions as:

$$f(x)=\frac{2^x-1}{x} \quad \text{and} \quad f(x)=\frac{x^{\sqrt{2}}-1}{x-1}, \quad \text{and}$$

letting x approach zero and one respectively. Since these functions cannot be simplified to give them values at zero and one, the student is helped to see the importance of the limiting process. It is also advisable to introduce functions in which the value of the function and its limit are different at some particular point. This will further clarify the difference between the limit of a function and its value at a point.

4. *Introducing the undergraduate to the language of abstract algebra*, by Professor Violet H. Larney, Kansas State College.

The author stated her belief that every undergraduate major in mathematics should be given an opportunity to familiarize himself with some of the terms and basic concepts of abstract algebra. Not only would the prospective graduate student be more adequately prepared to begin his graduate studies, but also the student who does not plan to enter graduate school would gain a greater degree of mathematical maturity, a better understanding of algebra in general, and an awareness of some of the interesting topics in modern algebra. The approach can be through either a semester introductory course in abstract algebra, or through a renovated traditional course (such as theory of equations) into which is injected the modern algebra point of view. The present paper suggested a possible means for introducing each of the several topics to which the undergraduate should be exposed; the major terms under consideration were groups, fields, integral domains, and matrices

5. *The spherical contour in precision tooling*, by Miss Agnes Nibarger, University of Wichita.

In making a contour on a tool to cut a sphere, the usual practice is to cut a groove in the tool with a grinding wheel whose periphery has the form of a torus. Since efficiency demands the cutting face to be a section of the groove taken at an angle less than a right angle, a distortion of the spherical surface results.

An improvement in cutting the contour by shaping the grinding wheel to a general torus is suggested. The general torus is defined as the surface formed by revolving a circle about a line not necessarily in its plane. The distance between the line and the plane of the circle is a variable depending upon the radius of the required sphere, the diameter of the grinding wheel, and the angle at which the cutting plane is taken across the groove.

6. *On the intersection of a straight line and an n-cell*, by Mr. Keith Moore, University of Kansas, introduced by the Secretary.

The points $x_0, x_1, \cdots, x_n$ of n-dimensional Euclidean space, $R^{(n)}$, will be called linearly independent if the vectors, (x_i-x_0), $(i=1, \cdots, n)$, are linearly independent. By the n-cell of $R^{(n)}$ with the linearly independent points $x_0, x_1, \cdots, x_n$ as vertices is meant the set of points $x=\sum_{i=0}^{n}a_ix_i$, where $\sum_{i=0}^{n}a_i=1$. The points for which $a_i>0$, for all i, can be shown to be interior points of the n-cell. The points for which at least one $a_i=0$ are boundary points. It is proved analytically that if x' is an interior point of an n-cell and if x'' is any other point of $R^{(n)}$, then the straight line through x' and x'' intersects the boundary of the n-cell in precisely two points.

LAURA Z. GREENE, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirtieth annual meeting of the Illinois Section of the Mathematical Association of America was held at the University of Illinois, Urbana, Illinois on Friday afternoon and Saturday forenoon, May 11 and 12, 1951. Professor W. T. McDaniel, Chairman of the Section, presided at all sessions.

There were ninety in attendance, including the following sixty-seven members of the Association:

H. G. Ayre, P. T. Bateman, D. R. Bey, A. H. Black, R. P. Boas, B. K. Brown, F. R. Brown, Josephine H. Chanler, B. B. Clark, D. R. Clutterham, J. J. Corliss, I. K. Feinstein, M. K. Fort, Jr., D. E. Freeland, A. E. Gault, N. A. Goldsmith, M. J. Gottlieb, R. T. Gregory, A. E. Hallerberg, M. C. Hartley, Corrine R. Hattan, L. L. Helms, F. E. Hohn, Dorothy S. Hooper, B. O. Hoyle, P. W. Ketchum, E. C. Kiefer, G. D. Kyle, L. L. Layton, Harry Levy, C. C. MacDuffee, C. T. McCormick, W. C. McDaniel, A. W. McGaughey, B. E. Meserve, H. J. Miles, E. B. Miller, C. N. Mills, Josephine M. Mitchell, G. E. Moore, M. G. Moore, Edna M. Norskog, F. S. Nowlan, Rufus Oldenburger, Daniel Orloff, J. W. Peters, W. L. Pickard, R. E. Pingry, E. W. Ploenges, Irma Reiner, Irving Reiner, T. E. Rine, L. A. Ringenburg, L. D. Rodabaugh, E. J. Schweppe, M. Anice Seybold, H. A. Simmons, Norma K. Stelford, R. C. Stephens, C. J. Stowell, E. H. Taylor, R. J. Teed, B. R. Ullsvik, L. R. VanDeventer, H. E. Vaughn, Alice K. Wright, R. K. Zeigler.

At the business meeting on Saturday morning the following officers were elected for the coming year: Chairman, S. S. Cairns, University of Illinois, Urbana, Illinois; Vice-Chairman, E. W. Schreiber, Western State College, Macomb, Illinois; Secretary-Treasurer, E. C. Kiefer, Millikin University, Decatur, Illinois. The annual meeting for 1952 will be held at Western State College, Macomb, Illinois, on Friday and Saturday, May 9–10, 1952.

The following program was presented:

1. *A remark on approximations to the length of the perimeter of an ellipse*, by Professor L. A. Ringenberg, Eastern Illinois State College.

This remark was suggested by a problem which appeared in the 1950 William Lowell Putnam Competition: *Two obvious approximations to the length L of the perimeter of the ellipse with semiaxes a and b are* $L_1=\pi(a+b)$ *and* $L_2=2\pi\sqrt{ab}$. *Which one comes nearer the truth when the ratio b/a is very close to* 1? A third approximation is $L_3=2\pi\sqrt{a^2+b^2/2}$. Using series expansions it is shown that $L_3>L>L_1>L_2$ and $L_3-L>L-L_1$ for any eccentricity.

2. *Inequalities*, by Professor R. P. Boas, Jr., Northwestern University.

The author discussed a group of inequalities for the coefficients of positive trigonometric polynomials and trigonometric series. He indicated how such problems arise in the study of crystal structure.

3. *The mathematics in Aristotle*, by Sister Marie Stephen, Rosary College, introduced by Sister Mariola, Rosary College.

This topic is interesting historically, for in it we find many of the concepts of irrationals, the continuum and infinite, debated and settled in terms of definitions prevalent in the day. The topic of the indivisible line is discussed at length to illustrate the genius of the philosopher in inspiring his students to pursue to the end his unfinished hints. Theophrastus proposes the arguments of the proponents of the indivisible line and then systematically refutes them in terms more philosophic than mathematical, showing that most of them either lead to absurdities or collide with estab-

lished mathematical theory including the concept of the continuous and the system of irrationals and incommensurables.

4. *A modified Dedekind cut technique for constructing the reals from the rationals*, by Professor L. D. Rodabaugh, Southern Illinois University.

In this paper the complete ordered field of real numbers is constructed from the minimal ordered field of rational numbers by means of a modified Dedekind cut technique. Each of the definitions of elements, relations and operations and of the constructions of identity and inverse elements is given in a single sentence. The immediate result is the construction of the real field at one stroke, so to speak, without the necessity of resorting either to footnoted amendments to the definitions or to the procedure of first constructing the positive reals and later extending to all reals by means of a number pair device.

5. *Defeatism in education*, by Professor C. C. MacDuffee, University of Wisconsin.

6. *The colleges and the national emergency*, by Professor E. W. Ploenges, Millikin University.

Present and pending legislation relative to the drafting of college students was discussed. No conclusions could be drawn, but several observations were made regarding male registration in the colleges next fall.

7. *The problem of the poorly prepared student (Panel Discussion)*, by Professor H. A. Simmons, Northwestern University, Professor H. G. Ayre, Western Illinois State College, Professor M. C. Hartley, University of Illinois, Navy Pier, Chicago, and Professor M. Anice Seybold, North Central College.

Chairman McDaniel called for remarks from several members before the panel got under way, as an introduction to the general topic. Then Professor Simmons, acting as panel chairman, introduced the panel. Sometime before these meetings Professor Simmons had sent a questionnaire to some twenty colleges regarding their experience with the preparation of students to pursue college mathematics. He read the answers from several and commented on them, then made a few remarks about students coming to Northwestern University.

Professor Ayre then took the floor and discussed the subject as follows:

There is a preponderence of evidence that the average college freshman is poorly prepared in mathematics. This does not necessarily mean that the student is dull or has a low I.Q. He may only be the victim of circumstances. Secondary education has reached a new high as to confusion of objectives. In many cases the high school diploma is little more than a belated birth certificate or perhaps a certification of a more or less satisfactory record of attendance at school for twelve years. The colleges have little choice in the matter of accepting the high school product and must start the students where they are and proceed. It is recommended that tests of mathematical competence be administered to incoming freshmen and the course adjusted accordingly. At first there should be a very careful presentation of elementary topics from a new viewpoint, if possible. The beginning freshman needs review, he needs encouragement, and he surely needs to establish confidence in his ability to master mathematics. Time spent in orienting the student and creating a will to learn will pay big dividends later. Many students may need to spend as much as sixteen to twenty quarter hours on pre-calculus mathematics. The speaker recommends that college credit be given for the work on elementary topics; however, it might be well to limit credit towards a mathematics major or minor to approximately sixteen quarter hours for work in pre-calculus mathematics.

Then Professor Hartley read a lengthy paper, the abstract of which follows. The problem of the poorly prepared student can be approached directly and indirectly. The direct approach can be broken down into five attacks. First the administration should establish for each freshman

course rigid prerequisites in terms of present ability. The administration should assign its best teachers to the classes populated by poorly prepared students. The pedagogical approach is the approach of the good teacher—the teacher who studies his students, adapts his instruction to their characteristics and is willing to pioneer in untried procedures. The scientific approach has two important aspects. First we need to know how to teach so that a process which has been learned incorrectly can be completely eliminated and then learned correctly. Second we need a complete reevaluation of our teaching methods. Common errors indicate a wide "learning gap" between the first presentation of a topic and the actual recitation a year or more later. The explication will be found in analyzing and experimenting with each phase of every problem of instruction. The poorly prepared student has no desire to make his knowledge as complete as possible, he likes his own methods, he does not accept the judgment of his teacher, he feels no need to study because previously he has been passed for good behavior and regular attendance. The psychological approach must deal with the breaking down and elimination of these poor attitudes and their associated habits, and with the building up of desirable ones. The curricular approach includes the establishment of courses adapted to the ability of the student, and might well include a series of short courses of not more than four weeks' duration.

The indirect approach to the problem can be broken down into two attacks: pressures can be brought by the accrediting agencies upon the secondary school so that a high school diploma will be a symbol of competence in the subjects pursued, and a guarantee that the minimum essentials of each course have been mastered. At the other extreme, parents can bring pressure upon the schools to turn out a better product. Students are not at fault. The system in which they are entangled is to blame!

Professor Seybold was the last speaker on the panel and closed with the following remarks: She reported on the mathematics preparation of freshmen who enter North Central College, stating that in the last three years the percentage of freshmen offering three or more years of high school mathematics has dropped from 46 to 40. For the students who are not properly prepared for the courses in which they are registered an optional laboratory hour is offered each week. According to a survey of twenty-one catalogs of colleges mostly in the state of Illinois, fourteen of these colleges were offering high school mathematics courses with or without college credit. The most commonly offered high school mathematics courses were intermediate algebra and solid geometry. It was urged that mathematicians take action in reporting to educators their stand on high school mathematics preparation.

E. C. KIEFER, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Hamilton College, Clinton, New York, on Saturday, May 5, 1951. The Chairman of the Section, President B. C. Patterson of Washington and Jefferson College, presided at the morning session; the Vice-Chairman, Professor C. W. Munshower of Colgate University, presided at the afternoon session. At the conclusion of the afternoon session tea was served.

Ninety persons attended the meeting, including the following seventy-five members of the Association:

E. B. Allen, H. P. Atkins, H. T. R. Aude, Frederick Bagemihl, D. C. Barton, M. R. Bates, R. W. Beals, Jr., R. A. Beaver, W. R. Beck, R. L. Beinert, Harry Birchenough, F. J. H. Burkett, E. A. Butler, J. D. Campbell, R. C. Clelland, Nancy Cole, E. J. Downie, Jean B. Feidner, C. W. Foard, A. H. Fox, J. E. Freund, W. H. Fuchs, H. M. Gehman, B. H. Gere, Lillian Gough, N. G. Gunderson, R. G. Hill, H. K. Holt, Anna M. Howe, R. E. Huston, J. R. F. Kent, D. E. Kibbey,

R. D. Larsson, Caroline A. Lester, J. V. Limpert, Charles Loewner, R. C. Luippold, A. W. Mall, Dis Maly, June M. McArtney, L. L. Merrill, Norman Miller, Harriet F. Montague, Mabel D. Montgomery, D. S. Morse, C. W. Munshower, E. E. Nash, W. V. Nevins, C. V. Newsom, W. N. Palmer, B. C. Patterson, Ruth M. Peters, Theresa L. Podmele, L. R. Polan, J. F. Randolph, C. E. Rhodes, P. C. Rosenbloom, M. F. Rosskopf, G. X. Saltarelli, Edith R. Schneckenburger, Wladimir Seidel, Esther Seiden, R. E. Shear, William A. Smith, M. R. Spiegel, Mary R. Steudle, Ruth W. Stokes, Mary C. Suffa, Nura D. Turner, J. D. van Alstyne, G. W. Walker, A. K. Waltz, J. F. Wardwell, F. C. Warner, Frances M. Wright.

The following officers were elected: Chairman, C. W. Munshower, Colgate University; Vice-Chairman, J. F. Randolph, University of Rochester; Secretary, N. G. Gunderson, University of Rochester. The Chairman was authorized to appoint a committee to study the question of sponsoring a mathematics contest for high school students. The 1952 meeting will be held at Hobart and William Smith Colleges, Geneva, New York, in May.

The following papers were presented:

1. *A doubly oscillating pendulum*, by Professor A. K. Waltz, Clarkson College of Technology.

The speaker considered the following problem: Replace the rod of a simple pendulum by a coiled spring of negligible weight. Let the spring swing under gravity as a simple pendulum, friction neglected, and at the same time oscillate longitudinally. If initially $\rho=l_0$, $\theta=\theta_0$, $(\dot\theta)^2=(\dot\theta)_0^2$, and $(\dot\rho)^2=(\dot\rho)_0^2$, find the polar equation of the centroid of the pendulum bob.

Resolving forces along and perpendicular to the radius vector (with notation as for simple pendulum):

(1) $$\ddot\rho - \rho(\dot\theta)^2 = g\cos\theta - k(\rho-\rho_0), \quad \rho_0 = \text{natural spring length},$$

(2) $$\frac{1}{\rho}\frac{d(\rho^2\dot\theta)}{dt} = -g\sin\theta, \qquad k = \text{spring constant}, m = 1.$$

When $\dot\theta$ times (2) is subtracted from the time derivative of (1), then 2ρ becomes an integrating factor yielding

(3) $$2\rho\ddot\rho - (\rho)^2 - 3(\rho\dot\rho)^2 + k\rho^2 = C.$$

The velocity function

(4) $$v^2 = (\rho\dot\theta)^2 + (\dot\rho)^2 = 2g\rho\cos\theta - k\rho^2 + 2k\rho_0\rho - C$$

is then found by eliminating $\ddot\rho$ from (1) and (3).

Now since equation (2) may be written

(2b) $$(\rho\dot\theta)^2 = -2g\int \rho^2\sin\theta d\theta$$

the following equation in ρ and θ is found by eliminating $(\dot\theta)^2$ from (4) and (2b) with the aid of $(\dot\rho)^2=(d\rho/d\theta)^2(\dot\theta)^2$:

(5) $$\rho^4(2g\rho\cos\theta - 2k\rho^2 + 2k\rho_0\rho - C) + 2g\left[\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2\right]\int \rho^2\sin\theta d\theta = 0.$$

This equation (5) together with initial conditions determine the coefficients in an assumed solution of the form $\cos\theta = \sum_{n=0}^{\infty} a_n\rho^n$. The resulting solution is valid except when $k\to\infty$. Then $\dot\rho\equiv 0$, $\rho=\rho_0$, equation (1) becomes nil, and the solution reduces to (2) with $\rho=\rho_0$, giving rise to the simple pendulum.

Each a_n (except a_0) contains the factor a_0^2-1. Hence when $a_0^2-1=0$, $(\dot{\theta})^2\equiv 0$; $\cos\theta\equiv 1$, equation (2) becomes nil, and the solution reduces to (1) with these values inserted giving the special case of a coiled spring vibrating vertically.

2. *The bivariate binomial distribution and applications*, by Professor J. E. Freund, Alfred University.

Professor Freund developed the distribution function of the bivariate binomial distribution as well as its moment generating function and then applied his results to problems in correlation and sampling inspection. Generalizing the discussion to the multivariate case, the speaker showed how the fourvariate binomial distribution can be employed as a convenient model to evaluate the effect of the "human element," *i.e.*, the bias which is introduced by the "observer," in the estimation of probabilities. This mathematical model was used, in particular, as the basis of a reply to criticisms raised by Bertrand Russell against the statistical concept of probability.

3. *Extremal problems treated by elementary methods*, by Professor W. H. Fuchs, Cornell University.

The speaker made a plea for the inclusion of elementary geometrical extremal problems in the classroom treatment of maxima and minima. He discussed several examples, in particular the rectangle of largest area circumscribed to a given quadrangle, and the least ring through which a regular tetrahedron can be slipped.

4. *Report of the committee on the relations of secondary school and college mathematics*, Professor M. F. Rosskopf, Syracuse University.

On the basis of its deliberations, the committee recommended that the Upper New York State Section (1) encourage the formation of secondary school honor societies in mathematics, (2) promote mathematics clubs in which both secondary school and college mathematics teachers would be members, (3) conduct mathematics contests for high school students. The concrete results of committee work are (1) appointment of a committee to supervise and to plan a mathematics contest in Buffalo, New York, to be held in the spring of 1952; (2) active participation in the formation of a state-wide organization of mathematics teachers, the Association of Mathematics Teachers of New York State.

5. *A theorem in measure theory and applications to statistics*, by Professor Esther Seiden, University of Buffalo.

This speaker gave a discussion on the existence of an unbiased test for testing a composite statistical hypothesis against a composite alternative. Conditions under which such a test exists were stated.

6. *Some classes of power series*, by Professor Frederick Bagemihl, University of Rochester.

Let K be the class of complex power series having the unit circle as circle of convergence. Denote by C the class of those elements of K which can be continued analytically beyond the unit circle, and by A the class of those elements of K which satisfy an algebraic differential equation, and let C', A' be the complements of C, A with respect to K. Then each of the classes CA, $C'A$, CA', $C'A'$ has the power of the continuum. Any element of K can be transformed into an element of $C'A'$ by merely multiplying certain terms by -1, and this can be done in continuum many ways. (This generalizes results of Polya and Ostrowski.) There exist continuum many elements of $C'A'$, each having all its coefficients rational and different from zero, such that, no matter which terms are multiplied by -1, the resulting series all belong to $C'A'$.

7. *Some inequalities in differential geometry*, by Professor Charles Loewner, Syracuse University.

Consider in the x, y plane a Jordan domain D and mark on its boundary curve B four points A_1, A_2, A_3, A_4 in their natural order. One might call D a pseudo-rectangle with the vertices A_1, A_2, A_3, A_4. Consider further all curves in D connecting the "opposite sides" of the pseudo-rectangle formed by the arcs A_1A_2 and A_3A_4 on B and call the greatest lower bound of their lengths a. Introduce finally a similar quantity b by using the other pair of opposite sides A_2A_3 and A_1A_4. The quantities a and b may be called the sides of the pseudorectangle B. The following inequality holds

$$A \geqq ab \tag{1}$$

with A representing the area of D. The equality sign holds for a euclidean rectangle. The establishment of this inequality gives rise to questions of the following general character: On a Riemannian, or more generally a Finslerian manifold M of dimension 2, one or several classes of curves $C_1, C_2, \cdots, C_n$ are given and in each class the lengths of the curves have a positive greatest lower bound $a_1, a_2, \cdots, a_n$ respectively. Can a lower bound for the area A of M depending only on the a_i be established? The following results were obtained: (2) The inequality (1) holds for an arbitrary Riemannian metric; (3) If more generally a Finsler metric with a convex symmetric indicatrix is used and the area is suitably defined, (1) has to be replaced by

$$A \geqq \frac{2}{\pi} ab.$$

Of particular interest is the case where only one class C of curves is considered, namely the totality of closed curves which cannot be contracted to a point. The following results were obtained: (4) If the underlying topological space is a torus then $A \geqq a^2\sqrt{3}/2$; (5) If the space is topologically a projective plane then $A \geqq 2a^2/\pi$. The last result is contained in the doctoral thesis of Mr. P. M. Pu.

8. *The sum of a generalized harmonic series and some related results*, by Professor M. R. Spiegel., Rensselaer Polytechnic Institute.

A physical problem arising in potential theory led to the mathematical one of finding the sum of a "generalized harmonic series" $1/a - 1/(a+b) + 1/(a+2b) - \cdots$ where a and b are integers such that $1 \leqq a \leqq b$. This sum is found to be equal to

$$\frac{\pi}{2b \sin(\pi a/b)} - \frac{2}{b} \sum_{k=1}^{[b/2]} \left\{\cos (2k-1) \frac{\pi a}{b}\right\} \left\{\log_e \sin (2k-1) \frac{\pi}{2b}\right\}$$

where $[x]$ denotes, as usual, the largest integer not greater than x. Thus, for example, if $a=1$, $b=6$, we have

$$1 - \frac{1}{7} + \frac{1}{13} - \cdots = \frac{\pi}{6} + \frac{\sqrt{3}}{6} \log_e (2+\sqrt{3}).$$

The generalized harmonic series can also be expressed in terms of the logarithmic derivative of the Γ function, and one may derive the expression

$$\Gamma'\left(\frac{a}{b}\right) \Big/ \Gamma\left(\frac{a}{b}\right) = 2\left\{\sum_{k=1}^{[b/2]} \left(\cos \frac{2k\pi a}{b}\right) \left(\log_e \sin \frac{k\pi}{b}\right)\right\} - \log_e 2b - \frac{\pi}{2} \cot \frac{\pi a}{b} - \gamma$$

where γ is Euler's constant and the other symbols have the same significance as before. From these results, many integrals, some of which arise in analytic number theory, may be evaluated in closed form.

N. G. GUNDERSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1–2, 1952.

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.
ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.
INDIANA, Indiana University, Bloomington, May 3, 1952.
IOWA, Coe College, Cedar Rapids, April 18–19, 1952.
KANSAS
KENTUCKY, University of Kentucky, Lexington, April 19, 1952.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Virginia Military Institute, Lexington, April 26, 1952.
METROPOLITAN NEW YORK, Hofstra College, Hempstead, New York, May 3, 1952.
MICHIGAN, University of Michigan, Ann Arbor, April 12, 1952.
MINNESOTA, College of St. Catherine, St. Paul, May 10, 1952.
MISSOURI, Lindenwood College, St. Charles, May 2, 1952.
NEBRASKA, University of Nebraska, Lincoln, May 3, 1952.
NORTHERN CALIFORNIA
OHIO, Ohio State University, Columbus, April 19, 1952.
OKLAHOMA
PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.
PHILADELPHIA, University of Delaware, Newark, November 29, 1952.
ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May 23–24, 1952.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN, University of Arizona, Tucson, April 11–12, 1952.
TEXAS, East Texas State Teachers College, Commerce, April 25–26, 1952.
UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May 10, 1952.
WISCONSIN, Milwaukee, May 10, 1952.

THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository presentations intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9 and 10, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. The most recent issues are:

No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.
No. 8. *Rings and Ideals* by N. H. McCoy, xii+216 pages.
No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.
No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

Prentice-Hall
COLLEGE TEXTS

MEANINGFUL MATHEMATICS

by H. S. KALTENBORN, Memphis State College

Designed to provide the mathematical part of a liberal education, the material in this text ranges from primitive number systems to calculus. As far as possible, a fresh start is made at the beginning of each chapter, encouraging the student to continue in spite of any possible failure to master some of the special techniques at an earlier stage. A large number of problems provide adequate drill on fundamental processes.

384 pages 5½″ x 8½″ Published October 1951

ANALYTIC GEOMETRY AND CALCULUS

by LYMAN M. KELLS, U. S. Naval Academy

This combination of calculus and analytic geometry in a single text results in (a) better assimilation of both subjects by students, (b) a saving of time, and (c) early equipment of students with basic analytic principles, and the powerful tools of calculus, derivatives, and integrals. A large number of carefully-graded problems provide drill work and numerous important applications.

608 pages 6″ x 9″ Published 1950

Send for your copies today

PRENTICE-HALL INC. 70 FIFTH AVE·NY 11·NY

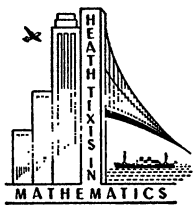
WILLIAM L. HART'S

College Trigonometry

A substantial treatment of plane and spherical trigonometry, of moderate length, incorporating distinctly collegiate viewpoints. Emphasizes analytic trigonometry, oriented for application in later mathematics; presents a mature and well-rounded treatment of numerical plane trigonometry; offers a satisfactory foundation in spherical trigonometry, including a reasonable number of elementary applications. *Text pages: Plane Trigonometry, 151; Complex Numbers and Appendix, 21; Spherical Trigonometry, 35; and Tables, 130. $3.50*

Elements of Analytic Geometry

A substantial course in plane and solid analytic geometry with terminology and content oriented with respect to later applications in calculus. Restricted emphasis on conic sections. Moderate length due to care in selection of content, not to brevity in details. Considerable attention to general terminology about variables, functions, and their graphs. *229 pages of text. $3.00*



ALFRED L. NELSON, KARL W. FOLLEY, MAX CORAL'S

Differential Equations

Comprehensiveness and emphasis on applications make this new text suitable both for students of pure mathematics and for engineering students. Methods for the solving of ordinary differential equations of the first and higher orders are given in careful and detailed form. Both symbolic and nonsymbolic methods are presented in the chapter on linear differential equations. Full treatment given to numerical approximations to the solution of differential equations. Sound exposition of Frobenius' method of solving linear differential equations with singularities. Partial differential equations of the first order and systems of such equations briefly treated. *In press*

D. C. HEATH AND COMPANY

Sales Offices: New York Chicago San Francisco
Atlanta Dallas *Home Office:* Boston

TWO NEW BOOKS—READY IN APRIL—

CALCULUS

By Atherton Hall Sprague, *Amherst College*

This mathematically sound textbook provides a logically complete course in the calculus. Presents Increment, Differential, and Derivative with exceptional rigor and thoroughness. Analytic proofs are accompanied by comprehensive and detailed explanations. *566 pages, 206 diagrams.*

THE NATURE OF NUMBER

An Approach to Basic Ideas of Modern Mathematics

By Roy Dubisch, *Fresno State College*

Especially written for those seeking a way to acquaint themselves with what modern mathematics is about, to gain an insight into its theory, and to familiarize themselves with the types of problem that present-day mathematicians are interested in. *159 pages, 23 figures.*

THE ANATOMY OF MATHEMATICS

By R. B. Kershner, *The Johns Hopkins University;* *and* L. R. Wilcox, *Illinois Institute of Technology*

Bridging the gap between the classic and modern approaches, this volume fills a long-existing need for a treatise on the axiomatic method. A valuable reference for use in those sciences which employ the results and techniques of abstract mathematics. *416 pages, 3 illustrations.*

READY NOW—

SYMBOLIC LOGIC

An Introduction

By Frederick Brenton Fitch, *Yale University*

A new and more practical approach to symbolic logic is offered in this well-written book. No previous course in logic is assumed. By easy steps the student gains experience in working with symbols and in actually constructing proofs for himself. *238 pages.*

THE RONALD PRESS COMPANY

15 East 26th Street • New York 10, N. Y.

DESCRIPTIVE GEOMETRY

A Pictorial Approach

By Harold Bartlett Howe, *Rensselaer Polytechnic Institute*

This up-to-date textbook employs the direct-method approach instead of the plane-trace method. Use is made of pictorial sketches rather than relying merely on textual explanation. Ability to sketch space pictures is gradually built up in progressive steps. *332 pages, 398 illustrations.*

ANALYTIC GEOMETRY

By Alfred L. Nelson, Karl W. Folley, *and* William M. Borgman; *all of Wayne University*

Planned for use in those courses where preparation for the calculus rather than the study of geometry is the paramount objective. Of maximum value to future students of the calculus, the basic sciences, and engineering. *215 pages, 110 illustrations, 4 tables.*

COLLEGE ALGEBRA

By Earle B. Miller, *Illinois College; and* Robert M. Thrall, *University of Michigan*

Provides the first-year student with a thorough grounding in the subject that will equip him for subsequent courses in mathematics. Avoids the complexity of the too advanced text and the sterility of the oversimplified presentation. *493 pages, 36 illustrations, 7 tables.*

PREPARATORY BUSINESS MATHEMATICS

By Lloyd L. Smail, *Lehigh University*

A successful textbook which gives college students in business administration adequate preparation for subsequent courses in mathematics of finance, in insurance, and in statistics. Contains a review of elementary algebra and selected topics from intermediate algebra, college algebra, and analytic geometry. *244 pages, 39 illustrations.*

Examination Copies Available

THE RONALD PRESS COMPANY

15 East 26th Street · New York 10, N. Y.

FOUR NEW TEXTS

Basic Skills in Mathematics

BY H. V. PRICE and L. A. KNOWLER, State University of Iowa. Includes arithmetic, properties of geometric figures, simple algebra, graphs, and elementary chapters on finance and statistics.

Mathematics of Finance

By E. D. MOUZON, Southern Methodist, and P. K. REES, Louisiana State University. Prerequisite: one semester of college algebra. Includes a brief introduction to life annuities and life insurance.

College Algebra and Plane Trigonometry

By J. H. ZANT, Oklahoma A. and M. An integrated organization of subject matter from the fields of college algebra and plane trigonometry, for a one-semester course. Examples illustrate each step.

Calculus

By T. L. WADE, Florida State University. A clearly presented and well-motivated treatment of the calculus. Techniques are introduced gradually and related to prerequisite courses.

GINN AND COMPANY HOME OFFICE: BOSTON

SALES OFFICES: NEW YORK 11 CHICAGO 16 ATLANTA 3
DALLAS 1 COLUMBUS 16 SAN FRANCISCO 3 TORONTO 5

ALGEBRA TEXTS by Raymond W. Brink

COLLEGE ALGEBRA

SECOND EDITION

Notable for its accuracy, logic and flexibility, this very complete, first-year text includes a review of high-school algebra. Contains exercises, study aids, diagrams and tables. *490 pp., $3.75*

ALGEBRA: COLLEGE COURSE

SECOND EDITION

Intended for students not requiring a review of high-school algebra, this book concentrates on the development of technical proficiency and basic understanding. *378 pp., $3.25*

INTERMEDIATE ALGEBRA

SECOND EDITION

Designed for college students who have had only one year of high-school algebra, this text presents in full the first 15 chapters of *College Algebra, 2nd Edition,* with problems and tables. *295 pp., $3.00*

APPLETON-CENTURY-CROFTS, 35 W. 32nd St., N.Y. 1, N.Y.

ALGEBRA FOR COLLEGE STUDENTS

By JACK R. BRITTON and L. CLIFTON SNIVELY. "Here is an algebra book suitable for use in both Intermediate Algebra and College Algebra and the authors are to be thanked for arranging chapters in an order suitable for this approach . . . very usable and very carefully written."—*Colorado Engineer. 529 pp. $4.00*

COLLEGE ALGEBRA

By LEWIS M. REAGAN, ELLIS R. OTT, and DANIEL T. SIGLEY. "The presentation of new material with the review of old is an excellent feature. I like the rigorous use of ideas and words which sets an example of proper mathematical precision for the student."—*Julia Wells Bowen, Connecticut College. 337 pp. $3.00*

FRESHMAN MATHEMATICS

By SLOBIN and WILBUR, revised by C. V. NEWSOM. The original text by Slobin and Wilbur has been almost completely rewritten by Professor Newsom. The organization of material—algebra, trigonometry, and analytical geometry presented together as a tandem course—has not been changed, however. Special attention is given to clarity of exposition and the logical development of one topic from another. The student is given a sound foundation in all three subjects without repetition or overlapping of material. *559 pp. $5.00*

RINEHART MATHEMATICAL TABLES, FORMULAS, AND CURVES

By HAROLD LARSEN. "This recently issued volume of mathematical tables will doubtless be found to be of much use to many students and others. It is difficult to think of any tables which are not included in it."—*Popular Astronomy.*

". . . the author has taken special care to present as accurate a table as possible, and the publisher has produced a pleasing design and appearance. The result is a handbook which will be more than well received by students, teachers and computors alike."—*American Mathematical Monthly. 264 pp. $1.75*

just published . . .

MATHEMATICS OF INVESTMENT

By PAUL R. RIDER and CARL H. FISCHER. A high-level treatment that does not go beyond the capability of the average student with an adequate background. Includes 160 pages of tables and new calculations of fractional low interest rates. *359 pp. $5.00*

TEACHING MATHEMATICS IN THE SECONDARY SCHOOL

By LUCIEN B. KINNEY and C. RICHARD PURDY. Provides a careful description of applications of present-day methods and materials, including visual aids as derived from actual classroom experience. *381 pp. $5.00*

RINEHART & CO.

232 madison ave. n.y. 16, n.y.

WILEY BOOKS

Published in April

a straightforward text that uses modern ideas to clarify and simplify conventional topics

ELEMENTARY ANALYSIS

By KENNETH O. MAY, *Professor and Chairman of the Department of Mathematics and Astronomy, Carleton College.* Specifically designed for an integrated beginning course, this book gives the student a good grasp of the logical nature of mathematics, the algebra of numbers, analytic geometry, and the simpler ideas of calculus by organizing all its topics around a few significant ideas.

Professor May develops the number system and algebra from the very beginning. He takes the student from the known and familiar integers to the more complicated and abstract ideas of real and complex numbers by stages, with the step from each stage to the next a natural and reasonable one. Starting with the basic laws of algebra for the integers, the author shows how the same laws continue to hold for the extended number systems. In this way these laws are firmly impressed on the student's mind and the logical structure of algebra becomes clear.

Each section of this new volume is a rounded discussion which combines theoretical soundness and rigor with careful motivation and extensive illustration. The student is constantly reminded of the meaning of what he is doing, its relation to other topics and its importance.

April 1952 **Approx. 600 pages** **Prob. $5.00**

THE DESIGN AND ANALYSIS OF EXPERIMENTS

By OSCAR KEMPTHORNE, *Professor of Statistics, Iowa State College.* A WILEY PUBLICATION IN STATISTICS, edited by Walter A. Shewhart. *1952. 631 pages. $8.50.*

NOMOGRAPHY AND EMPIRICAL EQUATIONS

By LEE H. JOHNSON, *Dean and Professor of Civil Engineering, Tulane University. 1952. 150 pages. $3.75.*

ADVANCED ANTENNA THEORY

By SERGEI A. SCHELKUNOFF, *Member of the Technical Staff, Bell Telephone Laboratories, Inc. One of the* WILEY APPLIED MATHEMATICS SERIES, edited by I. S. Sokolnikoff. *March 1952. 216 pages. $6.50.*

ANTENNAS: THEORY AND PRACTICE

By SERGEI A. SCHELKUNOFF *and* HARALD T. FRIIS, *Director of Radio Research, Bell Telephone Laboratories, Inc. One of the* WILEY APPLIED MATHEMATICS SERIES, edited by I. S. Sokolnikoff. *March 1952. 639 pages. $10.00.*

STATISTICAL THEORY WITH ENGINEERING APPLICATIONS

By A. HALD, *Professor of Mathematics, University of Copenhagen.* A WILEY PUBLICATION IN STATISTICS, edited by Walter A. Shewhart. *April 1952. Approx. 654 pages. $7.50.*

STATISTICAL TABLES AND FORMULAS

By A. HALD. A WILEY PUBLICATION IN STATISTICS, edited by Walter A. Shewhart. *April 1952. 97 pages. Prob. $1.50.*

Copies available on approval

JOHN WILEY & SONS, Inc.

440 Fourth Avenue New York 16, N.Y.

Important McGraw-Hill Books

GENERAL COLLEGE MATHEMATICS

By W. L. AYRES, C. G. FRY, and H. F. S. JONAH, Purdue University. In press

A new approach to freshman mathematics, designed to help the student in his application of mathematical topics to the biological and social sciences. Keyed primarily for those students who will take only one year of college mathematics, this text examines such subjects as ratios and percentages, linear and quadratic equations, trigonometry, interest and its application to installment buying, laws of growth, statistics, etc.

PLANE AND SPHERICAL TRIGONOMETRY

By LYMAN M. KELLS, WILLIS F. KERN, and JAMES R. BLAND, United States Naval Academy. Third Edition. 408 pages, with tables, $3.75

A complete revision of this leading text. While all the good features of the past edition have been retained, improvements were effected in the new edition by eliminating items which did not have a definite purpose, changing the logical as well as the pedagogical order, by simplification of major developments, and by the addition of new theory, new proofs, and new problems. *Plane Trigonometry*, a separate text, consisting of the first section of *Plane and Spherical Trigonometry*, is also available.

CONFORMAL MAPPING

By ZEEV NEHARI, Washington University. *International Series in Pure and Applied Mathematics.* 396 pages, $7.50

An excellent text designed for students with a good working knowledge of calculus. The potential and complex function theories necessary for a full treatment of conformal mapping are developed in the first four chapters. The theoretical and practical aspects of the subject are both thoroughly covered, and the discussion ranges from the fundamental existence theorems to the several techniques available for the conformal mapping of given geometric figures.

DIFFERENTIAL EQUATIONS

By ROBERT C. YATES, United States Military Academy. In press

This text is designed to prepare the student for work in modern engineering practice and theory and to present the basic mathematical tools necessary for analysis and solution of problems leading to differential equations. The book ranges from first order equations through general linear ordinary equations with constant coefficients, special equations mostly of a nonlinear character, numerical solutions and solutions by series, and then into the solution of the wave equation by separation of variables.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, Inc.

330 WEST 42nd STREET • NEW YORK 36, N. Y.

John F. Randolph • CALCULUS

This new text offers a flexible, well-proportioned, modern treatment of the theory and applications of calculus. The problems are numerous, practical, and well-illustrated. *Published in March—$5.00*

CONTENTS

1. Functions, Limits, Derivatives
2. The Definite Integral
3. Application of Derivatives
4. Transcendental Functions
5. Indefinite Integration
6. Solid Geometry
7. Partial Derivatives
8. Multiple Integrals
9. Further Applications
10. Approximations
11. Series

Appendix

A1. Proof of Limit Theorems
A2. Definition of e
A3. Inverse Functions
A4. Proof of l'Hospital's Rule II
A5. Cauchy Sequences
A6. Rearrangement of Series
A7. Calculus of Power Series

ELEMENTARY DIFFERENTIAL EQUATIONS

E. D. Rainville

Containing all the material of the author's *Short Course in Differential Equations,* this text provides a complete introduction to elementary differential equations in a full year's course for students who have had standard calculus. *Ready in April*

CONTENTS

1. Definitions, Elimination of Arbitrary Constants
2. Equations of the First Order and First Degree
3. Additional Methods for Equations of the First Order and First Degree
4. Orthogonal Trajectories
5. Hyperbolic Functions
6. Linear Equations with Constant Coefficients
7. Nonhomogeneous Equations: Undetermined Coefficients
8. Nonhomogeneous Equations: Operational Methods
9. Applications
10. Electric Circuits and Networks
11. Variation of Parameters and other Methods
12. Equations of Order One and Higher Degree
13. Special Equations of Order Two
14. The Power Series Method
15. Solutions Near Regular Singular Points
16. Additional Topics on Power Series Solutions
17. The Existence of Solutions
18. Numerical Methods
19. Partial Differential Equations
20. Fourier Series
21. Boundary Value Problems

The Macmillan Co.
New York 11, N.Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 59 NUMBER 5

CONTENTS

An Indeterminate Problem in Classical Mechanics . . DAVID GALE 291
Robinson's Constant 296
An Elementary Note on Powers of Quaternions. . .H. E. SALZER 298
On Certain Determinantal Equations. . . . . EVELYN FRANK 300
Mathematical Notes . . . . I. N. HERSTEIN AND J. E. ADNEY, ... C. F. MOPPERT, ELSIE CHURCH, J. M. ELKIN 309
Classroom Notes. . . . . ALFRED BRAUER, H. J. HAMILTON, G. B. HUFF AND . D. F. BARROW, T. M. APOSTOL, T. D. NAGLE, CLAIRE ADLER 319
Elementary Problems and Solutions 328
Advanced Problems and Solutions 332
Recent Publications 339
Clubs and Allied Activities. 343
News and Notices 347
Mathematical Association of America 355
May Meeting of the Indiana Section 357
June Meeting of the Pacific Northwest Section 360
October Meeting of the Oklahoma Section 363
Calendar of Future Meetings 364

MAY 1952

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	J. W. GIVENS	GEORGE SPRINGER
PATRICK DU VAL	L. M. KELLY	E. P. STARKE
HOWARD EVES	H. D. LARSEN	G. B. THOMAS
F. A. FICKEN	IVAN NIVEN	E. P. VANCE

EDITH R. SCHNECKENBURGER

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of Collegiate Mathematics)

OFFICERS OF THE ASSOCIATION

President, SAUNDERS MACLANE, University of Chicago
Honorary President, W. D. CAIRNS, Oberlin College
First Vice-President, F. L. GRIFFIN, Reed College
Second Vice-President, JEWELL H. BUSHEY, Hunter College
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo
Editor, C. B. ALLENDOERFER, University of Washington

Additional Members of the Board of Governors: E. B. ALLEN, C. F. BARR, T. A. BICKERSTAFF, J. C. BRIXEY, R. H. CAMERON, G. R. CLEMENTS, T. F. COPE, H. S. M. COXETER, W. M. DAVIS, J. M. EARL, G. M. EWING, L. R. FORD, TOMLINSON FORT, J. S. FRAME, A. E. GAULT, R. E. GILMAN, E. H. HANSON, M. R. HESTENES, AUGHTUM S. HOWARD, C. G. JAEGER, B. W. JONES, M. S. KNEBELMAN, R. E. LANGER, D. H. LEHMER, C. C. MACDUFFEE, W. E. MILNE, H. P. PETTIT, J. C. POLLEY, G. E. RAYNOR, E. B. ROESSLER, R. G. SANGER, F. H. STEEN, EARL WALDEN, MARIE J. WEISS, F. B. WILEY

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

AN INDETERMINATE PROBLEM IN CLASSICAL MECHANICS

DAVID GALE, Brown University

1. Introduction. We shall say that a physical experiment leads to an indeterminant result if upon reproducing the conditions of the experiment one obtains differing results, and this scatter is independent of the accuracy with which the experiment is conducted. In modern atomic physics one often performs experiments in which the exact outcome is indeterminate and the theory only predicts the probability of the occurrence of each possible outcome. We shall show here that such experiments may also be devised within the realm of the purely classical mechanics of collisions of particles and, in fact, that one of the popular freshman laboratory demonstrations, when slightly modified, leads to an indeterminate problem.

Anyone who has taken an elementary physics course has probably been exposed to the following demonstration. A set of balls is constrained to move in a straight line, perhaps by means of a long narrow trough in which they are free to roll. (To avoid rotation the experiment may also be performed with beads strung on a straight, stiff wire.) Initially m of the balls are in contact and at rest while the remaining n balls are also in contact but are moving with a velocity v toward the stationary balls. After collision one finds that the n balls at the front of the line are moving forward with velocity v, while the m balls at the back are now stationary. For this experiment it is necessary that all balls have the same mass. How does one account for this result? Well, clearly the laws of conservation of momentum and energy are satisfied. However, in the case of more than two balls there will be an infinite number of outcomes satisfying these conditions. For example, suppose there are three balls each with mass 1 and suppose that the first moves with velocity 1 toward the second and third which are in contact and stationary. Denoting the final velocities of the balls by v_1, v_2, and v_3, the following conditions must be satisfied:

$$\text{(I)}\qquad \begin{aligned} &\text{(a)}\quad v_1 + v_2 + v_3 = 1 \quad \text{(conservation of momentum)},\\ &\text{(b)}\quad v_1^2 + v_2^2 + v_3^2 = 1 \quad \text{(conservation of energy)},\\ &\text{(c)}\quad v_1 \leqq v_2 \leqq v_3 \qquad \text{(no ball can overtake another)}. \end{aligned}$$

Now consider the solutions of (I) as a subset of three-space. The intersection of (a) and (b) is a circle on the surface of the unit sphere while (c) determines a wedge shaped region, so the solutions form an arc of a circle. The actual solution is of course $v_1=0$, $v_2=0$, $v_3=1$, but another possibility would be, for instance, $v_1=-1/3$, $v_2=2/3$, $v_3=2/3$.

We might try to account for the result by a continuity argument. That is, let the balls be separated by a finite amount, analyze the successive two ball collisions, and take the limit as the distances between balls go to zero. It is easy to see that in the case of the classical demonstration described above this limit actually exists, for in each collision a ball gives up all its velocity to a

stationary ball. On the other hand, suppose the balls have different velocities or different masses. The following examples show that the continuity argument may break down completely, and we shall try to convince the reader that the outcome of these experiments will be indeterminate.

2. First example. Given three balls constrained to move on a straight line, the first with mass 1, the second with mass 3, and the third with mass 1. Initially the second ball is stationary while the first and third move toward it with unit speed from equal distances. From considerations of symmetry we would expect that after collision the center ball would remain stationary while the ones on the outside would move apart with unit speed. Suppose, however, that one of the balls, say the first one, strikes the center ball slightly before the third. Let us analyze the two ball collisions that will then occur. Recall that the outcome of a two particle straight line elastic collision is completely determined by the laws of conservation of energy and momentum. If the particles have masses m_1 and m_2 and the velocities before impact are u_1 and u_2, then the velocities after impact must satisfy the pair of equations:

$$\text{(II)} \qquad \begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2, \\ m_1u_1^2 + m_2u_2^2 &= m_1v_1^2 + m_2v_2^2. \end{aligned}$$

This pair of equations has two solutions, the trivial one $v_1 = u_1$ and $v_2 = u_2$, when no collision occurs, and the second solution in which we are interested,

$$\text{(III)} \qquad v_1 = \frac{2m_2u_2 - (m_2 - m_1)u_1}{m_1 + m_2}, \qquad v_2 = \frac{2m_1u_1 - (m_2 - m_2)u_2}{m_1 + m_2}.$$

Let us now apply (III) to the three ball situation described above. Denoting the velocities of the first, second and third balls by the subscripts 1, 2, and 3, we find that after the first collision the first ball attains a velocity $v_1 = -1/2$ and the second ball $u_2 = 1/2$. The second ball then hits the third ball and after collision the velocities are $v_2 = -1/4$ and $v_3 = 5/4$. The final velocities are thus,

$$\text{(IV)} \qquad v_1 = -1/2, \qquad v_2 = -1/4, \qquad v_3 = 5/4.$$

On the other hand, if the third ball strikes the second ball first the final speeds will be

$$\text{(IV)}' \qquad v_1 = -5/4, \qquad v_2 = 1/4, \qquad v_3 = 1/2.$$

Thus we see that the final result depends discontinuously on the initial positions of the balls. In the critical case where both balls hit simultaneously symmetry considerations give an answer, but it is clear that this symmetry can be upset by slightly altering the masses or initial velocities of balls one or three. What then should we expect as the outcome of our experiment? The only answer seems to be that no matter how we refine the experimental technique we may get either case (IV) or (IV)′ and thus our result will be indeterminate in the sense defined above.

The reader acquainted with mechanics may at this point raise an objection. In adopting the second of equations (II) we are making the assumption that the colliding bodies are "perfectly elastic" and hence no energy is lost in collision. In practice there is no such thing as a perfectly elastic body. Is it not this idealization which gives rise to the discontinuity? The answer is that even without the assumption of perfect elasticity the indeterminate result still holds, as we now show.

For the case of non-perfect elasticity equations (II) are replaced by,

$$\text{(V)} \qquad \begin{aligned} m_1u_1 + m_2u_2 &= m_1v_1 + m_2v_2, \\ v_1 - v_2 &= -e(u_1 - u_2), \qquad \text{where } 0 \leqq e \leqq 1. \end{aligned}$$

The quantity e is called the coefficient of restitution and depends on the material from which the particles are made. (For derivation of these equations see texts on particle mechanics, for instance, Lindsay, *Physical Mechanics*, pp. 350.) These equations become equivalent to (II) in the case where $e=1$. Equations (III) are now replaced by:

$$\text{(VI)} \qquad \begin{aligned} v_1 &= \frac{(m_1 - em_2)u_1 + m_2(1 + e)u_2}{m_1 + m_2}, \\ v_2 &= \frac{m_1(1 + e)u_1 + (m_2 - m_1e)u_2}{m_1 + m_2}. \end{aligned}$$

Consider now the above experiment where balls 1 and 3 have mass m_1 and ball 2 has mass m_2. Again assume that 2 is struck by 1 before it is struck by 3. Using (VI) we find that after the two collisions we get as final velocities,

$$\text{(VII)} \qquad \begin{aligned} v_1 &= \frac{m_1 - em_2}{m_1 + m_2}, \qquad v_2 = -\left(\frac{m_1(1 + e)}{m_1 + m_2}\right)^2, \\ v_3 &= \frac{m_1m_2(1 + e)^2 - (m_1 + m_2)(m_1 - m_2e)}{(m_1 + m_2)^2}. \end{aligned}$$

If we chose m_2 so large that $n_1 < em_2$, then $v_1 < 0$, $v_2 < 0$, and $v_3 > 0$. To guarantee that there will be no further collisions we must also know that $v_1 < v_2$, which can also be achieved by taking m_2 sufficiently large, since

$$\lim_{m_2 \to \infty} v_1 = -e \quad \text{and} \quad \lim_{m_2 \to \infty} v_2 = 0.$$

It is now clear that if our collisions occur in the reverse order we obtain

$$\text{(VII)}' \qquad v_1' = -v_3, \qquad v_2' = -v_2, \qquad v_3' = -v_1.$$

Thus in one case v_2 is negative, in the other positive, and we again observe the discontinuity and hence the indeterminacy.

We have shown that the simultaneous collision of three particles may lead to an indeterminate result. We shall now use this to give an example which

shows that the classical experiment with four balls may lead to an indeterminacy if the masses of the balls are suitably selected. To avoid having to work with long complicated formulae we shall pick definite numerical values for the masses of the balls. Also we shall for simplicity assume perfect elasticity although, in view of the previous example, it will be clear that this assumption is not necessary.

3. Second example. We are given four balls, B_1, B_2, B_3, and B_4, with masses 2, 1, 1, and 3 respectively. Balls 2, 3, and 4 are in contact and stationary, while ball 1 moves towards them with velocity 1. Let u, v, w, and x be the final velocities of the balls.

Assertion: The experiment may lead to one of two possible results.

Case I. $u=-1/3$, $v=0$, $w=2/3$, $x=2/3$.

Case II. $u=-11/27$, $v=10/27$, $w=4/9$, $x=2/3$.

Proof: Suppose balls 2 and 3 and balls 3 and 4 are separated by a distance ϵ. We must now analyze a series of two ball collisions which we shall number in chronological order. We assume that the first collision occurs at time $t_1=0$, and that the i-th collision occurs at time t_i. The symbols u_i, v_i, w_i, and x_i will denote the respective velocities after the i-th collision. Initially we have

$$u_0 = 1, \qquad v_0 = 0, \qquad w_0 = 0, \qquad x_0 = 0.$$

Collision 1. B_1 strikes B_2 at time $t_1=0$. Then

$$u_1 = 1/3, \qquad v_1 = 4/3, \qquad w_1 = 0, \qquad x_1 = 0.$$

Collision 2. B_2 strikes B_3 at time $t_2=\epsilon/(4/3)=3\epsilon/4$. Then

$$u_2 = 1/3, \qquad v_2 = 0, \qquad w_2 = 4/3, \qquad x_2 = 0.$$

We must now determine which collision will occur next, between balls 1 and 2 or balls 3 and 4. Since B_1 is now distance ϵ from B_2, and B_3 is distance ϵ from B_4, it is clear that 3 will strike 4 first since $w_2>u_2$.

Collision 3. B_3 strikes B_4 at $t_3=t_2+3\epsilon/4=3\epsilon/2$. Then

$$u_3 = 1/3, \qquad v_3 = 0, \qquad w_3 = -2/3, \qquad x_3 = 2/3.$$

We must now determine whether B_1 or B_3 will strike B_2 first. If B_1 strikes first, the collision occurs at time $t_4=t_1+\epsilon/(1/3)=3\epsilon$. If B_3 strikes first, the collision occurs at $t_4=t_3+\epsilon/(2/3)=3\epsilon$.

Conclusion: Both collisions occur at the same time! Thus we are again in an indeterminate situation similar to that of the first example and must resort to analyzing two separate cases according as B_3 or B_1 hits B_2 first.

Case I.

Collision 4. Between B_3 and B_2. Then

$$u_4 = 1/3, \qquad v_4 = -2/3, \qquad w_4 = 0, \qquad x_4 = 2/3.$$

Collision 5. Between B_2 and B_1. Then

$$u_5 = -1/3, \quad v_5 = 2/3, \quad w_5 = 0, \quad x_5 = 2/3.$$

Collision 6. Between B_2 and B_3. Then

$$u_6 = -1/3, \quad v_6 = 0, \quad w_6 = 2/3, \quad x_6 = 2/3.$$

Since now $u \leqq v \leqq w \leqq x$, there will be no more collisions.

Case II.

Collision 4′. Between B_1 and B_2.

$$u_{4'} = 1/9, \quad v_{4'} = 4/9, \quad w_{4'} = -2/3, \quad x_{4'} = 2/3.$$

Collision 5′. Between B_2 and B_3.

$$u_{5'} = 1/9, \quad v_{5'} = -2/3, \quad w_{5'} = 4/9, \quad x_{5'} = 2/3.$$

Collision 6′. Between B_1 and B_2.

$$u_{6'} = -11/27, \quad v_{6'} = 10/27, \quad w_{6'} = 4/9, \quad x_{6'} = 2/3. \quad \text{Q.E.D.}$$

(The skeptical reader is encouraged to check the author's calculations by verifying that the two conservation laws are satisfied at each collision.)

The above example illustrates another surprising fact. If we consider the case of the separated balls we have seen that the final velocities are discontinuous functions of the initial positions of the balls. That is, suppose that initially the distance between B_1 and B_2 is ϵ_{12}, and the distance between B_3 and B_4 is ϵ_{34}. Then we will get case I or case II according as $\epsilon_{23} > \epsilon_{34}$ or $\epsilon_{34} > \epsilon_{23}$, while the critical case, $\epsilon_{23} = \epsilon_{34}$, is indeterminate. It is also true that the final velocities are discontinuous functions of the masses of the balls. We leave it to the reader to convince himself that if $m_4 > 3$ we get case I, while if $m_4 < 3$ we get case II.

4. Summary. Let us try to summarize the significance of our results. We have seen that in what we have called the classical collision experiment, the usual laws of collision (with or without perfect elasticity) are insufficient to determine a unique outcome. There are then two alternative possibilities: first, that there exist some further physical laws which together with the classical ones will determine the result uniquely; second, that the indeterminacy actually exists in nature and that the experiment if repeated would give differing results. We have tried to argue that it is, in fact, the second alternative which holds. If allowed a slightly metaphysical comment we would put it this way. Philosophers talk of the law of "uniformity of nature" which states that given the same initial conditions the same results will follow. This "law" has had to be abandoned in the field of atomic physics. We suggest here that it may also fall down in the classical case.

ROBINSON'S CONSTANT

Editorial Note. A recent issue of this MONTHLY (Volume 58, No. 7, August-September, 1951) contained an article *A New Absolute Geometric Constant?* by Robin Robinson. This paper is a summary of the articles submitted to the editor dealing with the value of this constant.

It was proved that the number g_n of polygons of n sides (including "degenerate" cases) formed by a network of n lines (no three concurrent) is given by

$$g_{n+1} = ng_n + \binom{n}{2} g_{n-2},$$

where $g_1 = g_2 = 0$, $g_3 = 1$. Introducing u_n, where

$$g_n = u_n \cdot \tfrac{1}{2}(n-1)!,$$

this provides the recursion relation

$$u_{n+3} = u_{n+2} + \frac{1}{2n} u_n, \tag{1}$$

where $u_1 = u_2 = 0$, $u_3 = u_4 = u_5 = 1$. The author proved that

$$\lim_{n\to\infty} \frac{u_n^2}{n} = b,$$

where b is the constant 0.284,098, $\cdots$, and expressed an interest in finding a finite expression for b.

Thirteen readers submitted evidence that Robinson's constant b is given by

$$b = 4e^{-3/2}/\pi.$$

These readers were David Dickinson, Rufus Isaacs, Harry Pollard, R. M. Redheffer, John Riordan, H. F. Sandham, M. R. Spiegel, C. S. Sutton, G. Szekeres, F. Ursell, Morgan Ward, G. T. Williams, and Koichi Yamamoto.

The methods used were varied. Most treatments made use of a generating function, the one most commonly used being

$$f(x) = 2\sum_{n=3}^{\infty} \frac{g_n}{n!} x^n = \sum_{n=3}^{\infty} \frac{u_n}{n} x^n.$$

This series converges for $|x| < 1$. By use of the relation (1) it is readily shown that

$$x^2[f(x) + 2] = 2(1 - x)f'(x).$$

Since $f(0) = 0$, this leads to

$$f(x) + 2 = 2(1 - x)^{-1/2} \exp\left(-\frac{x^2 + 2x}{4}\right). \tag{2}$$

Yamamoto also derived this generating function directly from consideration of the geometric problem, without use of any recursion relation.

From this point on the various treatments diverged widely. Probably the neatest treatment (used by Redheffer, Ursell, and Williams), made use of the following theorem (see Titchmarsh, *Theory of Functions*, 2nd edition, p. 225):

If $f(x)=\sum_0^\infty a_n x^n$, $|x|<1$, and $g(x)=\sum_0^\infty b_n x^n$, $|x|<1$, where $\sum_0^\infty b_n$ diverges, and if $\lim_{n\to\infty}\{a_n/b_n\}=C$, then $\lim_{x\to 1}\{f(x)/g(x)\}=C$.

Applying this theorem to the generating function $f(x)$, with $g(x)=(1-x)^{-1/2}$, and using Stirling's formula to simplify the binomial coefficients b_n, we obtain

$$\lim_{x\to 1}\frac{f(x)}{(1-x)^{-1/2}}=\lim_{n\to\infty}\frac{u_n}{n}\sqrt{\pi n}$$

$$=\sqrt{\pi}\lim_{n\to\infty}\frac{u_n}{\sqrt{n}}=\sqrt{\pi b}.$$

But from (2) this limit also equals $2e^{-3/4}$, whence

$$b=4e^{-3/2}/\pi.$$

Robinson's other constant $B=\sqrt{\frac{1}{2}\pi b}$ is then $\sqrt{2}e^{-3/4}$, and

$$g_n\cong\sqrt{2}\,n^n e^{-n-3/4}.$$

The one treatment which had little resemblance to the foregoing was by Isaacs, and made ingenious use of contour integration. He set up the integral

$$u_n=\frac{e^{-3/4}}{\pi i}\int\frac{\exp(z^2-\frac{1}{4}z^4)}{z^2(1-z^2)^{n-2}}\,dz, \tag{3}$$

where the path of integration is a small circle taken *clockwise* about $z=1$. That (3) satisfies (1) is shown by substitution including an integration by parts. It is obvious that $u_1=u_2=0$; by computation of a residue, $u_3=1$. From (1) it follows that $u_4=u_5=1$. Thus an explicit formula for u_n is provided. By careful deformation of the path, in conjunction with suitable changes of variable, Isaacs showed that as $n\to\infty$ the integral in (3) becomes infinite with $2i\sqrt{n\pi}$. Hence the result $b=4e^{-3/2}/\pi$. This treatment is independent of Robinson's proof of the existence of the limit b.

Yamamoto went on to discuss other questions, verifying some of Robinson's conjectures regarding the behavior of $u_n^2/(n-i)$ for various constants i. He also pointed out that a similar recursion relation and similar asymptotic behavior occur in an apparently unrelated setting in Problem 45, page 110, of Pólya-Szegö: *Aufgaben und Lehrsätze* II, Berlin 1925.

AN ELEMENTARY NOTE ON POWERS OF QUATERNIONS*

H. E. SALZER, Computation Laboratory, National Bureau of Standards

1. A table of powers† of $z=x_0+i\rho$ can be readily used to obtain powers of quaternions of the form $X=x_0+x_1i+x_2j+x_3k$ whenever $x_1^2+x_2^2+x_3^2=\rho^2>0$, in virtue of the following result: If $z^n=a+ib$, then $X^n=a+\lambda(x_1i+x_2j+x_3k)$ where $\lambda=b/\rho$.

For example, to find $(7+8i+j+4k)^{13}$, since $8^2+1^2+4^2=9^2$, we look up $(7+9i)^{13}$ in the table of powers of z and find its real part to be 4065 65435 48992 and its imaginary part to be -3709 17313 23456. Then $(7+8i+j+4k)^{13}=4065$ 65435 48992-3709 17313 23456$\cdot(8i+j+4k)/9=4065$ 65435 48992-412 13034 80384 $(8i+j+4k)$.

2. The simplest and most direct demonstration of the above relation between z^n and X^n was noted by Dr. E. W. Cannon, as follows: Let $X=x_0+x_1i+x_2j+x_3k=x_0+\vec{x}$, where $\vec{x}=x_1i+x_2k+x_3k$ and $\rho^2=x_1^2+x_2^2+x_3^2=-\vec{x}\,\vec{x}$. By quaternion multiplication, $X^2=x_0^2-\rho^2+2x_0\vec{x}$. This suggests a possible parallel with the formation of powers of the complex number $x_0+i\rho$. In the cases $n=1$ and $n=2$ we see: (A) if $(x_0+i\rho)^n$ is designated by $a+ib$, then $X^n=a+(b/\rho)\vec{x}$. Assume the truth of (A) for some particular n. Then from the rules of quaternion multiplication,

$$X^{n+1}=\left(a+\frac{b}{\rho}\,\vec{x}\right)(x_0+\vec{x})=(ax_0-b\rho)+\frac{a\rho+bx_0}{\rho}\,\vec{x}.$$

But

$$(x_0+i\rho)^{n+1}=(a+ib)\cdot(x_0+i\rho)=(ax_0-b\rho)+(a\rho+bx_0)i.$$

This establishes relation (A) for $n+1$, and thus demonstrates it for all positive integers n.

3. The relation between z^n and X^n mentioned above can also be explained in the following way from more general considerations (noted by the writer): Every power X^n of a quaternion X is commutative with X.‡ On the other hand, all quaternions Y which commute with X are of the form $\mu+\lambda X$ where μ and λ are real numbers.§ For, let

* Preparation of this paper was sponsored in part by Office of Air Research, Air Materiél Command, USAF.

† The most extensive is the recent National Bureau of Standards, Applied Mathematics Series no. 8, Table of Powers of Complex Numbers, which gives the exact values of the first twenty-five powers of $z=x_0+i\rho$ for x_0 and ρ each ranging from 0 through 10 at unit intervals.

‡ In connection with powers of quaternions see I. Niven, The roots of a quaternion, this MONTHLY, vol. 49, 1942, pp. 386–388, and L. Brand, The roots of a quaternion, this MONTHLY, vol. 49, 1942, pp. 519–520.

§ This demonstration that $Y=\mu+\lambda X$ is a slight modification of the writer's original proof, which is due to the referee who noticed that the original proof was incomplete.

$$X = x_0 + x_1 i + x_2 j + x_3 k, \quad \text{and}$$
$$Y = y_0 + y_1 i + y_2 j + y_3 k.$$

The assumption $XY = YX$ can be expanded in terms of the x_r and the y_r, and the equating of coefficients of i, j, and k gives

$$x_2 y_3 = x_3 y_2, \qquad x_1 y_3 = x_3 y_1, \qquad x_1 y_2 = x_2 y_1.$$

Since not all of x_1, x_2, x_3 are zero, there is no loss of generality in assuming that $x_1 \neq 0$. Thus we have

$$Y = y_0 + y_1 i + y_2 j + y_3 k = \left(y_0 - \frac{y_1 x_0}{x_1}\right) + \frac{y_1}{x_1} X.$$

Conversely, it is obvious that $XY = YX$ when Y has the form $\mu + \lambda X$.

To establish the relation in the first paragraph, we know that if $X^n = a_n + \lambda_n(x_1 i + x_2 j + x_3 k)$ and $X^{n+1} = a_{n+1} + \lambda_{n+1}(x_1 i + x_2 j + x_3 k)$, then

$$\begin{cases} a_{n+1} = x_0 a_n - \rho^2 \lambda_n \\ \lambda_{n+1} = a_n + x_0 \lambda_n \end{cases} \tag{1}$$

with $a_0 = 1$, $\lambda_0 = 0$, or $a_1 = x_0$, $\lambda_1 = 1$. Relations (1) show that a_n and λ_n depend only upon x_0 and ρ, and by induction a_n and λ_n are seen to be identical functions of x_0 and ρ whether we start from the quaternion $X = x_0 + x_1 i + x_2 j + x_3 k$ or the special case of the complex number $z = x_0 + i\rho$. Thus in X^n, the a_n is the same as the corresponding a_n in z^n which is a, and the λ_n is the same as the corresponding λ_n in z^n which is b/ρ. This is the result stated in the first paragraph. Thus from the NBS *Table of Powers of Complex Numbers*, we can find the powers of all quaternions $x_0 + x_1 i + x_2 j + x_3 k$ where $x_0 = 0(1)10$ and (x_1, x_2, x_3), considered as points in three-space, lie upon any of the spheres with center at the origin, of radius 0, 1, 2, $\cdots$, 10.

4. The following alternative demonstration of the above relation between z^n and X^n which explains why it holds from even more general considerations, is due to Olga Taussky: The quaternion $X = x_0 + x_1 i + x_2 j + x_3 k$ has the well-known representation by the matrix

$$X = \begin{pmatrix} x_0 + i x_1 & x_2 + i x_3 \\ -x_2 + i x_3 & x_0 - i x_1 \end{pmatrix}$$

whose characteristic roots are $r = x_0 \pm \sqrt{-x_1^2 - x_2^2 - x_3^2}$. Since X satisfies a quadratic equation, $X^n = \mu + \lambda X$. Thus X^n, written as $a_n + b_1 i + b_2 j + b_3 k$ can be written as $\mu + \lambda x_0 + \lambda(x_1 i + x_2 j + x_3 k)$. But the characteristic roots of X^n are $r^n = a \pm ib$. Let r be denoted by $z = x_0 \pm i\rho$, $\rho^2 = x_1^2 + x_2^2 + x_3^2$. Hence $r^n = a \pm ib = \mu + \lambda x_0 \pm \sqrt{-\lambda^2(x_1^2 + x_2^2 + x_3^2)} = \mu + \lambda x_0 \pm i\lambda\rho$, or $a_n = \mu + \lambda x_0 = a$, and $\lambda = b/\rho$.

5. To generate successive powers of quaternions beyond the range of available tables of z^n, one might use (1) or a simple modification which is given below in (5) and (6), which are obtained from (1) by eliminating the a's and λ's. In-

cidentally, (5) and (6) might also be found more suitable for calculating successive powers of ordinary complex numbers, especially for the higher powers where one might wish to avoid cross-multiplications of real and imaginary parts.

From (1), we obtain

$$a_{n+1}\lambda_n - a_n\lambda_{n+1} = -\lambda_n^2\rho^2 - a_n^2, \tag{2}$$

$$a_{n+1} - x_0\lambda_{n+1} = -(\rho^2 + x_0^2)\lambda_n, \quad \text{and} \tag{3}$$

$$x_0a_{n+1} + \rho^2\lambda_{n+1} = (\rho^2 + x_0^2)a_n. \tag{4}$$

Formula (2) need be considered only as a checking relationship. From (1) and

$$a_n = x_0\lambda_n - (\rho^2 + x_0^2)\lambda_{n-1}, \tag{3'}$$

we have

$$\lambda_{n+1} = 2x_0\lambda_n - (\rho^2 + x_0^2)\lambda_{n-1}. \tag{5}$$

Also, from (1) and

$$x_0a_n + \rho^2\lambda_n = (\rho^2 + x_0^2)a_{n-1}, \tag{4'}$$

we have

$$a_{n+1} = 2x_0a_n - (\rho^2 + x_0^2)a_{n-1}. \tag{6}$$

Thus (5) and (6) lead immediately to the continued fraction

$$p_0 + \cfrac{p_1}{q_1 + \cfrac{p_2}{q_2 + \cfrac{p_3}{q_3 + \cdots}}}$$

where $p_0 = x_0$, $p_n = -(\rho^2 + x_0^2)$, $q_n = 2x_0$, for $n = 1, 2, 3, \cdots$, of which a_n/λ_n is the n^{th} convergent, considering $x_0/1$ as the first convergent.

ON CERTAIN DETERMINANTAL EQUATIONS*

EVELYN FRANK, University of Illinois

1. Introduction. In many applied problems it is necessary to know the nature of the roots of equations which are determinantal in form. It is the purpose of this paper to prove a number of known results in an extremely simple way by continued fractions with the use of the criteria of Grommer [3]and Frank [1], and to make these results more widely available. Theorems are also developed which extend known results to determinantal equations with complex elements and with fewer non-zero elements (cf. Theorems 3.2, 3.3, 4.6).

* Presented to the American Mathematical Society, September 4, 1951.

2. The equation $P(z)=0$. In this section the equation

$$P(z) = \begin{vmatrix} z & a_2 & 0 & \cdots & 0 & 0 \\ -c_1 & z & a_3 & \cdots & 0 & 0 \\ 0 & -c_2 & z & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & z & a_n \\ 0 & 0 & 0 & \cdots & -c_{n-1} & z \end{vmatrix} = 0 \tag{2.1}$$

is studied.

Recently M. J. Moore [6] proved the following theorem concerning equation (2.1) by the use of induction on a series of similar determinantal equations. It is now proved by continued fractions. The proof is carried out in detail since all subsequent theorems can be proved in a similar way by continued fractions.

THEOREM 2.1. *The equation* (2.1), *where* $a_j>0$, $c_j>0$, $j=1, 2, \cdots, n$, *has* n *pure imaginary roots if* n *is even, and one root zero and* $(n-1)$ *pure imaginary roots if* n *is odd.*

Proof. This proof is based on the expansion of a rational function into a continued fraction

$$\frac{f_1}{f_0} = \frac{1}{r_1 z + s_1} + \frac{1}{r_2 z + s_2} + \cdots + \frac{1}{r_n z + s_n},$$

where

$$f_0 = a_{00}z^n + a_{01}z^{n-1} + \cdots + a_{0n},$$
$$f_1 = a_{11}z^{n-1} + a_{12}z^{n-2} + \cdots + a_{1n},$$

are two polynomials of degrees n and $n-1$, respectively. Such an expansion is possible if and only if

$$D_j \neq 0, \qquad j = 0, 1, \cdots, n, \tag{2.2}$$

where $D_0=a_{00}$, and $D_1, D_2, \cdots, D_n$, are the first n principal minors of odd order (blocked off by lines) in the array

$$\begin{array}{c|cc|cc|c} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16}\cdots \\ a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05}\cdots \\ 0 & a_{11} & a_{12} & a_{13} & a_{14} & a_{15}\cdots \\ \hline 0 & a_{00} & a_{01} & a_{02} & a_{03} & a_{04}\cdots \\ 0 & 0 & a_{11} & a_{12} & a_{13} & a_{14}\cdots \\ \hline 0 & 0 & a_{00} & a_{01} & a_{02} & a_{03}\cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

where $a_{0j}=a_{1j}=0$ if $j>n$ (cf. [1], p. 147).

For n even, $n=2m$, equation (2.1) may be written in the form

$$P(z) = z^{2m} + \sum c_1a_2z^{2m-2} + \sum c_1a_2c_3a_4z^{2m-4} + \cdots + c_1a_2 \cdots c_{2m-1}a_{2m} = 0,$$

in which each summation represents the sum of all possible products of the type $c_1a_2c_3a_4 \cdots$ with ascending subscripts and with the c's and a's occurring alternately. Let $y=z^2$. Then

$$P(y) = y^m + \sum c_1a_2y^{m-1} + \sum c_1a_2c_3a_4y^{m-2} + \cdots + c_1a_2 \cdots c_{2m-1}a_{2m} = 0.$$

Form

$$P'(y) = my^{m-1} + (m-1)\sum c_1a_2y^{m-2} + \cdots + c_1a_2 \cdots c_{2m-3}a_{2m-2}.$$

The continued fraction expansion for $P'(y)/P(y)$ exists since the determinants D_j (2.2) are different from zero. The expansion may be obtained from the numbers in the following table by the algorithm given in [1] (table (2.4)):

$$a_{00} = 1 \qquad a_{01} = \sum c_1a_2 \qquad a_{02} = \sum c_1a_2c_3a_4 \qquad \cdots$$

$$a_{11} = m \qquad a_{12} = (m-1)\sum c_1a_2 \qquad a_{13} = (m-2)\sum c_1a_2c_3a_4 \cdots$$

$$b_{11} = \frac{\sum c_1a_2}{m} \qquad b_{12} = \frac{2\sum c_1a_2c_3a_4}{m} \qquad b_{13} = \frac{3\sum c_1a_2c_3a_4c_5a_6}{m} \qquad \cdots$$

$$a_{22} = \frac{2\sum c_1a_2c_3a_4 - \dfrac{(m-1)(\sum c_1a_2)^2}{m}}{m} \cdots$$

. . . . . .

$$r_1 = \frac{a_{00}}{a_{11}} = \frac{1}{m}, \quad s_1 = \frac{b_{11}}{a_{11}} = \frac{\sum c_1a_2}{m^2}, \quad r_j = \frac{a_{j-1,j-1}}{a_{j,j}}, \quad s_j = \frac{b_{j,j}}{a_{j,j}}, \quad j = 2, 3, \cdots, m.$$

Then

$$\frac{P'(y)}{P(y)} = \frac{1}{r_1y + s_1 + \dfrac{1}{r_2y + s_2 + \cdots + \dfrac{1}{r_my + s_m}}}, \tag{2.3}$$

where $r_1, r_3, \cdots$, are positive, and $r_2, r_4, \cdots$, are negative. Hence by the Grommer Theorem (cf. [2], Theorem 4.1, or [3]), all the roots of $P(y)=0$ are *real*. If $P(y)=0$ has multiple roots, $P'(y)$ and $P(y)$ have common factors which may first be removed by the euclidean algorithm.

The fact that all the roots of $P(y)=0$ are negative may be ascertained by a theorem of Frank (cf. [1], Theorem 4.1), if one forms

$$Q(y) = \sum c_1a_2y^{m-1} + \sum c_1a_2c_3a_4c_5a_6y^{m-3} + \cdots,$$

and then the expansion

$$\frac{Q(y)}{P(y)} = \frac{1}{d_1y + k_1 + 1} + \frac{1}{d_2y + k_2} + \cdots + \frac{1}{d_my + k_m} \tag{2.4}$$

by use of the algorithm (5.3) of [2]. It is found here that all the $d_j, j=1, 2, \cdots, m$, are *positive*, in fact, $d_1=1/\sum c_1a_2$, $d_2=(\sum c_1a_2)^2/[\sum c_1a_2\sum c_1a_2c_3a_4 - \sum c_1a_2c_3a_4c_5a_6]$, and $k_j=0$. Consequently, all the roots of $P(y)=0$ are *negative*.

Finally, since $y=z^2$, all the roots of $P(z)=0$ are pairs of conjugate pure imaginary numbers, as was to be shown.

For n odd, $n=2m+1$,

$$\begin{aligned} P(z) &= z^{2m+1} + \sum c_1a_2z^{2m-1} + \cdots + c_1a_2 \cdots c_{2m}a_{2m+1}z \\ &= z(z^{2m} + \sum c_1a_2z^{2m-2} + \cdots + c_1a_2 \cdots c_{2m}a_{2m+1}). \end{aligned}$$

Thus $P(z)=0$ has one root zero and the other roots are pairs of conjugate pure imaginary numbers, by the same argument as above.

It is remarked that *if* $a_j<0$, $c_j<0$, $j=1, 2, \cdots, n$. *Theorem* 2.1 *still holds true*, since the product c_ja_j always occurs in all of the above formulas.

If $a_j<0$ and $c_j>0$, or $a_j>0$ and $c_j<0$, $j=1, 2, \cdots, n$, the r_j in (2.3) are positive and negative according as j is odd or even, respectively. Hence all the roots of $P(y)=0$ are real by the Grommer Theorem. In the continued fraction expansion (2.4) of $Q(y)/P(y)$, the d_j are negative and by Theorem 4.1 of [1], the roots of $P(y)=0$ are positive. Consequently, the following theorem holds.

THEOREM 2.2. *If* $a_j<0$ *and* $c_j>0$, *or if* $a_j>0$ *and* $c_j<0$, $j=1, 2, \cdots, n$, *if* n *is even, the roots of equation* (2.1) *are all real and occur in pairs equal and opposite in sign; if* n *is odd, one root is zero and the remaining roots are all real and occur in pairs equal and opposite in sign.*

3. The equation $B(z)=0$. In this section the equation

$$B(z) = \begin{vmatrix} z+b_1 & a_2 & 0 & \cdots & 0 & 0 \\ c_1 & z+b_2 & a_3 & \cdots & 0 & 0 \\ 0 & c_2 & z+b_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & z+b_{n-1} & a_n \\ 0 & 0 & 0 & \cdots & c_{n-1} & z+b_n \end{vmatrix} = 0 \tag{3.1}$$

is studied.

M. J. Moore [6] proved the following theorem concerning equation (3.1) for $a_j>0$, $c_j>0$, by similar determinantal equations.

THEOREM 3.1. *The equation* (3.1), where $a_j>0$, $c_j>0$, *or* $a_j<0$, $c_j<0$, *and the* b_j *are real constants,* $j=1, 2, \cdots, n$, *has* n *real, distinct roots.*

Proof. On the removal of constant factors from certain rows and columns of the determinant (3.1) and on multiplication of certain rows and columns by i, $B(iz)$ may be written in the form

$$(3.2)\quad B(iz) = K\cdot\begin{vmatrix} d_1z+k_1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & d_2z+k_2 & -1 & \cdots & 0 & 0 \\ 0 & 1 & d_3z+k_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & d_{n-1}z+k_{n-1} & -1 \\ 0 & 0 & 0 & \cdots & 1 & d_nz+k_n \end{vmatrix},$$

where the d_j are *real and positive* and the k_j are *pure imaginary or zero*. Here $B(iz)/K'$ ($K'=K\cdot\prod_{j=1}^{n}d_j$) is a polynomial of the form (1.2) of [1], *i.e.*,

$$P(z)/K' = z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n,\ a_j=p_j+iq_j,\ j=1,\ 2,\ \cdots,\ n.$$

If one forms

$$Q(z)/K' = p_1z^{n-1}+iq_2z^{n-2}+p_3z^{n-3}+iq_4z^{n-4}+\cdots \qquad \text{(cf. (1.3) of [1])}$$

in order to set up the expansion (2.4), one finds $Q(z)=0$ since $B(iz)$ is of the form $H_n(z)$ (cf. (4.3) of [1]) and the d_j are real and positive. But then all the roots of $P(z)/K'=0$ lie on the imaginary axis and are distinct ([1], pp. 152–153). Consequently, all the roots of $B(iz)=0$ lie on the imaginary axis and the roots of $B(z)=0$ therefore lie on the real axis.

It may be remarked that Theorem 2.2 is a corollary of Theorem 3.1, where $b_j=0$ and $k_j=0$. It does not, however, include the statement concerning the distribution of the roots on the real axis.

Theorem 2.1 may also be proved in the same way. Here $b_j=0$ and (2.1) may be written in the form $B(z)=0$ with $k_j=0$, so that all roots of (2.1) are pure imaginary or zero.

It may also be remarked that if $a_j<0$, $c_j>0$, or $a_j>0$, $c_j<0$, the determinant (3.1) cannot be transformed into a determinant of the form (3.2), where the c_j are real and positive. Consequently, if $a_j<0$, $c_j>0$, or $a_j>0$, $c_j<0$, equation (3.1) will in general have complex roots.

The following theorem is a direct consequence of Theorem 4.1 of [1].

THEOREM 3.2. *The equation*

$$(3.3)\quad \begin{vmatrix} d_1z+k_1+1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & d_2z+k_2 & -1 & \cdots & 0 & 0 \\ 0 & 1 & d_3z+k_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & d_{n-1}z+k_{n-1} & -1 \\ 0 & 0 & 0 & \cdots & 1 & d_nz+k_n \end{vmatrix} = 0,$$

where the d_j are real constants and the k_j are pure imaginary or zero, $j=1,\ 2,\ \cdots,\ n$, has k roots with positive real parts and $(n-k)$ roots with negative real parts if k

of the d_j are negative and the remaining $(n-k)$ are positive, provided the determinants D_j (2.2) are different from zero.

There is no loss of generality when one considers the number of roots of equation (3.3) in $R(z)<0$ and $R(z)>0$, since any given half-plane may be transformed into $R(z)<0$ by a linear transformation. Thus, one may formulate this theorem as follows.

THEOREM 3.3. *The equation*

$$\begin{vmatrix} d_1a(z+m)+k_1+1 & -1 & 0 & \cdots & 0 & 0 \\ 1 & d_2a(z+m)+k_2 & -1 & \cdots & 0 & 0 \\ 0 & 1 & d_3a(z+m)+k_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & d_{n-1}a(z+m)+k_{n-1} & -1 \\ 0 & 0 & 0 & \cdots & 1 & d_na(z+m)+k_n \end{vmatrix} = 0,$$

where $|a|=1$, $m>0$, the d_j are real constants, and the k_j are pure imaginary, $j=1, 2, \cdots, n$, has k roots in the open half-plane $\bar{a}z+a\bar{z}+2m>0$, and $(n-k)$ roots in the open half-plane $\bar{a}z+a\bar{z}+2m<0$ if k of the d_j are negative and $(n-k)$ of the d_j are positive, provided the determinants D_j (2.2) are different from zero, i.e., provided the expansion (2.4) exists.

An upper bound for the moduli of the roots of equation (3.3) is the maximum modulus of all points on all the Cassinian ovals

$$\begin{aligned} |(d_1z+k_1+1)(d_2z+k_2)| &= \frac{1}{g_1}, \\ |(d_2z+k_2)(d_3z+k_3)| &= \frac{1}{(1-g_1)g_2}, \cdots, \\ |(d_{n-1}z+k_{n-1})(d_nz+k_n)| &= \frac{1}{(1-g_{n-2})g_{n-1}}, \end{aligned} \tag{3.4}$$

where $0<g_j<1$, $j=1, 2, \cdots, n-1$, (cf. [2], p. 897). The point with largest modulus on each oval is on the line joining its two fixed points.

For example, the determinantal equation

$$P(z) = \begin{vmatrix} 2.5z+1-1.25i & -1 & 0 \\ 1 & .059z+.028i & -1 \\ 0 & 1 & -56.874z+55.471i \end{vmatrix} = 0 \tag{3.5}$$

has two roots with negative real parts and one root with positive real part, by Theorem 3.2.

The Cassinian ovals (3.4) for $g_1=56/59$, $g_2=160/168$, are

$$C_1: \quad |(z + .4 - .5i)(z + .5i)| = 50/7,$$
$$C_2: \quad |(z + .5i)(-z + i)| = 25/4.$$

The points of maximum modulus of C_1 and C_2 are $(-1.213, 2.531)$ and $(0, 2.86)$, respectively. Therefore, 2.866 is the maximum modulus of all points of C_1 and C_2, and a very close upper bound to the largest modulus of all the zeros, namely, 2.59.

To obtain a lower bound for the moduli of the roots of (3.5), consider $P'(z) = z^3P(1/z)$. Then the Cassinian ovals (3.4) for $g_1 = .85/1.75$, $g_2 = 1.75/2.04$ are

$$C_3: \quad |(z + .04 - .46i)(-z - 1.7i)| = 1/.85,$$
$$C_4: \quad |(-z - 1.7i)(z - .3i)| = 10/9.$$

The maximum modulus of all points of C_3 and C_4 is 2.15. Hence a lower bound for the moduli of the roots of equation (3.5) is $1/2.15 = .47$.

Computation of the roots of (3.5) by the methods of [2] gives the approximate values as $-.2548+2.5780i$, $-.1684-2.5754i$, $.0232+.9974i$.

4. Characteristic equations. By the Grommer Theorem the roots of equation (3.1) are not necessarily real if more elements are non-zero, for example,

$$\begin{vmatrix} z & a_2 & b_2 \\ c_1 & z & a_3 \\ 0 & c_2 & z \end{vmatrix} = 0, \qquad a_j > 0,\ c_j > 0,\ j = 1, 2, 3.$$

Similarly, the roots of equation (2.1) are not necessarily pure imaginary or zero if more elements are non-zero. However, the following special case is in a certain sense an extension of Theorem 2.1 to non-zero elements, but it does limit the value of the elements. This equation is considered, for instance, by Kowalewski ([5], p. 139), who gives the theorem that $D(z) = 0$ can have no real roots different from zero (with no conditions on the a_{jj}).

THEOREM 4.1. *If, in the equation*

$$D(z) = \begin{vmatrix} a_{11} + z & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} + z & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} + z \end{vmatrix} = 0, \tag{4.1}$$

$a_{jk} = -a_{kj}$, $a_{jj} = 0$, and the a_{jk} are real constants, $j, k = 1, 2, \cdots, n$, all the roots of $D(z) = 0$ are pairs of conjugate pure imaginary numbers if n is even, and one root is zero and $(n-1)$ roots are pairs of conjugate pure imaginary numbers if n is odd.

The equation $D(-z) = 0$ (cf. (4.1)) is the *characteristic equation of the matrix* (a_{jk}), $j, k = 1, 2, \cdots, n$, and the negative roots of (4.1) are the *characteristic*

roots of the matrix.

Theorem 4.1 may be proved by the theorems of Grommer and Frank in exactly the same way as Theorem 2.1, since

$$D(z) = z^n + S_2 z^{n-2} + S_4 z^{n-4} + \cdots = 0,$$

where the S_j, $j=2, 4, \cdots$, are the sums of all j-rowed principal minors of the determinant $\|a_{jk}\|$, $j, k=1, 2, \cdots, n$, and are all positive by the properties of such determinants.

In a similar way, Theorem 4.2 may be shown.

THEOREM 4.2. *The equation* $D(z)=0$ (4.1) *has all roots real if the constants* a_{jk} *are real and* $a_{jk}=a_{kj}$, $j, k=1, 2, \cdots, n$.

The following theorems are somewhat more general. The first can be shown directly from the Grommer Theorem. Theorem 4.4 can be proved by an application of the Grommer Theorem to $D(iz)=0$, for, if the roots of $D(iz)=0$ are real, the roots of $D(z)=0$ are pure imaginary or zero.

THEOREM 4.3. *The equation* $D(z)=0$ (4.1) *has all roots real if the constants* a_{jk}, $j, k=1, 2, \cdots, n$, *are real, and if for every pair* j, k *of the* $(n-1)$ *indices* $2, 3, \cdots, n$,

$$a_{1j}a_{jk}a_{k1} = a_{j1}a_{kj}a_{1k}$$

and $a_{1j}a_{j1}$ *is positive* [7].

THEOREM 4.4. *The equation* $D(z)=0$ (4.1) *has all roots pure imaginary or zero if the constants* a_{jk}, $j, k=1, 2, \cdots, n$, *are real,* $a_{jj}=0$, *and if for* $j, k, m=1, 2, \cdots, n$,

$$a_{jk}a_{km}a_{mj} = -a_{kj}a_{mk}a_{jm}$$

and $a_{jk}a_{kj}$ *is negative* [7].

Under the conditions of Theorem 4.4, if $a_{jj}\neq 0$, $j=1, 2, \cdots, n$, the roots of (4.1) will in general be complex.

The following well-known theorem is a further extension of the above, and is easily shown by the continued fraction expansion for $D'(z)/D(z)$ from equation (4.1) and the Grommer Theorem. It has been proved in many other ways (cf., for example, Perron [10], p. 19). Bounds for the roots are also well known ([10], pp. 36 ff.).

THEOREM 4.5. *If* a_{jk} *and* a_{kj} *are conjugate complex constants and the* a_{jj} *are real constants,* $j, k=1, 2, \cdots, n$, *all the roots of equation* (4.1) *are real.*

If one forms the continued fraction expansion for $D'(iz)/D(iz)$ from (4.1), where a_{jk} and $-a_{kj}$ are conjugate complex constants and the a_{jj} are pure imaginary or zero, $j, k=1, 2, \cdots, n$, by the Grommer Theorem it is found that $D(iz)=0$ has all roots real and hence $D(z)=0$ all roots pure imaginary or zero.

Consequently the following theorem holds.

THEOREM 4.6. *The equation* (4.1) *has all roots pure imaginary or zero if* a_{jk} *and* $-a_{kj}$ *are conjugate complex constants and the* a_{jj} *are pure imaginary or zero,* $j, k = 1, 2, \cdots, n$.

Theorem 4.7 (cf. [10], p. 20) is a generalization of the above, and can be proved in the same manner by the continued fraction expansion for $R'(z)/R(z)$ and the Grommer Theorem.

THEOREM 4.7. *The equation*

$$R(z) = \begin{vmatrix} a_{11} + b_{11}z & a_{12} + b_{12}z \cdots a_{1n} + b_{1n}z \\ a_{21} + b_{21}z & a_{22} + b_{22}z \cdots a_{2n} + b_{2n}z \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1} + b_{n1}z & a_{n2} + b_{n2}z \cdots a_{nn} + b_{nn}z \end{vmatrix} = 0 \tag{4.2}$$

has real roots if $\sum_{\lambda=1}^{n}\sum_{\mu=1}^{n} a_{\lambda\mu}z_\lambda w_\mu$, $\sum_{\lambda=1}^{n}\sum_{\mu=1}^{n} b_{\lambda\mu}z_\lambda w_\mu$ *are Hermitian forms* (*i.e.*, $a_{\lambda\mu} = \bar{a}_{\mu\lambda}$, $b_{\lambda\mu} = \bar{b}_{\mu\lambda}$) *and the latter form is positive* (*or negative*) *definite.*

Still further generalizations of this theorem have been developed by M. Parodi for determinantal equations of order n with elements made up of polynomials of degree m [9].

The following theorem can be shown with the use of a theorem on determinants (cf., for instance, Hadamard [4] or Taussky [11]), namely, that the determinant with complex elements $\|a_{jk}\| \neq 0$, $j, k = 1, 2, \cdots, n$, if the elements are such that $|a_{jj}| > A_j$, where A_j denotes the sum of the moduli of the non-diagonal terms of the j-th row of the determinant $\|a_{jk}\|$. However, it can also be readily proved by the above methods, *i.e.*, the continued fraction expansion (2.4) and Theorem 4.1 of [1]. The latter, of course, depends on an expansion of the determinant first. There are a number of techniques for the development of such expansions (cf. Wayland [12]).

THEOREM 4.8. *If the* a_{jk}, $j, k = 1, 2, \cdots, n$, *are complex constants, a sufficient condition that equation* (4.1) *have only roots with negative real parts is that*

$$R(a_{jj}) > \sum_{k \neq j} |a_{jk}|.$$

A sufficient condition that equation (4.1) *have only roots in the lower half of the complex plane is that*

$$I(a_{jj}) > \sum_{k \neq j} |a_{jk}|.$$

The roots lie interior to the circles

$$|a_{jj} + z| = \sum_{k \neq j} |a_{jk}|.$$

Bibliography

1. E. Frank, On the zeros of polynomials with complex coefficients, Bull. Amer. Math. Soc., vol. 52, 1946, pp. 144–158.
2. E. Frank, The location of the zeros of polynomials with complex coefficients, Bull. Amer. Math. Soc., vol. 52, 1946, pp. 890–898.
3. J. Grommer, Ganze transzendente Funktionen mit lauter reellen Nullstellen, J. Reine Angew. Math., vol. 144, 1914, pp. 114–165.
4. J. Hadamard, Leçons sur la Propagation des Ondes, Hermann, Paris, 1903, pp. 13–14.
5. G. Kowalewski, Einführung in die Determinantentheorie, 3d ed., Chelsea, New York, 1948.
6. M. J. Moore, A simple test for the reality and sign of the roots of two determinantal equations of high degree, Math. Gaz., London, vol. 34, 1950, pp. 94–98.
7. T. Muir and W. H. Metzler, A Treatise on the Theory of Determinants, Longmans, Green & Co., New York, 1933.
8. M. Parodi, Sur une application d'un théorème de M. J. Hadamard, Bull. Sci. Math. (2), vol. 72, 1948, pp. 136–138.
9. M. Parodi, Sur quelques proprietés d'une équation algébrique, Bull. Sci. Math. (2), vol. 74, 1950, pp. 121–130.
10. O. Perron, Algebra, 3d ed., vol. 2, Gruyter, Berlin, 1951.
11. O. Taussky, A recurring theorem on determinants, this MONTHLY, vol. 56, 1949, pp. 672–676.
12. H. Wayland, Expansion of determinantal equations into polynomial form, Quart. App. Math., vol. 2, 1945, pp. 277–306.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tennessee.

A NOTE ON THE AUTOMORPHISM GROUP OF A FINITE GROUP

I. N. HERSTEIN, Cowles Commission and University of Chicago, and
J. E. ADNEY, Ohio State University

In this note a theorem is proved about the relationship of the order of a finite group to the order of its automorphism group. The role of this theorem is to exhibit certain prime divisors of the order of the automorphism group. Although the result is natural and easy to prove, we have not been able to locate it anywhere.

Let G be a finite group of order $o(G)$, and let $A(G)$ be the automorphism group of G of order $o[A(G)]$. The theorem we prove is:

THEOREM. *If p is a prime number, and $p^2 \mid o(G)$, then $p \mid o[A(G)]$.*

Since our proof involves a reduction to the case in which G is an Abelian group of prime power order, we first prove the

LEMMA. *If G is an Abelian group of order p^m, $m \geqq 2$, then $p \mid o[A(G)]$.*

Proof. If G is cyclic, $o[A(G)]=(p-1)p^{m-1}$ [1, p. 116], and since $m-1\geqq 1$, $p \mid o[A(G)]$. If, on the other hand, G is of type $(p, p, \cdots, p)$ the lemma is also true since $o[A(G)]=(p^m-1)(p^m-p)(p^m-p^2)\cdots(p^m-p^{m-1})$ [1, p. 112]. In any of the other possibilities for G, G is the direct product of cyclic subgroups $S_1, S_2, \cdots, S_t$ of orders $p^{m_1}, p^{m_2}, \cdots, p^{m_t}$ respectively, where, without loss of generality, we may assume that $t>1$ and that $m_1\geqq 2$. Since S_1 is cyclic of order p^{m_1}, $m_1\geqq 2$, it has an automorphism ϕ of order p. Every $a\in G$ can be written uniquely as $a=s_1s_2\cdots s_t$ where each $s_i\in S_i$. We define ϕ^* by $\phi^*(a)=\phi(s_1)s_2\cdots s_t$. Clearly ϕ^* is an automorphism of G. Moreover ϕ^* is of order p since ϕ is of order p. Thus $p \mid o[A(G)]$.

Now that the lemma has been established we are in a position to prove the aforementioned theorem. Suppose that $p^2 \mid o(G)$, and $p \nmid o[A(G)]$. Since $A(G)$ contains as a subgroup the group $I(G)$ of inner automorphisms, and since $I(G)$ is isomorphic to the group G/C, where C is the center of G, then the fact that $p \nmid o[A(G)]$ implies that $p \nmid o(G/C)$; thus S_p, the Sylow subgroup associated with the prime p, must be contained in the center. Hence S_p is a normal subgroup of G. Moreover, since $S_p\subset C$, S_p is in the center of its normalizer, so by a Theorem of Burnside [1, theorem 4, p. 139], G contains a normal subgroup N whose index in G is the order of S_p. It is then clear that $S_p\cap N=1$ and that $S_pN=G$. That is, G is the direct product of S_p and N. Since $p^2 \mid o(G)$, S_p is of order p^m with m at least 2. Since $S_p\subset C$, it is Abelian, so by our lemma, S_p possesses an automorphism ϕ of order p. Now every $a\in G$ can be represented uniquely as $a=sn$ where $s\in S_p$, $n\in N$ and where $sn=ns$. We define ϕ^* for G by $\phi^*(a)=\phi(s)n$. Clearly ϕ^* is an automorphism of G, and since ϕ is of order p, ϕ^* is also of order p. ϕ^* being in $A(G)$ then gives us that $p \mid o[A(G)]$, contrary to our assumption that $p \nmid o[A(G)]$. Thus the theorem is proved.

Bibliography

1. Zassenhaus, Hans: Theory of Groups (English Translation), New York, 1949.

DEDUCTION OF CARDANO'S FORMULA BY CONFORMAL MAPPING

C. F. MOPPERT, Basle, Switzerland

An algebraic function $w=a_0+a_1z+\cdots+a_nz^n$, $a_n\neq 0$, maps the simple z-plane onto an n-sheeted Riemann surface. Inversely, if we must find the values of z corresponding to a given w, the w-surface must be mapped onto the simple z-plane. The problem of solving an algebraic equation of this kind appears therefore related to a problem of conformal mapping.

We will illustrate this principle by solving the equation of third degree

$$z^3 - 3z - 2w = 0.$$

We shall get Cardano's formula in a slightly different form, that is to say

$$z = \sqrt[3]{w + \sqrt{w^2 - 1}} + \frac{1}{\sqrt[3]{w + \sqrt{w^2 - 1}}}. \tag{1}$$

It is unimportant which value is given to the square root, but the three values of the cube root will furnish the three values of z. Naturally, if we take a certain value of the square or the cube root in the first summand, the same value must be chosen in the second one. By evaluating the expression z^3-3z, we can verify at once that (1) is a solution.

Since any equation of third degree with simple roots can be transformed by a linear transformation into the equation $z^3-3z-2w=0$, our formula gives the solution for any equation of this kind.

The function $w=\frac{1}{2}(z^3-3z)$ maps the simple z-plane onto a three-sheeted Riemann surface. The equation $w'=\frac{3}{2}(z^2-1)=0$ has the simple roots $z=1$ and $z=-1$. Therefore the points $w(1)=-1$ and $w(-1)=1$ are simple branch-points of the surface. There are no other finite branch-points, but at $w=\infty$ there is a double branch-point. The only value of z for which $w(z)=\infty$ is $z=\infty$.

If we succeed in mapping the w-surface onto the z-plane in such a way that the simple points $w=\pm 1$ are transformed into the double points $z=\mp 1$, the simple point $w=\infty$ into the triple point $z=\infty$, and all other simple w-points conformally into simple z-points, then we will have solved the equation $z^3-3z-2w=0$.

But we will not begin with the mapping of the w-surface mentioned above, this surface not being simple enough. It is easier first to map a six-sheeted surface of the variable, say v, which is regularly branched,* onto the simple z-plane. At a later stage in the argument we will return to the three-sheeted w-surface. The surface of the variable v shall have the following properties: (a) it is six-sheeted and simply connected, (b) its only branch-points are at the points $v=-1$, $v=1$ and $v=\infty$, (c) at each of the points $v=-1$ and $v=1$ are three simple branch-points and at $v=\infty$ are two double branch-points. This surface will therefore be regularly branched.

Cutting the surface of the variable v along the real axis, we get 12 half-planes which we consider as curvilinear triangles with the vertices $v=-1$, $v=1$ and $v=\infty$ and with the angles π, π and π. Now we have only to map one of these triangles onto a certain curvilinear triangle with the angles $\pi/2$, $\pi/2$ and $\pi/3$ respectively, and then by reflection the whole v-surface will be mapped onto a simple v_5-plane. We shall map a lower half-plane of the v-surface onto the following triangle of the v_5-plane: two of its sides are parts of the positive real and positive imaginary axis, meeting at $v_5=0$. This is therefore one of the vertices of the triangle with an angle of $\pi/2$. We close this triangle by an arc of the circle

* A Riemann surface is called regularly branched, if for each fixed base point, every point of the surface lying over this base point is a branch-point of the same order; cf. Nevanlinna, R., Eindeutige analytische Funktionen, Grundlehren, vol. XLVI, Berlin, Springer 1936/Edwards, Ann Arbor, 1944, pp. 270–300. The present note was suggested by material in this book.

which has its center at $v_5 = -i/\sqrt{3}$ and passes through $v_5 = 1$. This triangle, which lies entirely in the first quadrant, has its two other vertices at $v_5 = i/\sqrt{3}$ and $v_5 = 1$ and the corresponding angles are $\pi/2$ and $\pi/3$ (Fig. 1).

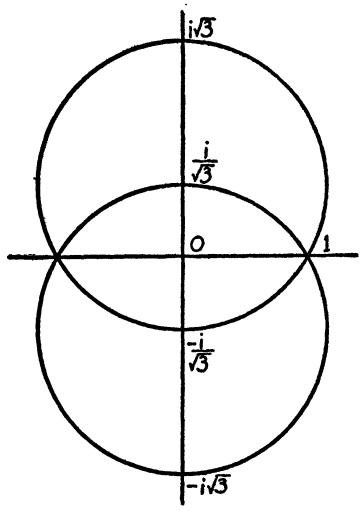


FIG. 1

It is easy to see that the transformations

$$v_1 = \frac{1 - v}{1 + v}, \qquad v_2 = \sqrt{v_1}, \qquad v_3 = \frac{1 + iv_2}{1 - iv_2},$$

$$v_4 = -\epsilon\sqrt[3]{v_3}, \qquad v_5 = \frac{\epsilon - v_4}{\epsilon + v_4}, \quad \text{where} \quad \epsilon = \frac{1}{2} + \frac{i}{2}\sqrt{3},$$

carried out one after the other, finally map the half-plane $\Im(v) < 0$ onto the triangle in question. Here it is presumed that we take $\sqrt{-1} = +i$ and $\sqrt[3]{-1} = -1$.

Now we put $v_6 = v_5^2$. By this transformation, each pair of the v_5-triangles, symmetrical with respect to the origin, goes into a single v_6-triangle. The triangle in the v_5-plane which we discussed before, is now transformed into the following one: the segments of the real axis $-\frac{1}{3} \leqq v_6 \leqq 0$ and $0 \leqq v_6 \leqq 1$ are two of its sides, the origin is one of the vertices with π as the corresponding angle. The third side is formed by an analytic curve, cutting the real axis in $-\frac{1}{3}$ and 1 at angles of $\pi/2$ and $2\pi/3$ respectively and lying in the upper half-plane (Fig. 2).

Between the v-surface and the v_5-plane there existed a 1-1 correspondence; more precisely, there existed a 1-1 correspondence between every one of the triangles in the v_5-plane (Fig. 1) and one of the half-planes of the v-surface.

In general there lie exactly six points over each base point of the v-surface, which map in a 1-1 manner onto the points of the v_5 plane. These six points can be grouped into three pairs in such a way that for each pair the corresponding

value of v_6 is the same. Thus there is a 1-1 mapping of the v_6-plane onto the Riemann surface which is obtained by identifying in the primitive v-surface the two half-planes which map into the same v_6 triangle.

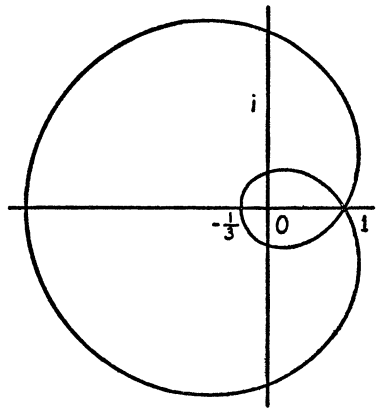


FIG. 2

After this identification is performed, the v-surface will have only three sheets. As both of the triple points $v_5=-1$ and $v_5=1$ are now transformed into the point $v_6=1$, this three-sheeted surface will have only one double branch-point, at $v=\infty$. Before that we had three image-points of $v=-1$ in the v_5-plane, *i.e.* the points $v_5=0$ and $v_5=\pm i\sqrt{3}$, each of which was to be counted as a double point. By $v_6=v_5^2$, the points $v_5=-i\sqrt{3}$ and $v_5=i\sqrt{3}$ go into the same point $v_6=-3$, which must also be counted as a double point, whereas the point $v_5=0$ remains at $v_6=0$, but changes from a double point into a simple point. Therefore, the new three-sheeted surface will have a simple branch-point and a simple sheet at the point $v=-1$. We can make the same observations about the image-points of $v=+1$, namely, there is a double point at $v_6=-\frac{1}{3}$ and a simple point at $v_6=\infty$.

The three-sheeted surface has therefore the following branch-points: a double one at ∞, and at $+1$ as well as at -1 there is a simple branch-point and a simple sheet. This surface is therefore identical with the surface of the variable w mentioned at the beginning.

Now we have only to use a linear transformation to send the point $v_6=1$ into $v_7=\infty$, the point $v_6=-\frac{1}{3}$ into $v_7=-1$, and the point $v_6=-3$ into $v_7=1$. This is done by the transformation $v_7=2(v_6+1)/(v_6-1)$. The three-sheeted w-surface is now mapped onto a simple plane, the image-point of its double branch-point is at $v_7=\infty$, and the image-points of its simple branch-points are at $v_7=1$ and $v_7=-1$. Therefore we can put $v_7=z$.

The transformations $v_1(v)$, $v_2(v_1)$, $v_3(v_2)$ and $v_4(v_3)$ yield $v_4=-\epsilon\sqrt[3]{v+\sqrt{v^2-1}}$, and the transformations $v_5(v_4)$, $v_6(v_5)$ and $v_7(v_6)$ yield $v_7=\epsilon^2v_4+(1/\epsilon^2v_4)$. So we have the final formula

$$z = \sqrt[3]{w + \sqrt{w^2 - 1}} + \frac{1}{\sqrt[3]{w + \sqrt{w^2 - 1}}}$$

as the solution of the equation

$$z^3 - 3z - 2w = 0.$$

A CERTAIN CUBIC TRANSFORMATION

ELSIE CHURCH, University of Kentucky

Take a fixed point $A_3(0, 0, 1)$ as origin, and take as the fixed fundamental base curve a cubic of the syzygetic pencil, $F = x_1^3 + x_2^3 + x_3^3 - 3\lambda x_1 x_2 x_3 = 0$. Points collinear with the origin and conjugate with respect to the base are said to be inverse. A line through the origin and a generic point $P'(x_1', x_2', x_3')$ will intersect the polar line of P' with respect to the cubic in a point $P(x_1, x_2, x_3)$. P and P' are conjugate points.

The only case considered here is the one in which A_3P' and the polar line of P' with respect to the base cubic F are distinct.

The polar line of P' with respect to F is also the polar line of the first polar of P' with respect to F, that is, the polar line of the polar conic of P' with respect to F. Thus to determine the cubic inverse of a point P', determine the polar conic of P', and use the ordinary construction for pole and polar with respect to the conic.

When the point $P(x_1, x_2, x_3)$, the intersection of A_3P' and the polar line of P', is found, a relation is established between x_1', x_2', x_3' and x_1, x_2, x_3 by the equations $x_1 : x_2 : x_3 = f_1 : f_2 : f_3$ where $f_1 = \lambda x_1'^2 x_2' - x_1' x_3'^2$, $f_2 = \lambda x_1' x_2'^2 - x_2' x_3'^2$, $f_3 = x_1'^3 + x_2'^3 - 2\lambda x_1' x_2' x_3'$, and thus there is determined an interesting cubic transformation. Since f_1, f_2, f_3 are rational cubic functions of x_1', x_2', x_3', the transformation is rational. Under this transformation, to a given point $P'(x_1', x_2', x_3')$ corresponds one point $P(x_1, x_2, x_3)$, but to a given point P will correspond not one point P' but a finite number, $n > 1$. The points P' corresponding to a given point P are determined by the intersections of the three curves whose equations are

$$\frac{f_1}{x_1} = \frac{f_2}{x_2} = \frac{f_3}{x_3}$$

or

(a) $x_2 f_1 - x_1 f_2 = 0$; (b) $x_3 f_1 - x_1 f_3 = 0$; (c) $x_3 f_2 - x_2 f_3 = 0$.

These curves have nine points of intersection since equation (a) is a linear combination of (b) and (c). Thus if $f_1 = 0, f_2 = 0, f_3 = 0$ have no common point of intersection, to a given point P will correspond nine points P', but if $f_1 = 0$, $f_2 = 0, f_3 = 0$ do have one common point of intersection, then since the position of this point is independent of the position of P, it may be disregarded, and to a point P will correspond only eight points P', *etc.* It can be proved that $f_1 = 0$,

$f_2=0, f_3=0$ intersect in the following seven points:

(1) $(0, 0, 1)$

(2) $[\lambda, (\lambda^3 + \lambda\sqrt{\lambda^4 - \lambda})^{2/3}, \lambda(\lambda^3 + \lambda\sqrt{\lambda^4 - \lambda})^{1/3}]$

(3) $[\lambda, (\lambda^3 - \lambda\sqrt{\lambda^4 - \lambda})^{2/3}, \lambda(\lambda^3 - \lambda\sqrt{\lambda^4 - \lambda})^{1/3}]$

(4) $[\lambda, (\lambda^3 + \lambda\sqrt{\lambda^4 - \lambda})^{2/3}\omega^2, \lambda(\lambda^3 + \lambda\sqrt{\lambda^4 - \lambda})^{1/3}\omega]$

(5) $[\lambda, (\lambda^3 + \lambda\sqrt{\lambda^4 - \lambda})^{2/3}\omega, \lambda(\lambda^3 + \lambda\sqrt{\lambda^4 - \lambda})^{1/3}\omega^2]$

(6) $[\lambda, (\lambda^3 - \lambda\sqrt{\lambda^4 - \lambda})^{2/3}\omega^2, \lambda(\lambda^3 - \lambda\sqrt{\lambda^4 - \lambda})^{1/3}\omega]$

(7) $[\lambda, (\lambda^3 - \lambda\sqrt{\lambda^4 - \lambda})^{2/3}\omega, \lambda(\lambda^3 - \lambda\sqrt{\lambda^4 - \lambda})^{1/3}\omega^2]$

where ω and ω^2 are cube roots of unity. Thus to each given point P will correspond two points P', and this cubic transformation sets up a (1, 2) correspondence.

The equations of the transformation which determines the two points P' corresponding to a given point P can be obtained by using equations (a), (b), (c). The equations of the transformation are:

$$x_1' : x_2' : x_3' = x_1(\lambda x_1x_2 \pm \sqrt{\lambda^2x_1^2x_2^2 - x_3(x_1^3 + x_2^3 - \lambda x_1x_2x_3)})$$
$$: x_2(\lambda x_1x_2 \pm \sqrt{\lambda^2x_1^2x_2^2 - x_3(x_1^3 + x_2^3 - \lambda x_1x_2x_3)}) : x_1^3 + x_2^3 - \lambda x_1x_2x_3.$$

Some very interesting results are obtained when λ is assigned any one of the cube roots of unity as its value. Under these conditions consider again the seven points (1), (2), (3), (4), (5), (6), (7). In all three cases (1) remains the same, (2) coincides with (3), (4) with (6), and (5) with (7). These coincidences determine the following nine points:

		Points (2), (3)	Points (4), (6)	Points (5), (7)
(S):	$\lambda = \omega$,	I$(1, \omega^2, 1)$,	IV$(1, \omega, \omega)$,	VII$(1, 1, \omega^2)$,
(R):	$\lambda = 1$,	II$(1, 1, 1)$,	V$(1, \omega^2, \omega)$,	VIII$(1, \omega, \omega^2)$,
(T):	$\lambda = \omega^2$,	III$(1, \omega, 1)$,	VI$(1, 1, \omega)$,	IX$(1, \omega^2, \omega^2)$.

These nine points lie by threes on each of nine lines. The points that lie in the same column are collinear, and if a point is chosen from each row and each column, those three points are collinear. The nine lines pass by threes through the vertices of the fundamental triangle.

In the original syzygetic pencil of cubics, the nine common points of intersection of all the cubics are their inflexion points. These nine inflexion points lie by threes on twelve lines—the Maclaurin lines. The Maclaurin lines intersect in twelve other points called the "critic centers" of the cubic F. The nine points determined by the coincidences when $\lambda=1$, ω, or ω^2, and the point (1) are ten of these critic centers.

A GENERAL RULE FOR DIVISIBILITY BASED ON THE DECIMAL EXPANSION OF THE RECIPROCAL OF THE DIVISOR

J. M. ELKIN, Chicago, Illinois

1. The following rule provides a general test for determining the divisibility of an integer $N=\sum_{i=0}^{k} a_i 10^i$ by any integer n:

1. Find p and q, where p is the number of terms preceding the period in the decimal expansion of $1/n$ and q is the number of terms in the period. They can be found directly or by applying the theorem which states that, if $n=n_0 2^\alpha 5^\beta$ where n_0 and 10 are relatively prime, then $p=\max(\alpha, \beta)$ and q is the exponent to which 10 belongs modulo n_0. (The number of digits in N must exceed $p+q$ for the rule to have any significance.)

2. Mark off the digits of N in groups of q beginning with the $(p-1)$th digit from the right. (Up to $q-1$ zeros may have to be placed before N to fill out the last group; this gives $[(k-p)/q]+1$ groups.)

3. Find the sum, S, of the q-digit numbers formed in 2.

4. Substitute S for the digits of N preceding the last p digits, obtaining the smaller number N'.

The rule states that if N' is divisible by n, so is N.

Proof. Since

$$N = \sum_{i=0}^{p-1} a_i 10^i + \sum_{j=0}^{[(k-p)/q]} \sum_{i=0}^{q-1} a_{p+jq+i} 10^{p+jq+i}$$

and

$$N' = \sum_{i=0}^{p-1} a_i 10^i + \sum_{j=0}^{[(k-p)/q]} \sum_{i=0}^{q-1} a_{p+jq+i} 10^{p+i},$$

$N-N'$ consists of a series of terms of the form

$$a_{p+jq+i}(10^{p+jq+i} - 10^{p+i}).$$

Since, in the decimal expansion of $1/n$, the remainder after $p+jq+i$ terms and after $p+i$ terms are equal,

$$10^{p+jq+i} \equiv 10^{p+i} \pmod{n}$$

and, therefore,

$$N \equiv N' \pmod{n}.$$

2. A simpler rule can be stated if a remainder of $n-1$ appears for the first time after r terms have been found in the decimal expansion of $1/n$. When this happens, $p=0$ and $q=2r$. Alternatively, if n is prime, and if $1/n$, which is purely periodic, has a period $2r$, the remainder after r terms is $n-1$. In such cases, the rule is to mark off the digits of N in groups of r beginning with the last digit, and to obtain N'' as the difference between the sums of the odd and even

groups, respectively, of the r-digit numbers. (The test has meaning only if the number of digits of N exceeds r). If N'' is divisible by n, so is N.

Proof. The truth of the first statement is seen from the fact that, if the remainder after r terms is $n-1$, the remainder after $2r$ terms is found by dividing $(n-1)10^r$ by n. But, since

$$10^r \equiv n - 1 \ (\mathrm{mod}\ n),$$
$$(n-1)10^r \equiv (n-1)^2 \equiv 1 \ (\mathrm{mod}\ n).$$

A remainder of 1 implies that $p=0$. Also, since a remainder of 1 could not have occurred before the $2r$-th term, $q=2r$. The truth of the second statement can be shown as follows: let the remainder after r terms be l, where $1<l<n$. Then

$$10^r \equiv l \ (\mathrm{mod}\ n)$$

and

$$l10^r \equiv l^2 \ (\mathrm{mod}\ n).$$

But

$$l10^r \equiv 1 \ (\mathrm{mod}\ n),$$

so that

$$l^2 \equiv 1 \ (\mathrm{mod}\ n).$$

Since n is prime and $l=1$ is excluded as a solution, this has only one solution $n-1$.

The rule can now be proved by writing:

$$N = \sum_{j=0}^{[k/r]} \sum_{i=0}^{r-1} a_{jr+i}10^{jr+i}$$

and

$$N'' = \sum_{j=0}^{[k/r]} \sum_{i=0}^{r-1} (-1)^j a_{jr+i}10^i.$$

$N-N''$ therefore consists of a series of terms of the form

$$a_{jr+i}(10^{jr+i} - (-1)^j 10^i).$$

But, since

$$10^r \equiv (n-1) \ (\mathrm{mod}\ n)$$
$$10^r \equiv -1 \ (\mathrm{mod}\ n)$$
$$10^{jr} \equiv (-1)^j \ (\mathrm{mod}\ n)$$
$$10^{jr+i} \equiv (-1)^j 10^i \ (\mathrm{mod}\ n)$$

it follows that

$$N \equiv N'' \pmod{n}.$$

Examples:

For $n=9$:

$$p = 0,\ q = 1, \text{ and } N' = a_0 + a_1 + a_2 + \cdots .$$

For $n=11$:

$$p = 0,\ q = 2, \text{ and } N' = (a_0 + 10a_1) + (a_2 + 10a_3) + \cdots$$

or

$$r = 1 \text{ and } N'' = a_0 - a_1 + a_2 - \cdots .$$

For $n=10^u+1$:

$$r = u \text{ and } N'' = (a_0 + 10a_1 + \cdots + 10^{u-1}a_{u-1}) - (a_u + 10a_{u+1} + \cdots + 10^{u-1}a_{2u-1}) + \cdots .$$

For $n=7$ or 13:

$$r = 3 \text{ and } N'' = (a_0 + 10a_1 + 100a_2) - (a_3 + 10a_4 + 100a_5) + \cdots .$$

The usual rule for 7,

$$N''' = (a_0 + 3a_1 + 2a_2) - (a_3 + 3a_4 + 2a_5) + \cdots$$

can easily be derived from this.

For $n=12$:

$$p = 2,\ q = 1 \text{ and } N' = a_0 + 10a_1 + 100(a_2 + a_3 + \cdots).$$

This can be reduced to

$$N''' = a_0 - 2a_1 + 4(a_2 + a_3 + \cdots).$$

For $n=2^\alpha 5^\beta$:

$$p = \max(\alpha, \beta),\ q = 0, \text{ and } N' = a_0 + 10a_1 + \cdots + 10^{p-1}a_{p-1}.$$

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

THE PROOF OF THE LAW OF SINES

ALFRED BRAUER, University of North Carolina

Most mistakes made by mathematicians consist in neglecting to consider a certain possibility. A proof or a theorem can become invalid if two roots of an equation become equal, if a determinant becomes zero, if the factorization in the considered number field is not unique, and so on. Often it is possible to complete the proof for the exceptional case, but sometimes the theorem is wrong. It is therefore of importance that the student learn as early as possible to check if a proof is really complete.

It is surprising that in a large number of textbooks on trigonometry the proof for the law of sines is incomplete.

In general, the proof is given by drawing one of the altitudes. Then two cases are considered, namely the case that each angle is less than 90° and the case that one of the angles is greater than 90°. Two figures illustrate the proof.

The authors of these books seem to believe that it is not necessary to prove the theorem for the case that one of the angles equals 90° since the trigonometric functions can directly be used for right triangles. But this is not correct.

Given two sides a and b, and the angle B opposite to the smaller side b. In this case we are not permitted to use the law of sines if this theorem is not proved for right triangles since it can be that the angle $A = 90°$. But we cannot use the trigonometric functions either, since we do not know that the triangle has a right angle.

In these books the law of sines is used and it is explicitly stated that A can be equal to 90° and that we have one and only one solution in this case. But it is not recognized that here a theorem is used in a case in which it is not proved.

If a, b, and B are given, but not numerically, we cannot determine whether the triangle is a right one or not. If the law of sines is not proved for right triangles, a complete solution must consist of two parts, namely assuming that no angle equals 90°, and assuming that one angle equals 90°. If a, b, and B are given numerically, it can happen that $\sin B = b/a$ if a 4-place table is used, but $\sin B \neq b/a$ if a 5-place table is used, and the method of solution would depend on the kind of table to be used.

This shows that it is necessary to prove the law of sines for right triangles. Of course, using continuity it is obvious that the theorem is correct for right triangles, too. But also the elementary proof is so simple that there is no reason to omit the proof of this case in any book or class.

CAUCHY'S FORM OF R_n FROM THE ITERATED INTEGRAL FORM

H. J. HAMILTON, Pomona College

Professor C. P. Nicholas might have remarked, in his happy article, *Taylor's Theorem in a first course* (this MONTHLY, vol. 58, no. 8, October, 1951, pp. 559–562), that Cauchy's form of the remainder, as well as Lagrange's, may be rather easily deduced from the iterated integral form. We need only to apply the geometrically obvious equality

$$\int_a^b \int_a^s \phi(t, s)\,dt\,ds = \int_a^b \int_t^b \phi(t, s)\,ds\,dt$$

repeatedly, "from the inside out," as indicated:

$$\begin{aligned}
R_n &= \int_a^x \int_a^y \cdots \int_a^z \int_a^u \underbrace{\int_a^v \int_a^w} f^{(n)}(t)\,dt\,\underbrace{dw}\,dv\,du \cdots dy \\
&= \int_a^x \int_a^y \cdots \int_a^z \underbrace{\int_a^u} \overbrace{\underbrace{\int_a^v} \int_t^v} f^{(n)}(t)\,\widetilde{dw}\,\underbrace{dt\,dv}\,du \cdots dy \\
&= \int_a^x \int_a^y \cdots \underbrace{\int_a^z} \overbrace{\underbrace{\int_a^u} \int_t^u} \int_t^v f^{(n)}(t)\,dw\,\widetilde{dv}\,\widetilde{dt}\,du \cdots dy \\
&\quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&= \overbrace{\int_a^x \int_t^x} \cdots \int_t^z \int_t^u \int_t^v f^{(n)}(t)\,dw\,dv\,du \cdots \widetilde{dy}\,\widetilde{dt} \\
&= \int_a^x f^{(n)}(t) \left\{ \int_t^x \cdots \int_t^z \int_t^u \int_t^v dw\,dv\,du \cdots dy \right\} dt \\
&= \int_a^x f^{(n)}(t)\,\frac{(x-t)^{n-1}}{(n-1)!}\,dt = f^{(n)}(\xi)\,\frac{(x-\xi)^{n-1}}{(n-1)!}\,(x-a), \quad a \leqslant \xi \leqslant x,
\end{aligned}$$

which is the desired form. (The last step, which is an application of the simplest mean-value theorem for integrals, is to be found in numerous textbooks, the integral which appears in it being a standard formula for R_n.)

A MINUTE THEORY OF RADICAL EQUATIONS

G. B. HUFF and D. F. BARROW, University of Georgia

A section on radical equations is usually found in freshman algebra texts in the chapter on quadratics. There is ordinarily no attempt to develop any theory for radical equations, although there is an admonition to test for "extraneous roots." (See this MONTHLY, vol. 55, p. 639, for an interesting comment on this latter notion.) The texts do not invariably emphasize that the interest in the problem arises from the fact that in these equations $\sqrt{R}$ is defined only for

R real and non-negative and represents the non-negative square root of R. Doubtless the average freshman does not notice the lack of a theory here, but the instructor may, especially when he tries to make up nice problems. In this note we wish to point out two things. Some theory may be obtained, as in the case of quadratic equations, by applying the methods of the book to literal problems. An intuitive geometric idea of what is going on may be obtained from the geometry of the parabola. In both cases the results seem simple enough to be interesting and are certainly useful in designing problems.

We will restrict the strictly algebraic attack to the radical equation

$$\sqrt{Ax+B}+\sqrt{ax+b}=c, \qquad A-a>0, \qquad c>0, \tag{1}$$

where A, B, a, b, and c are in some field K of real numbers. Due to the definition of the symbol $\sqrt{R}$, we seek real numbers, not necessarily in K, which are roots of (1). Let D be defined by:

$$D=(A-a)(Ab-aB)+Aac^2. \tag{2}$$

We shall prove the following:

THEOREM. *The equation* (1) *has no solution if* $D<0$. *If* $D\geqq 0$, *the equation* (1) *has two, one, or no solutions according as two, one, or none of the conditions* (*p*) *and* (*n*) *hold*:

$$\text{(p)}\quad Ac\geqq -\sqrt{D}\geqq ac; \qquad \text{(n)}\quad Ac\geqq \sqrt{D}\geqq ac.$$

By the usual device of eliminating radicals by squaring, we construct the *auxiliary equation*,

$$\begin{aligned}(A-a)^2z^2+2[(A-a)(B-b)-(A+a)c^2]z\\ +[(B-b)^2-2(B+b)c^2+c^4]=0,\end{aligned} \tag{3}$$

and compute its discriminant to be $16c^2D$, where D is given by (2). Since any root of (1) must by agreement be real and must also be a root of (3), then $D<0$ implies that (1) has no solutions of the sort we want.

If $D\geqq 0$, then the auxiliary equation (3) has the root

$$z_+=[(A+a)c^2-(A-a)(B-b)+2c\sqrt{D}\,]/(A-a)^2,$$

and a root z_- which is the same except for the sign on the term $2c\sqrt{D}$. As a first step in checking these as possible roots of (1), we compute that

$$Az_++B=(Ac+\sqrt{D})^2/(A-a)^2; \qquad az_++b=(ac+\sqrt{D})^2/(A-a)^2. \tag{4}$$

Similar formulas hold for Az_-+B and az_-+b. This gives a result that every freshman algebra teacher has checked countless times, namely:

If D *is non-negative and* z_0 *is a root of the auxiliary equation* (3), *then* Az_0+B *and* az_0+b *are non-negative and are in fact squares in* $K(\sqrt{D})$, *the field obtained*

by adjoining $\sqrt{D}$ *to* K.

If $K(\sqrt{D})$ is the field needed to solve the auxiliary equation, then the square roots found in checking are also in $K(\sqrt{D})$.

We now continue with the proof of the theorem by showing that z_+ is a root of (1) if and only if the condition (p) of the theorem holds. It is convenient here to write (p) in the form:

$$(p_1) \quad Ac + \sqrt{D} \geqq 0; \qquad (p_2) \quad -ac - \sqrt{D} \geqq 0.$$

It is clear from the formulas (4) and the convention in equation (1) that $A-a>0$, that (p_1) and (p_2) are sufficient to insure that z_+ is a root of (1). To prove that these conditions are also necessary, we show that if either one or both fail, then z_+ does not satisfy (1).

Suppose that neither (p_1) nor (p_2) is true. Then $-Ac-\sqrt{D}>0$ and $ac+\sqrt{D}>0$ and substitution of z_+ in (1) gives the false statement

$$(-Ac - \sqrt{D})/(A - a) + (ac + \sqrt{D})/(A - a) = c.$$

The argument is a little different if (p_1) holds and (p_2) fails. Now we have $Ac+\sqrt{D}\geqq 0$ and $ac+\sqrt{D}>0$ and substitution yields $(Ac+\sqrt{D})/(A-a)+(ac+\sqrt{D})/(A-a)=c$. This reduces to $ac+\sqrt{D}=0$, the desired contradiction. It follows, in the same way, that z_+ is not a root of (1) when (p_1) fails and (p_2) holds.

Similarly, z_- is a root of (1) if and only if the condition (n) holds. For $D>0$, this completes the proof of the second assertion of the theorem. When $D=0$, $z_+=z_-$, the inequalities (p) and (n) are the same, and equation (1) has one or no roots as this one relation is or is not satisfied.

Another method of discussing the roots of (1) may be found by means of analytic geometry. We shall deal only with the case where there are genuinely two radicals in equation (1), that is, both radicals actually contain x, and the ratio of the radicands is not independent of x. This means that we assume $Aa(Ab-aB)\neq 0$. We begin by plotting in the u, v-plane the graphs of

$$\text{(d)} \quad \sqrt{u} + \sqrt{v} = c, \qquad \text{(e)} \quad \sqrt{u} - \sqrt{v} = c, \qquad \text{(f)} \quad -\sqrt{u} + \sqrt{v} = c; \quad c > 0.$$

The graphs of these three equations are three arcs which form one complete parabola lying wholly in the first quadrant. The vertex of this parabola is the point $(c^2/4, c^2/4)$ and it is tangent to the axes at $(c^2, 0)$ and $(0, c^2)$. The graph of (d) extends from $(0, c^2)$ to $(c^2, 0)$ and contains the vertex. The graph of (e) runs from $(c^2, 0)$ to "infinity" and the graph of (f) runs from $(0, c^2)$ to "infinity." If a tangent line be drawn to the parabola at any point, the sum of the intercepts of this tangent line is c^2. Next, we consider in the u, v-plane the straight line whose parametric equations are

$$\text{(g)} \qquad u = Ax + B, \qquad v = ax + b,$$

where x is the parameter. If this line cuts the graph of (d), the value of the parameter x determined by this point will be a solution of (1). Under our assumption $Aa(Ab-aB)\neq 0$, the line (g) has two non-zero intercepts given by $(aB-Ab)/a$ and $(Ab-aB)/A$. Thus when equation (1) is given, these intercepts are easily computed and an inspection of the figure yields the following criteria.

Case I. *When both intercepts are positive and not greater than c^2, the equation* (1) *has two solutions if the sum of the intercepts is greater than c^2, just one solution if the sum of the intercepts is equal to c^2, and none if the sum of the intercepts is less than c^2.*

Case II. *When only one of the intercepts satisfies the condition of being positive and not greater than c^2, equation* (1) *has exactly one solution.*

Case III. *When neither intercept satisfies the condition of being positive and not greater than c^2, equation* (1) *has no solutions.*

It is clear that both methods give equally simple statements for equations like (1) with different arrangements of signs. With slight changes, both ideas apply to the radical equation $\sqrt{Ax+B}=cx+d$. We close by pointing out exactly how the above results may be used to construct examples to order. Suppose we want a radical equation whose auxiliary equation has rational roots of which only z_- is a root of (1). Having chosen, more or less at random $A=3$, $a=2$, and $c=2$, it is clear that $\sqrt{D}=5$ is a convenient solution of (n) which does not satisfy (p). Substitution of these values in (2) gives $3b-2B=1$. Any solution of this equation gives a problem of the desired sort. The particular solution $B=4$, $b=3$ yields $\sqrt{3x+4}+\sqrt{2x+3}=2$. If we want an auxiliary equation with irrational roots, a slightly different example arises by choosing $\sqrt{D}=\sqrt{23}$ for the same choice of A, a, and c. The condition on B and b is now $3b-2B=-1$, with $B=2$, $b=1$ as a simple solution. The reader may find it interesting to verify that z_- is indeed the only root of $\sqrt{3x+2}+\sqrt{2x+1}=2$. Both of these come under Case II of the geometric criteria.

TERM-WISE DIFFERENTIATION OF POWER SERIES

T. M. Apostol, California Institute of Technology

Let the function $f(x)$ be defined by a power series

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \tag{1}$$

in the convergence-interval $|x|<r$. The purpose of this note is to present an elementary proof of the fact that the derivative $f'(x)$ exists at every point in $|x|<r$ and that it may be obtained by differentiating the series in (1) term-by-term. The usual proofs of this fact make use of the concept of uniform convergence or else involve rearrangement of double series, and consequently

are not given in most beginning calculus texts. Hardy's *Pure Mathematics* does not treat differentiation or integration of power series since he avoids "the inclusion of anything that involves really difficult ideas." The proof given below makes use only of the mean-value theorem of differential calculus and it is felt that the proof is simple enough to be presented to students in sophomore calculus.

It is assumed that the existence of an interval of convergence has been established for power series and that it is known that power series converge absolutely inside the interval. We first prove:

THEOREM 1. *Let the series in equation* (1) *have the interval of convergence* $|x| < r$. *Then the series*

$$\sum_{n=0}^{\infty} n a_n x^{n-1} \tag{2}$$

has the same interval of convergence.

Proof. Let us first take a positive value of x, $0 < x < r$, and let h be a small positive number such as $0 < x < x+h < r$. Then the series for $f(x)$ and for $f(x+h)$ are each absolutely convergent and hence so is the series

$$\frac{f(x+h) - f(x)}{h} = \sum_{n=0}^{\infty} a_n \frac{(x+h)^n - x^n}{h}. \tag{3}$$

Applying the mean-value theorem to each term of this series we have

$$\frac{(x+h)^n - x^n}{h} = n\xi_n^{n-1}$$

where $x < \xi_n < x+h$. Hence the series in (3) is identical to the series

$$\sum_{n=0}^{\infty} n a_n \cdot \xi_n^{n-1}, \tag{4}$$

which must be absolutely convergent, since the series in (3) is. The series in (4) is no longer a power series but it dominates the power series (2) so that the series (2) must be absolutely convergent for this x. Hence the interval of convergence of (2) includes the interval $|x| < r$, and it cannot extend beyond this interval since (2) dominates (1). This proves the theorem.

If we apply the theorem to series (2) we obtain the following:

COROLLARY. *The series*

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

has the same interval of convergence as the series in (1).

The next theorem can be obtained as a consequence of Theorem 3 but a direct proof is simple and it also serves as an introduction to the proof of Theorem 3.

THEOREM 2. *The function defined by* (1) *is continuous for each* x *inside the interval of convergence.*

Proof. Let $0 \leqq |x| < r$ and choose x_1 so that $0 \leqq |x| < |x_1| < r$. Next choose h so that $0 \leqq |x+h| < |x_1| < r$. We will prove that

$$\lim_{h \to 0} |f(x+h) - f(x)| = 0.$$

Proceeding as in the proof of Theorem 1, we have

$$f(x+h) - f(x) = h \sum_{n=0}^{\infty} n a_n \xi_n^{n-1}, \tag{5}$$

where each number ξ_n lies between x and $x+h$. By Theorem 1, the series

$$\sum_{n=0}^{\infty} n a_n x_1^{n-1}$$

is absolutely convergent and since $|\xi_n| < |x_1|$, equation (5) leads to the inequality

$$0 \leqq |f(x+h) - f(x)| \leqq |h| \sum_{n=0}^{\infty} n |a_n| |x_1|^{n-1}. \tag{6}$$

The series multiplying $|h|$ in (6) represents a number independent of h so that if we now let $h \to 0$ the theorem follows.

THEOREM 3. *If the function* $f(x)$ *is given by the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

in the interval $|x| < r$, *then for any* x *inside this interval the derivative* $f'(x)$ *exists and is given by the power series*

$$f'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

Proof. Take x and then x_1 so that $0 \leqq |x| < |x_1| < r$. Let

$$F(x) = \sum_{n=0}^{\infty} n a_n x^{n-1}.$$

This series converges absolutely in $|x| < r$ by Theorem 1, and our aim is to prove that $f'(x)$ exists and equals $F(x)$ in this interval. Proceeding as in the previous theorem we choose h so that $|x+h| < |x_1| < r$ and obtain

$$F(x) - \frac{f(x+h) - f(x)}{h} = \sum_{n=0}^{\infty} na_n(x^{n-1} - \xi_n^{n-1}), \tag{7}$$

where each number ξ_n lies between x and $x+h$. Now we apply the mean-value theorem again to obtain

$$x^{n-1} - \xi_n^{n-1} = (n-1)\eta_n^{n-2}(x - \xi_n)$$

where each number η_n lies between x and ξ_n. Hence we have

$$F(x) - \frac{f(x+h) - f(x)}{h} = \sum_{n=0}^{\infty} n(n-1)a_n\eta_n^{n-2}(x - \xi_n), \tag{8}$$

the series on the right of (8) being absolutely convergent since it is identical to the one in (7). But we have $|x-\xi_n| < |h|$ and $|\eta_n| < |x_1|$ so that we have

$$0 \leqq \left| F(x) - \frac{f(x+h) - f(x)}{h} \right| \leqq |h| \sum_{n=0}^{\infty} n(n-1)\,|a_n|\,|x_1|^{n-2}, \tag{9}$$

the series in (9) being convergent by the corollary to Theorem 1. Letting $h \to 0$ in (9) we see that $f'(x)$ exists and equals $F(x)$.

For the sake of completeness we include

THEOREM 4. *Let*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

have the interval of convergence $|x| < r$. *Let*

$$G(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}.$$

Then the series for $G(x)$ *has the same interval of convergence and*

$$G'(x) = f(x).$$

Proof. Since the power series for $G(x)$ has *some* interval of convergence, Theorem 3 tells us that it must be the same as the interval for $f(x)$ and that $G'(x) = f(x)$.

A METHOD FOR THE SOLUTION OF THE GENERAL CUBIC

T. D. NAGLE, University of Bridgeport

Given the general cubic

$$f(x) = x^3 + bx^2 + cx + d, \qquad d \neq 0, \tag{1}$$

By the transformation $x = 1/y$, we replace the equation $f(x) = 0$ by the equation $g(y) = 0$ where

A MODERN TRICK

CLAIRE ADLER, New York University

There are old methods for showing that

$$\sqrt[3]{2+\sqrt{5}}+\sqrt[3]{2-\sqrt{5}}=1,$$

but this one seems extremely simple:

Let

$$\sqrt[3]{2+\sqrt{5}}=a$$

$$\sqrt[3]{2-\sqrt{5}}=b$$

and

$$a+b=x.$$

Then

$$x^3=a^3+b^3+3abx=-3x+4 \tag{1}$$

or

$$x^3+3x-4=0 \tag{2}$$

and the only real root of (2) is

$$x=1.$$

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1016. *Proposed by Norman Anning, University of Michigan*

Find the element of likeness in: (a) simplifying a fraction, (b) powdering the nose, (c) building new steps on the church, (d) keeping emeritus professors on campus, (e) putting B, C, D in the determinant

$$\begin{vmatrix} 1 & a & a^2 & a^3 \\ a^3 & 1 & a & a^2 \\ B & a^3 & 1 & a \\ C & D & a^3 & 1 \end{vmatrix}.$$

E 1017. *Proposed by Harry Furstenberg, New York City*

On the common secant AB of two intersecting circles, O and O', are chosen any two points, C and D, outside of either circle. The tangents CQ and CS are drawn to O' and O, respectively, on one side of AB and the tangents DR and DT are drawn to O' and O on the other side of AB. Prove that QR and ST intersect on AB.

E 1018. *Proposed by B. B. Misra, Ravenshaw College, Cuttack, India*

AB is a line segment, which is divided by the point O in the ratio $m:n$. P is a variable point (r, θ) referred to O as pole and OB as initial line. Subject to the conditions

$$0 \leqq r \leqq a < \min(AO, OB), \qquad 0 \leqq \theta \leqq \pi,$$

find the minimum value of $m^2/AP+n^2/BP$, and also find the position of P for which this minimum value is attained.

E 1019. *Proposed by A. E. Livingston and L. H. Wegner, University of Oregon*

Let $\{f_n(x)\}$ be a sequence of functions continuous on the closed interval $[a, b]$, and let $\lim_{n\to\infty} f_n(x)=f(x)$ uniformly on the open interval (a, b). Then there exists a function $F(x)$ satisfying: (1) $F(x)$ is continuous on $[a, b]$, (2) $F(x)=f(x)$ for $x\epsilon(a, b)$, (3) $\lim_{n\to\infty} f_n(a)=F(a)$ and $\lim_{n\to\infty} f_n(b)=F(b)$.

E 1020. *Proposed by John Disch, Cleveland, Ohio*

A perfectly flexible inextensible chain of uniform density is hanging from two supports at equal heights above the surface of a liquid. If the chain sags into the liquid find the dip in the chain in terms of those factors that affect the problem.

SOLUTIONS

E 967 [1952, 43]. *Correction by J. H. Braun, Washington, D. C.*

In the solution of part (c) replace $[1-5\pi^2/3n^2]$ by $[1-8\pi^2/5n^2]$. For, from (2), it follows that

$$S_n < \pi[1 - (\pi^2/15n^2)(25 - 2\pi^2/n^2)].$$

But, for n an integer greater than 4, we have $25-2\pi^2/n^2>24$, whence

$$S_n < \pi[1 - 8\pi^2/5n^2].$$

Pythagorean Triangles

E 986 [1951, 631]. *Proposed by W. R. Ransom, Tufts College*

Show that there are fifty right triangles whose sides are integers less than 100.

Solution by C. W. Trigg, Los Angeles City College. The hypotenuses of all primitive right triangles are given by $c=m^2+n^2$, where m and n are relatively

prime positive integers of different parity, and $m>n$. We determine every $c<100$ and compute the number (f) of multiples of c less than 100. Thus

n	1	1	1	1	2	2	2	2	3	3	4	4	4	5	5	6
m	2	4	6	8	3	5	7	9	4	8	5	7	9	6	8	7
c	5	17	37	65	13	29	53	85	25	73	41	65	97	61	89	85
f	19	5	2	1	7	3	1	1	3	1	2	1	1	1	1	1

Since $c>a=m^2-n^2$, $c>b=2mn$, and $\Sigma f=50$, there are fifty right triangles with integer sides less than 100.

Also solved by H. T. R. Aude, Leon Bankoff, H. H. Berry, E. R. Bowersox, J. H. Braun, W. E. Briggs, W. E. Buker, P. L. Chessin, D. R. Clutterham, R. J. Cormier, J. E. Darraugh, F. F. Dorsey, William Douglas, J. C. Eaves, Herbert Emich, C. V. Fronabarger, R. A. Gambill and J. D. Neff jointly, Vern Hoggatt, H. B. Hoyle, Jr., Sam Kravitz, Sidney Kravitz, R. Lariviere, Myles McConnon, Jack Minker, Prasert Na Nagara, Elizabeth Napp, Leola Odland, F. D. Parker, C. F. Pinzka, A. P. Rhodes, Azriel Rosenfeld, Charles Salkind, P. T. Schaefer, E. M. Scheuer, R. Z. Vause, Jr., G. W. Walker, and the proposer. Late solutions by W. O. Carter, Barbara Clark, Roxie Eldridge, Dick Hartung, N. C. C. Hu, P. K. Kloeppel, Jane Lassiter, Imogena Lowden, Donald Morrison, F. W. Stallard, Peter Terwey, Jr., W. R. Van Voorhis, W. T. Wolf.

Berry showed that there are 652 Pythagorean triangles with sides less than 1000.

Clutterham solved the problem on the *ORDVAC*, an electronic digital computer that the University of Illinois is building for the Army and which is now in its final testing stages. The vanishing of the function $a^2+b^2-c^2$ was tested for all integral values of a, b, c: $1\leqq a\leqq b\leqq c<100$. The actual computing time was about 8 minutes with another 7 minutes required for the printing of the results.

Douglas called attention to William Sell, *Integral Solutions of* $A^2+B^2=C^2$, this MONTHLY, Oct. 1936, p. 481.

Grid Points in a Strip

E 987 [1951, 631]. *Proposed by W. L. Stamey and J. L. Zemmer, Jr., University of Missouri*

Prove that the interior of the strip, on a cartesian grid, bounded by two arbitrarily close parallel straight lines, contains either no points with integer coordinates or infinitely many such points.

Solution by F. Bagemihl and W. Seidel, University of Rochester. Assume that the strip is not parallel to the y-axis (otherwise the result to be proved is trivial). Then $\epsilon>0$, β, and μ exist, such that the strip is the set of points (x, y) for which $|(y-\mu x)-\beta|<\epsilon$. If μ is irrational, then, according to Kronecker's theorem (see,

e.g., Hardy and Wright, *An Introduction to the Theory of Numbers*, p. 363), this inequality has infinitely many pairs of integral solutions, x, y. If μ is rational, say a/b, and the lattice point (x_0, y_0) satisfies this inequality, then so do the lattice points (x_0+bn, y_0+an), $n=0, \pm1, \pm2, \cdots$.

Also solved by T. M. Apostol, I. A. Dodes, S. H. Gould, Vern Hoggatt, David Mandelbaum, C. S. Ogilvy, Gustave Rabson, and Azriel Rosenfeld.

A Set of Numbers Not All Prime

E 988 [1951, 632]. *Proposed by Leo Moser, Texas Technological College*

Let a, b, x_0 be positive integers and let $x_1=ax_0+b, \cdots, x_n=ax_{n-1}+b$. Prove that x_n cannot be prime for all n.

Solution by N. G. Gunderson, University of Rochester. It is easily proved by induction that $x_n=x_1+(x_1-x_0)(a^n-a)/(a-1)$. Let p be any prime dividing x_1. By Fermat's Theorem p divides $(a^p-a)/(a-1)$, and therefore p divides x_p. But $x_p>x_1\geqq p$, and so x_p is composite.

Also solved by P. M. Anselone, John Christopher, Herbert Emich, C. V. Fronabarger, Emil Grosswald, Prasert Na Nagara, L. W. Neustadt, Joseph Oppenheim, Azriel Rosenfeld, W. Seidel, Peter Terwey, Jr., J. A. Tierney, R. Z. Vause, Jr., and the proposer.

Scissors with a Constant Cutting Angle

E 989 [1951, 632]. *Proposed by C. W. Bruce, Tennessee Polytechnic Institute*

If the angle between the blades of scissors is less than the angle of friction of the metal on the cloth being cut, there will be a little tendency for the cloth to move forward as the scissors are being closed. How can a pair of scissors be constructed so that one blade is straight and there is a constant angle between the cutting edges?

Solution by G. W. Walker, Buffalo, N. Y. For simplicity, place the pivot so that the line of the straight cutting edge will, if continued, pass through it. Then, in any functioning position, the angle between the blades will be the angle between the tangent to the curved cutting edge at the point of action and the radius vector from the pivot to that point. This angle is constant if the curved cutting edge is an equiangular spiral having the pivot as its pole.

Also solved by I. A. Dodes, F. F. Dorsey, Sidney Kravitz, H. L. Lee, David Mandelbaum, J. D. Neff, C. O. Oakley, C. S. Ogilvy, Azriel Rosenfeld, and the proposer. Late solution by Leon Bankoff (who designed a scissors in which both blades are straight).

Spheres of Droz-Farny

E 990 [1951, 632]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

If a sphere with center at the orthocenter of an orthocentric tetrahedron intersects the lines joining the midpoints of the edges of the tetrahedron, then the 24 distances of the vertices from the intersections on those lines joining the

midpoints of the edges issuing from these vertices are all equal. (The analogous property for the triangle was announced by Steiner and proved by Droz-Farny. See Johnson, *Modern Geometry*, p. 256.)

Editorial Note. The proof of the analogous property for the triangle as given in Johnson's book carries over almost verbatim for the tetrahedron.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4488. *Proposed by George Grossman, DeWitt Clinton High School, New York City*

Using the symbolic representation

$$c^{(3)} \equiv 8m_0 + 12m_1 + 6m_2 + 1m_3$$

to indicate that a cube of dimensionality 3 possesses 8 vertices (dimensionality 0), 12 manifolds of dimensionality one, etc., then a tesseract (hypercube of dimensionality four) can be represented symbolically by

$$c^{(4)} \equiv 16m_0 + 32m_1 + 24m_2 + 8m_3 + 1m_4.$$

If a hypercube of dimensionality n is represented by

$$c^{(n)} \equiv a_{n0}m_0 + a_{n1}m_1 + \cdots + a_{nn}m_n,$$

determine the coefficients a_{ij}.

4489. *Proposed by D. J. Newman, Harvard University*

Sum the series

$$\sum_{n=1}^{\infty} \frac{(n/e)^{n-1}}{n!}.$$

4490. *Proposed by L. M. Blumenthal, University of Missouri*

Let S be a set of more than $2n$ pairwise distinct (non-null) vectors of Euclidean n-space E_n. If each vector of E_n is expressible as a linear combination of vectors of S with non-negative coefficients, then S contains a subset R of at most

$2n-1$ vectors such that R has this same property. (This is an extension of no. 4395 [1952, 46]).

4491. *Proposed by C. S. Venkataraman, Trichur, South India*

If ω denotes the Brocard angle of a triangle ABC, prove that

(i) the sides are equal when $\cot \omega = \sqrt{3}$.

(ii) the square of the lengths of the sides, a^2, b^2, c^2, are in arithmetic progression when $\cot \omega = 3 \cot B$.

4492. *Proposed by G. Lumer, Instituto de Matematica y Estadistica, Montevideo, Uruguay*

A real function $f(x)$ is said to be locally increasing at the point x_0 if there exists a neighborhood U of x_0 such that for $x \in U$,

$$\frac{f(x) - f(x_0)}{x - x_0} \geqq 0.$$

Show that there exist functions $f(x)$, defined and continuous on a closed interval $[a, b]$, locally increasing on a set D, everywhere dense on $[a, b]$, but such that $f(b) < f(a)$.

SOLUTIONS

Orthocentric Tetrahedron Bordered by Isosceles Tetrahedrons

4417 [1950, 691]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

(1) The necessary and sufficient condition that an orthocentric tetrahedron $T \equiv ABCD$ can be bordered by isosceles tetrahedrons $T_a \equiv A_1BCD$, $T_b \equiv B_1CDA$, $T_c \equiv C_1DAB$, $T_d \equiv D_1ABC$ is that all the face angles of T be acute. (2) The tetrahedrons T and $T_1 \equiv A_1B_1C_1D_1$ are orthologic. (3) The lines AA_1, BB_1, CC_1, DD_1 are four hyperbolic lines. (4) The sum of the squares of the areas of the faces of T_a, T_b, T_c, T_d is equal to the sum of the squares of the products of the opposite edges of T.

Solution by the Proposer. The following hints contain all the essentials of the argument.

(1) The face angles of any isosceles tetrahedron are all acute.

(2) The perpendiculars from A_1, B_1, C_1, D_1 on the opposite faces of T go through the points symmetrical to the orthocenters of these faces with respect to their circumcenters. Hence they are concurrent at a point symmetrical to the orthocenter H of T with respect to the circumcenter O of T. It follows that the perpendiculars from A, B, C, D on the faces $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$ of T_1 are also concurrent.

(3) The lines AA_1, BB_1, CC_1, DD_1 intersect OH.

(4) In an orthocentric tetrahedron the sum of the squares of the areas of the faces is equal to

$$\tfrac{1}{4}[(aa')^2 + (bb')^2 + (cc')^2],$$

where $a = BC$, $a' = AD$, *etc.*

Editorial Note. The translator, W. E. Byrne, supplies the following references from N. A. Court, *Modern Pure Solid Geometry.*

Art. 212. Theorem. The four altitudes of an orthocentric tetrahedron are concurrent.

Art. 213. Definition. The point common to the four altitudes of an orthocentric tetrahedron is called the orthocenter of the tetrahedron.

Art. 216. Theorem. The altitudes of an orthocentric tetrahedron pass through the orthocenters of the respective faces of the tetrahedron.

Art. 290. Definition. A tetrahedron in which each edge is equal to its opposite will be referrred to as an isosceles tetrahedron.

Art. 309. Theorem. The foot of the altitude dropped from a vertex of an isosceles tetrahedron upon the corresponding face is the symmetric of the orthocenter of this face with respect to its circumcenter.

Determinant Evaluation

4418 [1950, 692]. *Proposed by A. C. Aitken, Edinburgh, Scotland*

Evaluate the determinant

$$(1) \qquad a_n = \begin{vmatrix} b_1 & -1 & 0 & \cdots & 0 \\ b_2 & b_1 & -2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-1} & b_{n-2} & b_{n-3} & \cdots & 1-n \\ b_n & b_{n-1} & b_{n-2} & \cdots & b_1 \end{vmatrix} \qquad \text{where} \qquad (2) \quad b_n = \frac{n^n}{n!}.$$

Solution by J. S. Frame, Michigan State College. In general, to obtain a relation between the coefficients a_n and b_n in (1), without taking account of (2), we may set $a_0 = 1$, $a_1 = b_1$, and define the two functions y and z by the power series

$$(3) \qquad y = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}, \qquad z = \sum_{n=1}^{\infty} b_n \frac{x^n}{n}.$$

Upon expanding the determinant (1) by cofactors of its last row we obtain the recurrence relation

$$(4) \qquad n\frac{a_n}{n!} = a_0 b_n + \frac{a_1}{1!} b_{n-1} + \frac{a_2}{2!} b_{n-2} + \cdots + \frac{a_{n-1} b_1}{(n-1)!},$$

which is equivalent to the differential equation $y'=yz'$, whose solution, satisfying $y=1$ when $x=z=0$, is

$$y = e^z. \tag{5}$$

Hence, using the coefficients b_n defined in (2), our problem is to find the coefficient a_n of $x^n/n!$ in the expansion of e^z when

$$z = \sum_{k=1}^{\infty} \frac{(kx)^k}{k \cdot k!}. \tag{6}$$

If in (6) we make the substitution

$$x = te^{-t}, \tag{7}$$

we find

$$\begin{aligned} \frac{z}{t} &= \sum_{k=1}^{\infty} \frac{(kt)^{k-1}e^{-kt}}{k!} = \sum_{m,k=1}^{\infty} (-1)^{m-k}\binom{m}{k}\frac{(kt)^{m-1}}{m!} \\ &= 1 + \sum_{m=2}^{\infty} \frac{t^{m-1}}{m!}\left[\sum_{k=0}^{m} (-1)^{m-k}\binom{m}{k}k^{m-1}\right]. \end{aligned} \tag{8}$$

The expression in brackets vanishes for $m>1$, since it is the coefficient of $t^{m-1}/(m-1)!$ in the expansion of $(e^t-1)^m$. Hence the right member of (8) reduces to 1, and we have $z=t$ and from (7) $x=z\,e^{-z}$. It follows from (5), (3), and (2) that

$$y = e^z = z/x = \sum_{n=0}^{\infty} b_{n+1}x^n/(n+1) = \sum_{n=0}^{\infty} (n+1)^{n-1}x^n/n! \tag{9}$$

By comparing coefficients of $x_n/n!$ in (3) and (9) we obtain the required formula for the determinant a_n in (1), namely,

$$a_n = (n+1)^{n-1}. \tag{10}$$

Also solved by N. J. Fine, G. P. Henderson, O. E. Stanaitis, G. Szegö, and the Proposer.

Editorial Note. Fine obtains recurrence formula (4) above (with the values of b_k introduced) and recognizes it as a familiar formula for $u_n \equiv (n+1)^{n-1}$. In fact, recalling a result due to Abel,

$$(x+y+nz)^n = y\sum_{j=0}^{n}\binom{n}{j}[n+(n-j)z]^{n-j}(y+jz)^{j-1},$$

and placing $x=y=z=1$, he has

$$u_{n+1} = \sum_{j=0}^{n}\binom{n}{j}(n-j+1)^{n-j}u_j.$$

If j is replaced by $n-k$, it is seen that u_n satisfies the same recurrence formula that a_n does. Since the initial conditions are also the same, $a_n = u_n$ for all n.

The Proposer obtains the evaluation by induction, using the determinant $\Delta(t)$ formed from the given determinant by replacing each $k^k/k!$, in the last row only, by $t^{k-1}/(k-1)!$. Then $\Delta(n)$ is the given determinant. It is shown that $\Delta(t)$ equals $(t+1)^{n-1}$ by obtaining

$$\Delta(n-1) = n^{n-1},\ \Delta'(n-2) = (n-1)^{n-1},$$
$$\Delta''(n-3) = (n-1)(n-2)^{n-2}, \cdots, \Delta^{(n-1)}(0) = (n-1)!$$

Stanaitis finds the proposed determinant as a special case of

$$\begin{vmatrix} c_1 & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 2c_2 & c_1 & -2 & \cdot & \cdot & \cdot & \cdot \\ 3c_3 & 2c_2 & c_1 & -3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = n! \sum \frac{c_1^{e_1} c_2^{e_2} \cdots c_n^{e_n}}{e_1! e_2! \cdots e_n!},$$

where $$e_1 + 2e_2 + \cdots + ne_n = n.$$

See Muir-Metzler, *A Treatise on the Theory of Determinants*, 1930, p. 719.

Concerning a Theorem of McCoy

4419 [1950, 692]. *Proposed by L. G. Jones, University of Oregon and Alex Rosenberg, University of Toronto*

In a paper, *Remarks on Divisors of Zero*, this MONTHLY [1942, 286–295], N. H. McCoy stated and proved the following theorem: Let $f(x)$ be an element of $R[x]$, where R is a commutative ring. Then $f(x)$ is a divisor of zero in $R[x]$ if and only if there exists a nonzero element c of R such that $cf(x) = 0$.

Show that, if R is not assumed to be commutative, the theorem no longer holds.

Solution by Louis Weiner, University of Chicago. A counter example is provided if for R we choose the set of all 2 by 2 matrices. We set

$$f(x) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then if $\mathbf{c}f(x) = 0$, where $\mathbf{c} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we must have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & a \\ c & c \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} b & b \\ d & d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

whence $a = c = 0$, $b = d = 0$. However if

$$g(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix},$$

we have $g(x)f(x)=0$.

Note. The theorem also fails if R is not assumed to be associative; for let R be the Jordan algebra with basis $1, u_1, u_2$ where $u_1^2=u_2^2=1$, $u_1u_2=u_2u_1=0$, and let $f(x)=u_1x+u_2$. Then if $\mathbf{c}f(x)=0$ where $\mathbf{c}=a+bu_1+cu_2$, we have $\mathbf{c}u_1=au_1+b=0$, $a=b=0$; $\mathbf{c}u_2=au_2+c=0$, $c=0$; $\mathbf{c}=0$. However, if $g(x)=u_2x-u_1$, then $f(x)g(x)=-u_1^2+u_2^2=-1+1=0$.

Also solved by I. H. Rose and by each of the Proposers.

Summation of Tangents

4420 [1951, 42]. *Proposed by F. S. Acton, National Bureau of Standards, Institute for Numerical Analysis*

Evaluate

$$g(m)=\sum_{j=0}^{m-1}(-1)^j\tan\left[\frac{\pi}{2}\left(\frac{2j+1}{2m}\right)\right].$$

I. *Solution by J. B. Rosser, Cornell University, and the Proposer.* Consider the function $f(z)=-2m\tan z/\cos 2mz$. This has poles along the real axis,

$$\begin{cases}\text{from the cosine, at } 2mz=(j+\frac{1}{2})\pi, j=0, 1, \cdots.\\ \text{from the tangent, at } k\pi/2, k=1, 3, \cdots.\end{cases}$$

If we evaluate $[\int_C f(z)\,dz]/2\pi i$, where C is the rectangle in the z-plane with vertices $\pm iy_0$, $\pi\pm iy_0$, the cosine residues give rise to twice $g(m)$ because of the antisymmetry of both tangent and sine with respect to $\frac{1}{2}\pi$. The residue at $\frac{1}{2}\pi$ is $-2m/-\cos 2m(\frac{1}{2}\pi)$ or $2m(-1)^m$.

Because of the periodicity of both tangent and cosine, $f(z)$ is identical for $x=0$ and $x=\pi$, and the contributions along the horizontal parts of C may be made to vanish by removing these sides of the rectangle to $\pm\infty$, causing the whole integral to become zero.

Thus $2g(m)+(-1)^m2m=0$ or $g(m)=(-1)^{m+1}m$.

II. *Solution by F. Underwood, University of Nottingham, England.* Since $\tan\theta=-\tan(\pi-\theta)$, $g(m)$ may be written

$$g(m)=\sum_{k=0}^{m-1}\tan\frac{(4k+1)\pi}{4m}.$$

The equation

$$\tan\alpha=\tan m\theta=\frac{{}_mC_1t-{}_mC_3t^3+\cdots}{1-{}_mC_2t^2+\cdots},\qquad t=\tan\theta, \tag{1}$$

is satisfied if $\tan\theta=\tan[(k\pi+\alpha)/m]$. Hence (1), with $\alpha=\pi/4$, is an equation whose roots are given by the m terms whose sum is $g(m)$. Now equation (1) reduces to

$$t^m+mt^{m-1}+\cdots+(-1)^{m/2}=0,$$
$$t^m-mt^{m-1}+\cdots+(-1)^{(m+1)/2}=0,$$

according as m is even or odd. Accordingly, the sum of the m roots, namely $g(m)$, equals $-m$ or $+m$, respectively. Both cases are included in the result: $g(m)=(-1)^{m+1}m$.

Also solved by Jewell H. Bushey, C. E. Lemke, Max LeLeiko, and O. E. Stanaitis.

Irreducible Polynomials

4421 [1951, 42]. *Proposed by N. C. Arkeny, Princeton University*

Prove that there are infinitely many polynomials of degree n with integer coefficients, the first being unity, which have n real roots and are irreducible in the field of rational numbers.

I. *Solution by William Scott, University of Kansas.* Let p be any prime, and let

$$f(x) = p + \sum_{j=1}^{n} (x - 3jp).$$

Then for $j=0, 1, \cdots, n$,

$$f((3j+1)p) \begin{cases} > 0 & \text{if } n-j \text{ is even,} \\ < 0 & \text{if } n-j \text{ is odd.} \end{cases}$$

Thus $f(x)$ has n real roots. It is irreducible by Eisenstein's criterion.

II. *Solution by P. T. Bateman and L. Schoenfeld, University of Illinois.* If $a_1, a_2, \cdots, a_n$ are any integers such that $a_k \geqq a_{k+1}+2$ $(k=1, \cdots, n-1)$, then

$$f(x) = (x-a_1)(x-a_2)\cdots(x-a_n) - 1$$

is irreducible and has n roots. The irreducibility is well known and easy to prove (Pólya and Szegö, *Aufgaben and Lehrsätzė aus der Analysis*, Part VIII, Problem 121.) To see that $f(x)$ has n real roots suppose $n \geqq 2$ and choose integers $b_0, b_1, \cdots, b_n$ such that

$$b_0 > a_1 > b_1 > a_2 > \cdots > a_{n-1} > b_{n-1} > a_n > b_n.$$

Then if $0 \leqq 2j \leqq n$ the product $(b_{2j}-a_j)(b_{2j}-a_2)\cdots(b_{2j}-a_n)$ is a positive integer greater than unity and thus $f(b_{2j})>0$. Also if $1 \leqq 2j+1 \leqq n$ clearly $f(b_{2j+1})<0$. Hence $f(x)$ has n real roots.

Also solved by Helen C. Arens, Alfred Brauer (two solutions), N. J. Fine, J. B. Kelly, A. H. Kruse, D. R. Morrison, Leo Moser, Morris Newman, P. A. Samet, Olga Taussky-Todd, J. L. Ullman, V. J. Varineau, Morgan Ward, and the Proposer.

The n Kings Problem

4422 [1951, 42]. *Proposed by Jacques Dutka, Rutgers University*

n elements numbered 1, 2, $\cdots$, n, stand in order. In how many ways can

they be rearranged in a line so that no two are together which originally stood together?

Solution by the Proposer. Reference is made to I. Kaplansky, *The Asymptotic Distribution of Runs of Consecutive Elements*, Annals of Math. Statistics, vol. 16 (1945) pp. 200–203. If r denotes the number of instances in which a succession $i, i+1$ or $i+1, i$ occurs in a permutation of $1, 2, \cdots, n$, then the corresponding probability is given by Kaplansky as

$$P(n, r) = \frac{2^r e^{-2}}{r!}\left[1 - \frac{r^2 - 3r}{2n} + \frac{r^4 - 8r^3 + 9r^2 + 22r - 16}{8n(n-1)}\right] + O(n^{-3}).$$

The distribution of r is asymptotically a Poisson distribution, and therefore the answer to the proposed problem may be written as

$$n!P(n, 0) = n!\left\{e^{-2}\left[1 - \frac{16}{8n(n-1)}\right] + O(n^{-3})\right\}.$$

For example, if $n=10$, $n!\,P(n, 0)$ is almost $(10!)(0.132)$.

Note by Leo Moser, University of Alberta. This problem was proposed previously as E 777 [1948, 160]. As noted there, a solution is found by I. Kaplansky in his paper *Symbolic Solution of Certain Problems in Permutations*, Bull. Amer. Math. Soc., vol. 50 (1944), pp. 911–913. This solution, however, is in symbolic form and is not easily expressed by a simple explicit formula. As Kaplansky points out, the problem may be formulated as follows: In how many ways can n kings be placed on an n by n chess board, one on each row and column, so that no two attack each other?

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any other editors or officers of the Association.

Introduction to Number Theory. By Trygve Nagell. New York, John Wiley and Sons Co. 1951. 309 pages. $5.00.

The literature of elementary number theory contains a vast number of topics which the beginning student may study with profit and pleasure. It is therefore not surprising that almost all of the excellent texts which are available on number theory contain important material not to be found in the other texts. The

present book is no exception in this respect. With the characteristic enthusiasm of a devotee of the queen of mathematics the author has provided the reader with a brief introduction to some of the most important achievements of elementary number theory. Among the several results and methods published for the first time in book form is Selberg's elementary proof of the prime number theorem.

An idea of the scope of the book may be obtained from the chapter titles: 1. Divisibility; 2. On the distribution of primes; 3. Theory of congruences; 4. Theory of quadratic residues; 5. Arithmetical properties of the roots of unity; 6. Diophantine equations of the second degree; 7. Diophantine equations of higher degree; 8. The prime number theorem.

The book gives a comprehensive account of classical number theory and includes a number of recent results. Throughout the text historical comments and statements of modern results are liberally interspersed; these are certain to vivify the student's understanding of the subject. There are 180 exercises. Many of these problems are by no means routine or elementary in nature and will challenge the ingenuity of the most ambitious student.

Some of the more notable features of the book are the following. Included in Chapter 1 is Niven's recent (1947) proof of the irrationality of π. In Chapter 3 the important remainder theorem of Aubry-Thue and its generalization by Scholz are proved. The name of Aubry is, however, not mentioned. These theorems are employed in Chapter 6 to derive in an elegant manner a number of theorems about quadratic partitions of primes. In particular the classical result that every prime p of the form $4n+1$ is the sum of two squares is thus established. Chapter 5 contains an interesting introduction to the Gauss sums and cyclotomic polynomials. A theorem of Bauer on the prime divisors of certain polynomials is used to prove that if n is an integer, there are infinitely many primes which are $\equiv -1 \pmod{n}$.

Chapters 6 and 7 on Diophantine analysis constitute an admirable introduction to a difficult subject. After presenting a concise and careful exposition of the classical theory the author, with a master's hand, guides the reader directly to the fringes of unexplored territory. Here theorems associated with the names of Kummer, Maillet, Mordell, Siegel, Skolem, Thue, Weil and the author himself may be found. An outstanding result of Vandiver in connection with Fermat's last theorem is cited. Unfortunately, there is no reference to the name of Vandiver.

On page 52 the statement is made that the Mersenne number $M_{127}=2^{127}-1$ is the largest number known at present to be a prime. This number which enjoyed its position of sovereignty during the entire first half of the present century has this year toppled from its throne. Professor D. H. Lehmer has kindly informed the reviewer that some indefatigable computers have recently unearthed larger primes. In Cambridge, England, the Edsac machine has been used to verify by the converse of Fermat's theorem the primality of the number $180M_{127}^2+1$. In Allier, France, A. Ferrier has used a desk calculator to identify

the number $(2^{148}+1)/17$ as a prime. One recalls with amusement that in 1811 Peter Barlow wrote in his "Theory of Numbers" that " . . . the Mersenne number $M_{31}=2^{31}-1$ is the largest number known at present to be prime, and probably the greatest that ever will be discovered; for, as they are merely curious without being useful, it is not likely that any person will attempt to find one beyond it."

The last chapter is the author's *pièce de résistance.* Here is presented for the first time in any book Selberg's sensational elementary proof of the prime number theorem. The basis of the proof is the following asymptotic formula obtained by Selberg in 1948:

$$\sum_{p \leqq x} (\log p)^2 + \sum_{pq \leqq x} \log p \log q = 2x \log x + O(x),$$

where p and q run over the primes. This formula is itself a consequence of the prime number theorem; Selberg's ingenious proof is, however, completely elementary. From the formula there are several ways to deduce the prime number theorem. The author follows an exposition given by van der Corput and based on notes of some lectures given by Erdös in Amsterdam in 1948. For the sake of historical accuracy the reviewer regrets that the author does not even allude to the role played by Erdös in Selberg's discovery of his first proof. In his original proof Selberg made use of the following result by Erdös: For an arbitrary, positive fixed number δ, there exists a $K(\delta)>0$ and an $x_0=x_0(\delta)$ such that for $x>x_0$, there are more than $K(\delta)x/\log x$ primes in the interval from x to $x+\delta x$.

The appearance of this book has unquestionably solved for many teachers of number theory the perplexing problem of finding a text suited to the needs of the classroom.

A. L. WHITEMAN
University of Southern California

Les Récréations Mathématiques (Parmi les Nombres Curieux). By Victor Thébault. Gauthier-Villars, Paris, 1952. 6+298 pages. 2500 fr.

The author has provided in this volume a compendium of about a hundred fifty arithmetical problems of types already familiar to readers of *Mathesis* and of the Problem Departments of this MONTHLY. A large number of the problems and much of the discussion have been published previously. In particular pp. 188–195 duplicates two recent articles in this MONTHLY, [1949, 443–448], [1951, 101–103], although the results are considerably extended in the pages which follow.

As the title indicates, the problems are mostly recreational in character and deal with curiosities among the integers. Specially featured are: squares, cubes, products, *etc.*, which have unusual digit patterns, or can be written with the digits 0, 1, · · · , 9 used just once each; numbers all of whose powers have the same set of terminal digits; sums of selected elements in addition and multiplication tables; numerical triangles satisfying various requirements; and different

systems of notation in which such properties may be exhibited. For some of the problems fairly complete solutions are provided, for others, briefer hints, and for many more, mere indications of the results. The methods employed are entirely elementary, mainly arithmetic and elementary algebra (and trigonometry where applicable), together with a little number theory (through Fermat's theorem and the solution of the Pell-Fermat equation). The solutions are frequently very ingenious; ingenious, too, is the discovery of so many unusual problems which will yield to the available methods of treatment.

The author makes an eloquent appeal for the significance and use of mathematical recreations both because of the high place accorded them by many of the leading mathematicians and also because of their undoubted value as an instrument for fostering a love and appreciation for mathematics in the developing student mind.

The value of the book would have been enhanced by better organization of material. While it is true that assorted problems are difficult to index and provide with cross-references, yet more could have been done to help the reader locate problems of any desired type. A large number of misprints appear, although they seldom obscure the argument and will confuse the reader for only a moment or two.

Altogether this is a book well worth reading, and it will be an unusual reader who does not return again and again to renew his acquaintance with some problem that holds out a special appeal to him.

Several tables are provided, such as the squares, in the decimal system, having all distinct digits and the squares of six or less digits in all systems of notation from 2 to 12 inclusive. A short presentation of the theory of the Pell-Fermat equation and a note on numerical triangulation by M. A. Buquet are included as an appendix.

E. P. STARKE
Rutgers University

NEW BOOKS RECEIVED

Conformal Mapping. By Zeev Nehari. New York, McGraw-Hill Book Company, 1952. viii+396 pp. $7.50.

Business Mathematics. By W. F. Cassidy and C. C. Robusto. New York, Prentice-Hall, 1952. vii+109 pp. $4.75.

The Design and Analysis of Experiments. By Oscar Kempthorne. New York, John Wiley and Sons, 1952. xix+631 pp. $8.50.

Statistical Calculation for Beginners. By E. G. Chambers. New York, Cambridge University Press, 1952. 168 pp. $2.50.

Mathematical Tables, vol. 1, 3rd ed., Circular and Hyperbolic Functions. By the British Association for the Advancement of Science, issued by the Royal Society. England, Cambridge University Press, 1952. 40+72 pp. $3.50. (Available at American branch of Cambridge University Press, New York.)

Classical Mechanics. By D. E. Rutherford. Edinburgh and London, Oliver and Boyd, 1951. (Available at Interscience Press, New York.) viii+200 pp. 10/6 net.

Partial Differentiation. By R. P. Gillespie. Edinburgh and London, Oliver and Boyd, 1951. (Available at Interscience Press, New York.) viii+105 pp. 6/—net.

Nomography and Empirical Equations. By L. H. Johnson. New York, John Wiley and Sons, 1952. ix+150 pp. $3.75.

A University Algebra. By D. E. Littlewood. London, England, William Heinemann, Ltd., 1950. 8+292 pp. 25s net.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of special features, student papers, bibliographies of program topics, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

Mathematics Club, Columbia College

The *Mathematics Club of Columbia College* presented the following programs during 1950–51:

A theorem of Kronecker, by Mr. Cornelius W. Langley

The gamma function, a series of two lectures by Professor Claude Chevalley

Topology, a series of three lectures by Mr. Marvin P. Epstein

The Laplace transform, by Professor Walter Strodt

Euclidean geometry, by Professor E. R. Lorch

Mathematical models in statistics, a series of two lectures by Mr. Daniel Slotnick.

Officers elected for 1951–52 are: President, Elliot Auerbach; Vice-President and Secretary, David Berkowitz; Vice-President and Treasurer, Charles Kaufman.

Mathematics Club, University of Buffalo

The *Mathematics Club of the University of Buffalo* held six regular meetings during 1950–51. The following papers were presented:

An approximation method for determining dielectric constants, by Roy Hendricks, Cornell Aeronautical Laboratories

A discussion of problem 4401, the American Mathematical Monthly, and a generalization, by Donald O. McKay

A construction problem, by Eleanor S. Krivian

Paradoxes of the infinite as encountered in conditionally convergent series, by Joseph E. Kist

Numbers on a circle, by Michael J. Peinkofer

Mind-reading tricks with numbers, by Donald O. McKay.

An award from the American Chemical Rubber Co. was given to Pearl M. Snitker for outstanding work in freshman mathematics.

Other events included a special program for high school students, a Christmas dinner, and a picnic in June.

Officers elected for 1951–52 are: President, Leonard Schaer; Vice-President, David Kline; Secretary, Irma Rosen; Treasurer, Beverly Dale; Refreshments Committee, Rita Mochan and Jane Riley.

Kappa Mu Epsilon, Central Michigan College

The *Michigan Beta* chapter of *Kappa Mu Epsilon* at *Central Michigan College* held six meetings during 1950–51. The following papers were presented:

Mathematics and games of chance, by Lloyd Trinklein

Mathematics and religion, by Wayne Atkins.

Officers elected for 1951–52 are: President, Richard Little; Vice-President, Paul Buchholz; Treasurer, Donald McPhee; Secretary, Joseph Pipoly.

Pi Mu Epsilon, University of Alabama

The *Alabama Alpha* chapter of *Pi Mu Epsilon* held four program meetings during 1950–51, in addition to a Christmas party, a spring picnic, and two combined business meetings and initiation ceremonies. The following papers were presented:

Methods of reducing curves to unit curves in integration, by Mr. Herbert Diekhans

Statistics, common sense and epistemology, by Professor O. L. Lacey

The origin of mathematics, by Mr. Hassel Palmer

Theorem of Helly, by Mr. B. Vanderburg.

For the third consecutive year, the chapter sponsored a campus-wide competitive examination in mathematics. Prize winners were Donald A. Swenson, Walter B. Mitchell, Billy Letson, J. J. Samuels, and Robert Woltman.

The officers elected for 1951–52 are: Director, Donald A. Swenson; Vice-Director, Ying Victor Wu; Secretary, Betty Campbell; Treasurer, Louis Jaffe; Librarian, Dr. J. D. Mancill; Social Chairman, Elizabeth Cathey.

Pi Mu Epsilon, Michigan State College

The *Michigan Alpha* chapter of *Pi Mu Epsilon* held thirteen meetings during 1950–51, including business meetings, program meetings, initiations, the annual winter banquet, and the spring picnic. Topics presented were as follows:

Dissection problems on squares, by Dr. Stewart

Applications of mechanics to geometry, by Dr. L. M. Kelly

Soap film problem, by James Powell

Mathematics as a career, panel discussion by Dr. Stewart, Dr. Powell, Dr. Wells, Dr. Katz, and Dr. Herzog

The problem of the queens, by Dr. Herzog

Education in Puerto Rico, by Dr. Grove

The modern theory of dimension, by Dr. Nordhaus

Series and sequences, by Professor George Piranian, University of Michigan.

The L. C. Plant Awards, presented annually to the students who have in the past year contributed the most to mathematics through scholarship, interest in mathematics, and help to the mathematics department, were awarded to Charles Parker and R. Douglas Behr.

Officers for 1951–52 are: President, Paul Buben; Vice-President, Douglas Behr; Secretary, Henry Leonard; Treasurer, Charles Parker; Permanent Secretary, James Powell; Faculty Advisors, Dr. L. M. Kelly and Dr. J. E. Powell.

Mathematics Club, Stanford University

The following papers were presented to the *Stanford Mathematics Club* in 1950–51:

Report on the International Congress, by Professor G. Szegö

Some elementary topics in number theory, by Professor C. C. Olds

Some elementary games, by Professor D. Blackwell

Applications of calculus of variations to the vibrating membrane, by Mr. Joseph Carter

Some topics on the theory of large numbers, by Professor J. G. Van der Corput

Some theorems in Boolean algebra, by Mr. Gordon Hedrick

Mechanical linkages, by Mr. William Firey

The Bush differential analyzer, by Mr. Lloyd Bell.

Kappa Mu Epsilon, Southwestern Louisiana Institute

The *Louisiana Beta* chapter of *Kappa Mu Epsilon* was installed at *Southwestern Louisiana Institute*, Lafayette, on May 22, 1951, with Professor Z. L. Loflin acting as installing officer.

The officers are: President, Tuney Arceneaux; Vice-President, Luiz Figueiredo; Secretary, Elizabeth Ann Roberts; Treasurer, Charles Breaux; Corresponding Secretary and Faculty Sponsor, Merlin M. Ohmer.

Undergraduate Mathematics Club, University of Rochester

The following talks were given at the meetings of the *Undergraduate Mathematics Club of the University of Rochester* during 1950–51:

Mathematics and card tricks, by Robert Wadsworth

Topology, by Manuel Paxia

Mathematics in music, by Paul Schaefer

Mathematical education in Germany, by Dietic Gaier, Göppingen, Germany

I. B. M. machines, by Kathleen Berger

Careers in mathematics, by Charles Meadow.

The award for the best talk was given to Paul Schaefer.

Mathematics Club, Oberlin College

The following papers were presented during 1950–51 at meetings of the *Mathematics Club of Oberlin College:*

Problemes de Rencontre, by Benson Scheff

Cryptology, by Marna Kunstman

Something new under the sun, by James Moore

The meaning of curves, by Myra Parker

Infinity and trans-finite numbers, by Jeanne Taylor

Mathematical methods in economics, by Professor Kenneth Roose

Bridge, by Robert Whitney

Communication theory, by Richard Hartong

Perspective in art, by Professor Seymour Slive.

Professor Frank Ebersole spoke at the annual spring banquet on *Symbolic Logic.*

Officers elected for 1951–52 are: President, Robert Whitney; Vice-President, Edward Assmus; Secretary-Treasurer, Marion Goss; Publicity Chairman, John White; Social Chairmen, Elizabeth Doan and Louise Mayne; Faculty Advisor, Professor Alan Anderson.

Kappa Mu Epsilon, Central Missouri State College

The *Missouri Beta* chapter of *Kappa Mu Epsilon* held regular monthly meetings during 1950–51. The following talks were presented by members of the chapter:

Proofs for $\lim_{x\to 0} (\sin x)/x = 1$, by Donald Jones

Congruences, by Leonard Molotsky

Trigonometric functions expressed in terms of exponential functions, by Wendell McGuire

Numbers that mean too much, by Margaret Honn

Pythagorean number triples, by Kyriakos Lypirides

History and methods of computing pi, by Martin Tempel

The gamma function, by Keith Stumpff

Non-uniform motion of a piston, by Charles Sigrist

Report on the eighth biennial convention of Kappa Mu Epsilon, by Keith Stumpff.

Mr. Robert Jones, a student guest speaker, gave an illustrated talk on *Geometry in architecture.* Mr. F. W. Straulman of the Missouri Inspection Bureau was the speaker at the annual banquet; his talk was entitled *Fire insurance rating.*

Officers elected for 1951–52 are: President, Charles Sigrist; Vice-President, Donald Jones; Treasurer, Wendell McGuire; Faculty Sponsor and Corresponding Secretary, Professor Loren W. Akers.

Pi Mu Epsilon, University of California

The following talks were given in 1950–51 at the six program meetings of the *California Beta* chapter of *Pi Mu Epsilon:*

Large scale high-speed computers, by Professor D. H. Lehmer

Inner and outer development of mathematics, by Professor Abraham Seidenberg

Algebraic identities involving sums of squares, by Professor Raphael M. Robinson

Codification of geometry, by Dr. Ting K. Pan

Post algebra, by Mr. Ralph Willoughby

Gambling and mathematics, by Professor Michel Loève.

The officers elected for 1951–52 are: Director, William Noh; Vice-Director, Vivian Koblick; Secretary, Susan Chakmakjian; Treasurer, Newman H. Fischer Jr.; Librarian, Sarah Hallum.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

MEETING OF MATHEMATICS DIVISION OF A.S.E.E.

The Annual Meeting of the American Society for Engineering Education will be held at Dartmouth College, Hanover, New Hampshire, on June 23–27, 1952. The program of the Mathematics Division is as follows:

First Session, June 23, 2:00 P.M., "Operational Calculus for Engineering Students," by I. F. Ritter; "The Straight-Ahead Transport Equation," by J. E. Wilkins, Jr.; "What is Knowledge?", by G. C. Vedova; "The Study of a Cantilever Beam Under an Air Blast," by R. R. Stoll;

Second Session, June 24, 2:00 P.M., "Dual Numbers and Vectors," by W. E. Restemeyer; "Teaching With Color," by W. C. Krathwohl; "Mathematics, Machines and Error Analysis," by K. S. Miller;

Third Session, June 25, 2:00 P.M., Business Meeting of the Division; "The Teaching Aids Reviewing Committee Reports on Mathematics," by E. A. Whitman; "The General Electric Mathematics Fellowship Program for Secondary School Teachers," by E. B. Allen; "Engineering Applications of the Mathematical Theory of Probability," by Samuel Goldberg; "An Integrated Approach to Basic Mathematics," by W. R. Van Voorhis.

Dr. R. S. Burington will speak at the dinner on June 25 on the subject:

"What Approaches Would I Use, Were I to Teach Mathematics Again?"

For further information write to: Professor H. M. Gehman, chairman of the Mathematics Division, University of Buffalo, Buffalo 14, New York; or, to Professor J. H. Zant, secretary of the Division, Oklahoma Agricultural and Mechanical College, Stillwater, Oklahoma.

STATISTICAL SUMMER SESSION

The Department of Statistics and the Statistical Laboratory in cooperation with the Department of Mathematics and the Department of Industrial Engineering of the Virginia Polytechnic Institute will conduct a special statistical summer session, July 29 to August 15, 1952. The program will be for graduate students, research workers, and technicians in government and industry. Special offerings will be given in the statistics of taste testing, bio-assay, sampling and in engineering research and production. For further details write the Department of Statistics, Virginia Polytechnic Institute, Blacksburg, Virginia.

MATHEMATICS INSTITUTE AT THE UNIVERSITY OF HOUSTON

The University of Houston, Texas, will hold its Second Mathematics Institute on July 28–31, 1952. This Institute is being sponsored by the College of Education, the Mathematics Department of the College of Arts and Sciences, and the College of Business Administration.

Specialists in the field of mathematics teaching who will participate in the Institute program are: Dr. Edwina Deans, elementary supervisor, Arlington, Virginia; Dr. H. F. Spitzer, principal of the University Elementary School and associate professor of education, University of Iowa; and Dr. J. R. Mayor, chairman of the Department of Education and associate professor of mathematics, University of Wisconsin. Mr. Martin Wright, associate professor of mathematics, University of Houston, will address one of the general sessions. A mathematics laboratory for teachers of junior and senior high school mathematics will be directed by Miss Ida May Bernhard, supervisor, College Laboratory School, Southwest Texas State Teachers College, San Marcos, Texas.

All meetings will be held in air-conditioned rooms; residence can be secured in air-conditioned dormitories.

For further information, write Joyce Benbrook, Box 554, University of Houston, Houston 4, Texas.

MATHEMATICS INSTITUTE AT LOUISIANA STATE UNIVERSITY

Louisiana State University will hold its third annual session of the Mathematics Institute on June 15–21, 1952.

There will be discussion groups in algebra, geometry, arithmetic, junior high school mathematics and enrichment materials. Also, there will be a geometry laboratory. The discussions will be led by experts in these fields. In addition there will be lectures given by outstanding people in mathematics and related fields.

Excellent rooms and meals will be provided on the campus at reasonable rates. Plan a June trip to interesting Louisiana spending one week to your profit and enjoyment on the beautiful campus of Louisiana State University.

For programs and other information write to: Dr. H. T. Karnes, Director, Department of Mathematics, Louisiana State University, Baton Rouge 3, Louisiana.

SYLLABUSES FOR COURSES IN MODERN NUMERICAL ANALYSIS

Graduate courses in modern numerical analysis are being given with the assistance of the National Bureau of Standards personnel at the American University, Washington, D. C. and at the University of California in Los Angeles. Syllabuses for these courses may be obtained by writing to: Chief of the Computation Laboratory, National Bureau of Standards, Washington 25, D. C., or to Director of Research, Institute for Numerical Analysis, National Bureau of Standards, U.C.L.A., Los Angeles 24, California.

THE JOURNAL OF RATIONAL MECHANICS AND ANALYSIS

The publication of a new journal entitled *Journal of Rational Mechanics and Analysis* was announced recently. The Editors request that subscriptions and inquiries regarding the *Journal* be sent to them at the following address: Graduate Institute for Applied Mathematics, Indiana University, Bloomington, Indiana. The subscription price is $18.00 per volume; for those engaged in teaching or research the rate is $6.00 per volume.

SUMMER COURSES

The following institutions announce advanced courses in mathematics for the summer of 1952:

Boston University. May 26 to July 5: Professor Scheid, numerical methods; Professor Johanson, differential equations. July 7 to August 16: Professor Giever, elementary theory of numbers: Dr. Browder, introduction to modern mathematics; Dr. Pennington, Cambridge University, divergent series.

New York University, School of Education. June 30 to August 8: Dr. Dodes, application of mathematics, teaching aids in mathematics education; Professor Kinsella, current trends in mathematics curricula, testing and evaluation in mathematics education.

Northwestern University. June 20 to August 23: Lecturer Wyler, non-Euclidean geometry; Professor Boothby, introduction to analytic topology, engineering mathematics II; Professor Springer, differential equations, functions of a complex variable; Professor Wescott, theory of statistics; Professor Zelinsky, algebra of matrices and quadratic forms, introduction to modern algebra; Professor Scott, definite integrals, seminar in analysis; Professor Simmons, theory of equations; Professor Davis, history and teaching of mathematics, nonlinear problems of the physical sciences; Professor Moulton, engineering mathematics I.

University of California, Berkeley, Department of Mathematics. June 23 to August 2: Professor McDonald, coordination in mathematics for secondary schools.

University of California, Berkeley, Statistical Laboratory. June 23 to August 2: Professors Neyman and Fix and Mr. LeCam, statistical methods of searching for causal relationships; Dr. Kallianpur, elements of probability and statistics; Professor Neyman, individual research leading to higher degrees. August 4 to September 13: Professor Bates, second course in probability and statistics, applications of mathematical statistics; Professor Neyman, individual research leading to higher degrees.

University of Illinois. June 13 to August 9: Staff, differential equations, advanced calculus, fundamental concepts of mathematics, introduction to higher geometry, introduction to higher analysis, measure and integration, mathematical methods in engineering and science; Professor Atchison, algebraic geometry; Professor Bourgin, topological spaces; Professor Brahana, linear algebra; Professor Ketchum, theory of functions of a complex variable.

University of Nebraska. June 2 to July 25: Professor Doole, advanced Euclidean geometry; Professor Camp, elementary differential equations; Professor Ribeiro, solid analytic geometry; Professor Basoco, topics in analysis.

University of South Carolina. June 10 to August 12: Professor Lytle, statistics, theory of equations; Professor Weber, college geometry; Professor Hedberg, higher algebra; Professor Williams, functions of a complex variable.

PERSONAL ITEMS

Professor J. E. Davis, Drexel Institute of Technology, and Professor Marguerite Lehr of Bryn Mawr College were appointed to serve as delegates of the Association at the meeting of the American Academy of Political and Social Science which was held in Philadelphia, Pennsylvania, on April 18–19, 1952.

Professor J. M. Earl of the University of Omaha represented the Association at the inauguration of President C. H. Canfield of Tarkio College on March 27, 1952.

Brooklyn College announces the following: Assistant Professor A. W. Landers has been promoted to an associate professorship; Mr. Leon Herbach has been appointed to an instructorship.

Gonzaga University reports: Assistant Professor D. R. Ryan has been promoted to an associate professorship; Instructors J. J. Murray and G. M. Simoneau have been promoted to assistant professorships; Mr. J. A. Ryan, previously a student at the University, has been appointed to an instructorship.

Haverford College announces the following appointments: Assistant Professor R. C. James, formerly of the University of California at Berkeley, to an associate professorship; Dr. A. F. Strehler of the University of Chicago to an instructorship.

Kenyon College announces: Dr. D. T. Finkbeiner, formerly an instructor at Yale University, has been appointed to an associate professorship; Professor

W. R. Transue is on leave of absence and is in Italy on a Fulbright grant.

At Purdue University: Associate Professor Michael Golomb has been promoted to a professorship; Assistant Professor Sam Perlis has been promoted to an associate professorship; Mr. E. L. Klinger has been promoted to an assistant professorship; Dr. Meyer Jerison, previously research instructor at the University of Illinois, Dr. J. H. B. Kemperman, formerly research worker at Mathematisch Centrum, Amsterdam, Dr. C. R. Putnam, previously a member of the Institute for Advanced Study, and Assistant Professor H. E. Teicher of the University of Delaware have been appointed to assistant professorships.

Union College announces the following: Dr. D. S. Morse, chairman of the Department of Mathematics, has been appointed to the recently established Marie Louise Bailey professorship in mathematics; Associate Professor C. T. Male, Sr., has been promoted to a professorship; Mr. W. C. Stone has been appointed to an assistant professorship.

United States Naval Postgraduate School, now located at Monterey, California, makes the following announcements: Associate Professors W. E. Bleick, Joseph Giarratana, A. B. Mewborn, and C. C. Torrance have been promoted to professorships; Assistant Professors F. D. Faulkner, Walter Jennings, and F. M. Pulliam have been promoted to associate professorships; Dr. T. E. Oberbeck, formerly with the Operations Evaluation Group, Division of Industrial Cooperation, Massachusetts Institute of Technology, has been appointed Associate Professor of Mathematics and Mechanics.

At the University of California at Berkeley: Associate Professor A. L. Foster has been promoted to a professorship; Assistant Professor E. L. Lehmann, who is on leave of absence during 1951–52, has been promoted to an associate professorship; Instructors Evelyn Fix and Elizabeth L. Scott have been promoted to assistant professorships; Dr. Jan Kalicki and Dr. A. V. Martin have been appointed to visiting assistant professorships; Dr. Gopinath Kallianpur has been appointed Lecturer; Dr. Harley Flanders has been appointed to an instructorship; Professor B. A. Bernstein has retired with the title of Professor Emeritus; Assistant Professors S. P. Diliberto and J. L. Hodges are on leave of absence during 1951–52.

University of Chicago announces the following: Dr. Erwin Kleinfeld, previously an assistant at the University of Wisconsin, has been appointed to an instructorship; Professor Andre Ammann of the University of Geneva, Switzerland, Dr. George Austin, previously National Research Council Fellow, Dr. Leila A. Dragonette of the University of Pennsylvania, Professor Lino Gutierrez of the University of Habana, Mr. Mischa Cotlar, Guggenheim Fellow from the University of Buenos Aires, and Dr. Israel Herstein of the Cowles Commission have been appointed Research Associates in Mathematics; Associate Professor P. R. Halmos is on a year's leave of absence and is spending it at the University of Montevideo, Institute of Mathematics and Statistics, as Visiting Lecturer under the auspices of the United States Department of State; Professor M. H. Stone is Visiting Lecturer for the American Mathematical Society during 1952–

53; Assistant Professor I. E. Segal is on a year's leave of absence and is on a Guggenheim Fellowship at the Institute for Advanced Study; Assistant Professor E. H. Spanier was at the Institute for Advanced Study during the Autumn Quarter of 1951.

University of Cincinnati announces the promotion of Associate Professor Meyer Salkover to a professorship and the promotions of Assistant Professors J. L. Baker and W. E. Restemeyer to associate professorships.

At the University of Kentucky: Instructor M. S. Davis has been promoted to an assistant professorship; Graduate Assistants S. B. Vanaman, Jr., and W. M. Zaring have been promoted to part-time instructorships; Mr. H. W. Burnette of Alabama Polytechnic Institute and Mr. J. B. Wells, Jr., have been appointed to part-time instructorships.

University of Tennessee announces the appointments of Mr. H. S. Davis and Mr. R. D. Sheffield to teaching assistantships.

University of Wisconsin makes the following announcements: Dr. C. W. Curtis, previously a graduate student at Yale University, and Dr. E. R. Immel have been appointed to instructorships.

Mr. L. J. Abbeduto is teaching in the Hardin County Schools, Kenton, Ohio.

Associate Professor G. F. Allison of the University of New Brunswick is now at Ryerson Institute, Toronto, Canada.

Dr. E. L. Arnoff has a position as Aeronautical Research Scientist with the National Advisory Committee for Aeronautics, Cleveland, Ohio.

Dr. Sholom Arzt of New York University has a position as Mathematician at the Applied Physics Laboratory of Johns Hopkins University, Silver Spring, Maryland.

Assistant Professor Miriam C. Ayer of Wellesley College has received an appointment as Mathematician at Sandia Corporation, Albuquerque, New Mexico.

Dr. H. G. Ayre, professor of mathematics and director of the General College Division at Western Illinois State College, has been promoted to Head of the Department of Mathematics and is continuing in the position as Director of the General College Division.

Assistant Professor Frederick Bagemihl has resigned from the University of Rochester.

Mr. L. E. Berg is teaching in Fairmount, Indiana.

Assistant Instructor R. N. Bradt of the University of Kansas has been appointed to a graduate assistantship at Stanford University.

Mr. Robert Bryant, formerly an instructor at Miami University, has accepted a position at the Naval Research Laboratory, Washington, D. C.

Mr. C. M. Callahan, formerly a teaching fellow at Alabama Polytechnic Institute, has been appointed to an instructorship at Lincoln Memorial University.

Mr. R. E. Carroll of Emory University has a position as Dynamics Engineer at Bell Aircraft Corporation, Niagara Falls, New York.

Mrs. Virginia L. Chatelain has been appointed to an instructorship at Adela Hale Business Career School, Hutchinson, Kansas.

Professor Emeritus A. B. Coble of the University of Illinois has been appointed Visiting Professor at the University of North Carolina for the winter and spring quarters of 1951–52.

Mr. D. L. Daly of the University of Kentucky has been appointed to an instructorship at the University of Maine.

Mr. D. J. Ewy has been appointed to an instructorship at Fresno State College.

Dr. W. T. Fishback of the University of Vermont has been promoted to an assistant professorship.

Mr. M. P. Guhsé, formerly a graduate student at the University of Massachusetts, is teaching in St. Lucie County High School, Ft. Pierce, Florida.

Professor Vaclav Hlavaty of Indiana University has been appointed to the staff of the Graduate Institute for Applied Mathematics of the University.

Mr. J. S. Hokanson of Ohio University has been appointed to an instructorship at Ripon College.

Dr. J. H. Hornback has been appointed to an assistant professorship at the University of Alabama.

Mr. D. B. Houghton has been promoted to the position of Acting Chief of the Analysis Section, Franklin Institute, Philadelphia, Pennsylvania.

Acting Professor S. W. Howell of Yankton College has been promoted to the position of Professor of Mathematics and Physics and Head of Department of Mathematics.

Mr. D. B. Jordan, formerly a teaching fellow at Polytechnic Institute of Brooklyn, has a position as Junior Engineer at Sylvania Electric Products, Incorporated, Bayside, New York.

Mr. R. J. Lambert of Iowa State College has a position as Mathematician in the Defense Department, Washington, D. C.

Dr. D. H. Lehmer, formerly professor of mathematics at the University of California at Berkeley, has been appointed Director of Research of the Institute for Numerical Analysis, National Bureau of Standards, Los Angeles, California.

Miss Margaret M. Linney, formerly a student at Regis College, is now Engineering Aide at the General Electric Company, Schenectady, New York.

Dr. W. G. Lister has been awarded a post-doctoral AEC Fellowship and has been spending the academic year 1951–52 at Yale University.

Assistant Professor R. W. Long of Washington and Jefferson College has been appointed Lecturer at the University of Pittsburgh.

Dr. Nathaniel Macon, who was a Fulbright Fellow at the University of Amsterdam, the Netherlands, during 1950–1951, has been appointed to an assistant professorship at Alabama Polytechnic Institute.

Mr. Paul Meier has accepted a position at the Forrestal Research Center, Princeton, New Jersey.

Mr. D. F. Mela has received an appointment as Mathematician in the Operations Evaluation Group, Navy Department, Washington, D. C.

Mr. A. J. Mortola, formerly a tutor at City College of the City of New York, is teaching at St. Peter's College, Jersey City, New Jersey.

Associate Professor F. C. Mosteller of Harvard University has been promoted to a professorship.

Professor S. B. Myers of the University of Michigan is serving as Acting Head of the Department of Mathematics during the second semester of 1951–52 while Professor T. H. Hildebrandt is on leave of absence.

Associate Professor P. B. Norman of Wagner Memorial Lutheran College has accepted a position as Project Engineer, Curtiss-Wright Corporation, Carlstadt, New Jersey.

Miss Frances A. Norton, formerly an Instructor at Alabama Polytechnic Institute, has a position with the Hayes Aircraft Corporation, Birmingham, Alabama.

Dr. F. D. Parker of Case Institute of Technology has been appointed to an assistant professorship at St. Lawrence University.

Mr. V. E. Roberts, formerly a mathematics teacher in Fairbanks High School, Alaska, has been appointed to an instructorship at the University of Alaska.

Mr. G. B. Robinson, previously a teaching fellow at Cornell University, has been appointed to an instructorship at the University of Connecticut.

Mr. L. L. Ross of Ohio Northern University has accepted a position with Babcock and Wilcox Company, Barberton, Ohio.

Miss Jean E. Sammet, formerly an assistant at the University of Illinois, has a position as a computer for the Metropolitan Life Insurance Company, New York City.

Mr. R. C. Seber of Rockford College has been appointed to an assistant professorship at Wisconsin State College.

Mr. H. A. Seebald has a position as Methods Engineer with Remington Rand, Incorporated, New York City.

Assistant Professor Edward Silverman of Kenyon College is a staff member of Sandia Corporation, Albuquerque, New Mexico.

Mr. Sam Stallard, formerly lecturer at the University of British Columbia, has been appointed Lecturer at the University of New Brunswick.

Instructor Irwin Stoner of the University of Minnesota is now Assistant Scientist at the Rosemount Research Center of the University.

Instructor R. T. Tear of Rensselaer Polytechnic Institute has been recalled to active service in the United States Air Force.

Instructor J. G. Wendel of Yale University has accepted a position with the Rand Corporation, Santa Monica, California.

Mr. H. W. Wessely, previously a student at Illinois Institute of Technology,

has a position as Physicist with the Gaertner Scientific Corporation, Chicago, Illinois.

Assistant Professor G. M. Wing of the University of California at Los Angeles has been on leave of absence during 1951–52 and has been a staff member of the Los Alamos Scientific Laboratory, New Mexico.

Mr. R. G. Woodle, Jr., of the Missouri School of Mines and Metallurgy has been appointed to an instructorship at the University of Kansas.

Dr. R. A. Worsing of Iowa State College has joined the Physical Science Division of Boeing Airplane Corporation, Seattle, Washington.

Dr. F. H. Young, who was an AEC Research Fellow at the University of Oregon, is now at the Bureau of Ordnance, United States Navy, Washington, D. C.

Professor Emeritus W. B. Fite of Columbia University died on March 1, 1952. He was a charter member of the Association.

Professor F. G. Graff of Oberlin College died on February 22, 1952.

Dr. C. O. Lampland of the Lowell Observatory died on December 14, 1951. He was a charter member of the Association.

Professor V. S. Lawrence, Jr., of Virginia Polytechnic Institute died on February 20, 1952.

Professor Emeritus C. L. Poor of Columbia University died on September 27, 1951.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE EARLE RAYMOND HEDRICK LECTURES

Professor Tibor Rado of the Ohio State University will deliver a series of three expository lectures entitled "Derivatives and Jacobians" as the EARLE RAYMOND HEDRICK LECTURES at the meeting of the Association in East Lansing, Michigan on September 1 and 2, 1952. These lectures have been named in honor of Professor Earle Raymond Hedrick, one of the three organizers of the Association and its first President.

The Board of Governors has established the EARLE RAYMOND HEDRICK LECTURES in an effort to promote expository lectures and publications as significant activities of the Association.

Professor Rado's first lecture will be given at 4 P.M. on Monday, September 1. This lecture, to be devoted to a treatment of Jacobians, is intended to be an exposition which will be readily understood by anyone who is familiar with

advanced calculus and vector analysis as they occur in undergraduate courses.

The second lecture will be given at 9 A.M. on Tuesday, September 2. The first half of this lecture will be a continuation of the first lecture. The second half will be a description of relative Cech cohomology groups as the essential topological tool that is needed in the n-dimensional case. Professor Rado states that he has found a simple treatment of cohomology theory, and he hopes through these lectures to introduce this theory to the analysts for use in their work.

The third lecture will be given at 7:30 P.M. on Tuesday, September 2. This lecture, on a more advanced level than the other two, will be concerned with the transformation theory of multiple integrals in terms of the generalized Jacobians treated in the other two lectures.

Professor Rado states that the material of the three lectures will be published as a monograph of about two hundred pages. The manuscript will be ready for the printer in January, 1953.

The EARLE RAYMOND HEDRICK LECTURES for 1952 have been arranged by a special committee consisting of G. C. Evans, J. C. Oxtoby, and G. B. Price, chairman. At a later date this committee will recommend to the Board of Governors of the Association policies which will govern these lectures in future years.

PREPARATION OF PROBLEM AND SOURCE MATERIALS FOR THE MATHEMATICAL TRAINING OF SOCIAL SCIENTISTS

As readers of this MONTHLY probably know, a Committee on the Mathematical Training of Social Scientists has been at work for some time. The Committee includes representatives of twelve learned societies including the Mathematical Association of America.

As the result of a suggestion from this Committee, the Social Science Research Council is now sponsoring a small group to work during the summer of 1952. This group will attempt to compile from the literature of the various social sciences lists of problems, extracts from sources, and references to sources that illustrate varieties of uses of mathematics in the social sciences. These compilations are expected to serve a number of important ends—e.g., to provide mathematicians with material for use in texts and courses designed for social scientists, to indicate the general dimensions of the mathematical training appropriate for students of the social sciences now and in the future, and to facilitate the study of mathematics by social scientists for whom organized courses are not available.

This Committee believes that the group referred to would find it most helpful if it could start with a wide variety of suggestions from the various areas concerned. A general appeal for such suggestions is hereby made. They should be sent to Professor William G. Madow, Chairman, Committee on the Mathematical Training of Social Scientists, University of Illinois, Urbana, Illinois, as soon as possible.

Although the Committee does not wish to limit the suggestions to specific types of material, it would prefer greater emphasis on materials relating to the use of mathematics in the social sciences themselves than on those relating to statistics, since the materials necessary for statistics are better known. Finally, the Committee would appreciate learning where programs of mathematical training intended for social scientists are now in existence or in process of development, and where mathematics at the level of the calculus or higher is required for undergraduate or graduate degrees in the social sciences or may be substituted for another requirement for a degree in a social science.

THE MAY MEETING OF THE INDIANA SECTION

The twenty-eighth annual meeting of the Indiana Section of the Mathematical Association of America was held at Butler University, Indianapolis, Indiana, on Saturday, May 5, 1951. Two sessions were held at which Professor H. E. Crull of Butler University, Chairman of the Section, presided.

There were fifty-five in attendance including the following forty members of the Association:

W. C. Arnold, W. L. Ayres, Juna L. Beal, Stanley Bolks, C. F. Brumfiel, G. E. Carscallen, K. W. Crain, H. E. Crull, M. W. De Jonge, W. E. Edington, P. D. Edwards, J. L. Ericksen, R. A. Gambill, E. L. Godfrey, Michael Golomb, S. H. Gould, G. H. Graves, J. R. Hadley, J. F. Heyda, M. W. Keller, E. L. Klinger, Jacob Korevaar, P. J. McCarthy, Gladys B. McColgin, P. T. Mielke, J. T. Montgomery, P. W. Overman, M. O. Peach, J. C. Polley, Arthur Rosenthal, Sister Gertrude Marie, J. D. Speas, R. B. Stone, Anna K. Suter, A. D. Talkington, Jane A. Uhrhan, F. J. Wagner, M. S. Webster, K. P. Williams, H. E. Wolfe.

The following officers were elected: Chairman, M. W. Keller, Purdue University; Vice-Chairman, Florence Long, Earlham College; Secretary-Treasurer, J. C. Polley, Wabash College.

The chairman announced the election of Professor J. C. Polley of Wabash College to the Board of Governors as Sectional Governor for the three year period July 1, 1951, to June 30, 1954.

Professor P. D. Edwards, Chairman of the Committee on Awards, reported that the committee had awarded five Association medals, four for high mathematical achievement in the Indiana science talent search and one to the top ranking contestant in the final comprehensive contest in mathematics of the Indiana high school achievement program sponsored by Indiana University.

The annual meeting of 1952 will be held at Indiana University, Bloomington, Indiana, at a date to be announced later.

The following papers were presented:

1. *The geometry of ideal gas flows*, by Mr. J. L. Ericksen, Indiana University.

The Mach number, m, of a stationary, isoenergetic ideal gas flow in the plane was shown to satisfy a cubic equation of the form $\sum_{n=0}^{n=3} f_n(m^2-1)^n$, where the functions f_n are expressible in terms of the adiabatic exponent, the curvature of the streamlines, the curvature of the orthogonal trajectories of the streamlines, and certain derivatives of these curvatures. It is possible to obtain analytic conditions to determine whether or not a given congruence of curves can be streamlined for such a flow.

2. *Continuation of the sequence addition, multiplication, exponentiation, . . .*, by Professor S. H. Gould, Purdue University.

Since multiplication is addition with equal summands and exponentiation is multiplication with equal factors, it is natural to define a fourth operation, namely, exponentiation with equal exponents, and then a fifth, and so forth. The higher operations have been neglected on the ground that they are neither associative nor commutative. In the present note these properties were replaced by corresponding inequalities, leading to generalization of some familiar theorems. A compact notation was devised and unsolved problems were suggested. It was pointed out that the notation makes it easy to write numbers which are strikingly large.

3. *Q scores, a correlation study*, by Sister Gertrude Marie, Marian College.

The aim of this paper was to evaluate the mathematics scores of ten consecutive classes of college women in the National Sophomore Testing Program as a measure of mathematical ability and/or achievement. These scores were correlated with: (1) *Q* scores and *I.Q.*'s as given by the American Council of Education Psychological Examination; (2) science, reading comprehension, and total scores on the Cooperative General Culture and English Tests of the National Sophomore Testing Program; (3) high school and college mathematics achievement as indicated by number and kind of courses taken and class marks received.

4. *Linear dependence and the Wronskian*, by Professor M. S. Webster, Purdue University.

The purpose of this paper was to discuss the theorem "The functions of a set are linearly dependent if, and only if, their Wronskian is identically zero." Although the theorem is in general false, it is found in several texts including two published in the period 1940–1950. Peano and Bôcher recognized that the theorem is not valid and Bôcher published several correct theorems related to the false one. Theorems of this type and a necessary and sufficient condition involving the Gramian were discussed.

5. *Student and teacher*, by Professor G. H. Graves, Purdue University.

Since specialization implies cooperation, mathematicians and educators should work in close collaboration. The graduate student is particularly the victim of misunderstanding between these two groups. The relation between teacher and student is a personal one, not purely professional. The teacher must be profoundly interested in the development of the student's capabilities. When a student fails, the teacher fails with him. Suggestions were made regarding use of class meetings, conferences, psychological aids to study, administration of tests, and the subject of grading. The opinion that the teacher's objectives should include making himself unnecessary was expressed.

6. *The use of a fourth property of a right triangle in teaching mathematics*, by Professor E. L. Godfrey, Defiance College.

The following theorem is a natural addition to the corollaries concerning the lengths of the sides and the perpendicular to the hypotenuse of a right triangle which are included in the standard textbook presentation of Euclid's eighth proposition of book six. Corollary: The rectangle contained by the sides of a right triangle is equal to the rectangle contained by the base and the perpendicular drawn from the vertex of the right angle to the base. The formula for the altitude suggested by this property was shown to be useful frequently in college mathematics. Since the proof is obvious from a consideration of areas instead of similar triangles, it was suggested as a suitable and useful exercise to introduce in junior high school courses in arithmetic, algebra and/or general mathematics.

7. *On the zeros and asymptotic behavior of Bessel functions*, by Professors

Michael Golomb and Jacob Korevaar, Purdue University, and presented by Professor Korevaar.

In this paper completely elementary methods were used to derive the common results concerning the real and non-real zeros and the asymptotic behavior of Bessel functions; some new results were given on the location of the non-real zeros. No particular representations of the Bessel functions by power series, definite integrals or otherwise were used. For each problem a form of Bessel's differential equation was used which best suited the specific problem. The results then followed readily from the definition of Bessel functions as solutions of the differential equation and were therefore applicable to any (real-valued) solution of this equation, whether it be a Bessel function of the first kind, of the second kind, or a linear combination of such functions.

8. *Newtonian pattern of analysis*, by Professor M. O. Peach, University of Notre Dame.

The concepts of Newtonian mechanics are often extended to phenomena which are essentially non-mechanical. When this is done it should be done consciously and with precision if pitfalls are to be avoided. The author raised the general question: To what extent and under what conditions can non-mechanical phenomena be analyzed according to the pattern of Newtonian mechanics? Precise conditions were laid down for the introduction of the concept of force. A generalized definition of force, applicable to both mechanical and non-mechanical phenomena, was given. Necessary and sufficient conditions for the validity of the parallelogram law for adding forces and of Newton's third law were deduced. An application of these ideas to the theory of dislocations was briefly sketched. The incomplete nature of the answer given to the above question was emphasized, and several directions in which research of potential mathematical interest could be undertaken were pointed out.

9. *A theorem on improper integrals in abstract spaces*, by Professor P. T. Mielke, Wabash College.

In this paper the notion of improper integral is that used by H. Hahn and A. Rosenthal in *Set Functions*, the University of New Mexico Press, 1948. The theorem proved is one of several contained in the author's doctoral thesis, *Improper Integrals in Abstract Spaces*. The space E is to be considered a general space without any particular topological or metric structure; M a σ-field of subsets of E, $\phi(M)$ a totally additive set function in M which is assumed complete for ϕ. The function $f(x)$ is a point function, ϕ-measurable on a set $A \in M$. Theorem: If $\phi(A)$ is infinite and $f(x)$ is ϕ-summable on A, then, given $\epsilon>0$, there exists a $\bar{\delta}>0$ such that for every δ-scale $\{z_i\}$ with $\delta \leqq \bar{\delta}$

$$\left| L(f, \phi, \{z_i\}, A_\delta) - (A) \int f d\phi \right| < \epsilon$$

where $A_\delta = A[|f(\hat{x})| > \delta]$ and $L(f, \phi, \{z_i\}, A_\delta)$ is any of the four Lebesgue sums for f, ϕ, $\{z_i\}$, and A_δ.

Professor J. B. Irwin of the Department of Astronomy of Indiana University delivered by invitation an address entitled *The Possibilities of the Electronic Computer in Astronomical Research*. In this he discussed the nature and construction of recently developed computers and presented certain problems in astronomy to illustrate the possible use of the electronic computer in astronomical research.

J. C. POLLEY, *Secretary*

JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The fifth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at the State College of Washington, Pullman, Washington, on June 15, 1951, in conjunction with the four hundred seventy-first meeting of the American Mathematical Society. Professor A. F. Moursund, Chairman of the Section, invited Professors R. M. Winger and R. D. James to share the duties of presiding at the afternoon session.

There were fifty persons in attendance, including the following thirty-six members of the Association:

H. A. Antosiewicz, T. M. Apostol, R. W. Ball, J. P. Ballantine, R. A. Beaumont, J. S. Biggerstaff, Z. W. Birnbaum, J. L. Botsford, J. L. Brenner, L. G. Butler, Harold Chatland, P. A. Clement, K. L. Cooke, C. M. Cramlet, D. B. Dekker, Howard Eves, K. S. Ghent, J. W. Green, F. L. Griffin, Mary E. Haller, I. M. Hostetter, H. H. Irwin, R. D. James, J. M. Kingston, M. S. Knebelman, R. B. Leipnik, J. J. Livers, A. T. Lonseth, R. E. Lowney, L. H. McFarlan, A. F. Moursund, D. C. Murdoch, Ivan Niven, L. J. Paige, J. R. Vatnsdal, R. M. Winger.

A business meeting was held in the evening at which the following officers were elected: Chairman, Professor R. D. James, University of British Columbia; Vice-Chairman, Professor A. S. Merrill, University of Montana; Secretary-Treasurer, Professor J. M. Kingston, University of Washington.

It was decided to hold the next meeting of the Section at the University of Oregon, Eugene, Oregon, on June 20, 1952, and to make it, if possible, a joint meeting with the Society. A committee consisting of Professors Murdoch (Chairman), Eves, and Leipnik was appointed to investigate the possibility and advisability of conducting mathematical contests among high school students in this area. Professors Niven (Chairman) and Brenner, and Dr. Ball were appointed as a Program Committee to organize next year's program in the direction of providing greater usefulness to high school teachers. Additional problems in mimeographed form were made available by the Committee on a Problem Book in Mathematics.

The afternoon session consisted of the following eight fifteen minute papers and an invited hour address.

1. *The separation of a proper fraction into two unit fractions*, by Professors Howard Eves, Oregon State College, and H. Ohbayashi, Nagoya University, Japan. Presented by Professor Eves.

This paper summarizes much existing information on this interesting problem and considers some applications to conjectures concerning the method of solution of the problem by the ancient Egyptians. A proof is given of a result of Ohbayashi to the effect that a proper fraction p/q, $(p, q) = 1$, is separable into two unit fractions if and only if there exists an integer x such that $x \mid q'$ and $x^2 \equiv -q' \pmod{p}$, where $q' \mid q$. Distinct values of x lead to different unique separations.

2. *Remarks on definition and notation used in calculus*, by Professor R. B. Leipnik, University of Washington.

Although clear, correct definitions and consistent notations are available for the concepts introduced in elementary calculus, they are often avoided. Some unconventional definitions and no-

tations for functions, derivatives, differentials, etc., are described, and their advantages discussed, with the hope that they might be more widely adopted.

3. *Equivalence relations and canonical forms*, by Professor R. A. Beaumont and Dr. W. R. Ball, University of Washington. Presented by Dr. Ball.

The authors suggest that the idea of an equivalence relation on a set S and the corresponding partitioning of S should be emphasized as a fundamental algebraic concept in upper division algebra courses. A study of groups of transformations comes early in such a course, and since every group of transformations on a set S defines an equivalence relation, the latter concept may be introduced in a natural way at that time. The discussion of the representatives of equivalence classes leads to a simple and precise explanation of what is meant by canonical forms. Topics such as the reduction of matrices to canonical forms, congruence of integers, cosets of a group, and residue classes of a ring are unified by a careful study of equivalence relations, and the study itself can be used as an example of the process of abstraction. Numerous examples of equivalence relations from elementary mathematics may be given which clarify the earlier material and explain the concept to the student.

4. *Some recent work in the theory of discrete groups*, by Professor J. L. Brenner, State College of Washington.

This is an expository summary of some definitions and results.

5. Invited address: *On estimating a probability distribution from a finite number of observations*, by Professor Z. W. Birnbaum, University of Washington.

The probability distribution of a random variable X is known as soon as the distribution function $F(x) = \text{Prob}\{X \leqq x\}$ is given. In practical applications it often happens that $F(x)$ is not known, but one has the means to obtain a set of N independent observations (a sample of size N) of X. The practical problem then arises: is it possible to utilize the information contained in such a finite sample to obtain an estimate of $F(x)$? In 1933, A. Kolmogorov gave a rather surprising answer to this question. His results initiated a number of further publications, and the present paper is a report on developments in this field, including some very recent findings.

6. *Transversal families of curves*, by Professor L. H. McFarlan, University of Washington.

The transversality condition of the calculus of variations in the plane is applied to pairs of one parameter families of curves in order to obtain a generalization of the idea of orthogonal families of curves. Given a one parameter family of curves, an integrand function is found such that each member of the given family is transversal to every member of a second one parameter family of curves.

7. *Modular invariants associated with resistance networks*, by Professor C. M. Cramlet, University of Washington.

A graph of V vertices, and B connecting branches, supplies a topological space S for a finite geometry. The chain, circuit, coordinate system (C S) are defined geometric objects. A proper (C S) is a set of $N = B - V + 1$ linearly independent proper circuits (single closed loops). Any two (C S) are related by a transformation matrix $t \in T$ having elements $0, 1, -1$ and determinant ± 1. Products of the elements of T generate a group G, matrices of integers with determinant ± 1. Additional structure is endowed S by associating a resistance matrix $r_{\alpha\beta}$ and mesh currents i^{α} with a

preferred ($C\,S$) and extending those to extended ($C\,S$) of G by the assumption that these are covariant and contravariant respectively. *A property of S* is defined by the vanishing of a representation of G. In a similar manner a geometry may be defined by replacing G by the matrices of G reduced modulo 2. The symmetric tensor $r_{\alpha\beta}$, $N=2$, has a complete set of three independent invariants,

$$I = r_{11} + r_{12} + r_{22}, \qquad J = r_{12}^2 + r_{11}r_{22}, \qquad K = r_{12}(r_{12} + r_{11})(r_{12} + r_{22}).$$

For $N=3$ an invariant of degree 2, two of degree 3 and three of degree 5 constitute a complete set.

8. *A basis for pseudo-vectors*, by Professor I. M. Hostetter, Oregon State College.

Let A and B be points of an euclidean 3-space and a and b real scalars. The sum $aA+bB$, $a+b\neq 0$, is defined after Grassmann as the point C of weight $a+b$ which divides the segment AB in the ratio $b:a$. $A+(-1)B=A-B$ is defined as the free line vector BA, $A\times B$ as the glissant bound to the line through A and B. The outer product of a glissant and a free vector, $(A\times B)\times(C-B)=(A\times B)\times C$, is defined as an areal vector bound to the plane through $A\times B$ which is parallel to $C-B$.

If G_1, G_2, G_3 are three glissants of unit length forming an equilateral triangle, any glissant U in the plane may be written in terms of this basis as $U=4/3\epsilon^{ijk}p_iq_jG_k$, where p_i and q_j are the signed perpendicular distances from the sides of the triangle to the initial and terminal points of U in any position on its line of action. Similarly, if B_1, B_2, B_3, B_4 are the outward drawn areal vectors bound to the plane of and determined by the faces of a tetrahedron whose edges are of unit length, any areal vector W bound to a plane may be expressed in terms of the basis B_1 by $W=3/4\sqrt{6}\;\epsilon^{hijk}p_hq_ir_jB_k$, where p_h, q_i, r_j are the signed perpendicular distances from the planes of the tetrahedron to the vertices of any triangular representation of W.

9. *The reduction formulas for* $\int \sin^m\theta\cos^n\theta\,d\theta$, by Professor J. P. Ballantine, University of Washington.

The reduction formulas for $\int s^mc^n d\theta$ are rendered almost symmetrical, and hence more easily memorized, by the introduction of

$$\mu = -(m+n+1).$$

$$\int s^mc^n d\theta = \pm\frac{s^{m'}c^{n'}}{D} + \frac{N}{D}\int s^{m+\Delta m}c^{n+\Delta n}d\theta,$$

where $(\Delta m,\ \Delta n,\ \Delta\mu)$ is some permutation of $(+2,\ -2,\ 0)$. Hence 6 formulas. $m'=1+$smaller of $(m,\ m+\Delta m)$, $n'=1+$smaller of $(n,\ n+\Delta n)$. The sign of the term $s^{m'}c^{n'}/D$ is the same as the sign of $\Delta m-\Delta n$. $D=m+1$ if $\Delta m=2$, $D=n+1$ if $\Delta n=2$, $D=-(\mu+1)$ if $\Delta\mu=2$. $N=m-1$ if $\Delta m=-2$, $N=n-1$ if $\Delta n=-2$, $D=-(\mu-1)$ if $\Delta\mu=-2$.

If one of the parameters, m, n, or μ, is odd and positive, make a substitution in which $\cos\theta$, $\sin\theta$, or $\tan\theta$, respectively, is a new variable. If all the positive parameters are even, reduce them by applying the reduction formulas.

The reduction formulas fail when $D=0$, that is, when the parameter being increased starts at -1. Hence they fail when $(m,\ n,\ \mu)$ is any permutation of $(1,\ -1,\ -1)$ or $(-1,\ 0,\ -0)$. In the case of $(1,\ -1,\ -1)$, a substitution can be made.

The rules for the reduction formulas for $\int \sinh^m t\cdot\cosh^n t\,dt$ are the same, except that the sign before $s^{m'}c^{n'}/D$ is always plus.

J. MAURICE KINGSTON, *Secretary*

THE OCTOBER MEETING OF THE OKLAHOMA SECTION

The annual meeting of the Oklahoma Section of the Mathematical Association of America was held in connection with the convention of the Oklahoma Education Association in Oklahoma City on Friday, October 12, 1951. Professor C. M. Pirrong, Chairman of the Section, presided.

Sixty-four persons attended the meeting, including the following thirty-nine members of the Association:

E. F. Allen, R. V. Andree, Arthur Bernhart, J. C. Brixey, Sarah M. Burkhart, R. L. Caskey, Y. W. Chen, N. A. Court, R. B. Deal, R. C. Dragoo, H. F. Fehr, B. T. Goldbeck, A. A. Grau, E. V. Greer, O. H. Hamilton, Claire A. Harrison, J. O. Hassler, E. E. Heimann, W. N. Huff, L. W. Johnson, J. T. Krattiger, G. D. Kyle, J. E. LaFon, Eunice Lewis, H. W. Linscheid, R. D. McDole, Dora McFarland, G. E. Meador, R. R. Murphy, C. M. Pirrong, D. P. Richardson, Herbert Scholz, H. W. Smith, T. C. Smith, C. E. Springer, Vivian Spurgeon, G. R. Vick, J. H. Zant.

At the business session the following officers were elected: Chairman, L. W. Johnson, Oklahoma A. and M. College; Vice-Chairman, Truman Wester, Central State Teachers College; Secretary, R. V. Andree, University of Oklahoma.

The program consisted of the following five papers.

1. *On the incenter of a triangle*, by Professor E. F. Allen, Oklahoma A. and M. College.

This paper studied the projection of the incenter of a triangle under a projection which carried the circumscribing circle of the triangle into a central conic.

2. *Geometric figures and p-rings*, by Professor A. A. Grau, University of Oklahoma.

The p-rings may be considered as a generalization of Boolean algebras, which may be identified with 2-rings. The structure theorems of Stone and McCoy-Montgomery were reviewed and the following properties of a finite p-ring R were discussed: The elements x of R may be represented by n-tuples $(\xi_1, \cdots, \xi_n)$ of elements ξ_i in the residue class ring of the integers mod p. Let $\phi(\xi) = m$, the integer such that $0 \leqq m < \frac{1}{2}(p-1)$ and either $m \equiv \xi$ or $m \equiv \rho - \xi \pmod{p}$. Then $\rho(x,y) \equiv \sum_{i=1}^{n} \phi(x_i - y_i)$ is a metric over R. An *edge* is a pair of points x, y such that $\rho(x, y) = 1$. The metric thus associates with R a geometric configuration consisting of points and edges. The property of 2-rings, that given any 3 points x_1, x_2, x_3 there exists a unique point x where $f(x) = \sum_{i=1}^{3} \rho(x, x_i)$ takes on its minimum value, does not hold in a p-ring where p is odd for three given points or, more generally, for n given points.

3. *Property of points whose pedal triangles are similar*, by Professor G. D. Kyle, Arkansas State A. M. and N. College.

By a rotational process a clinant was developed and a series of points were found having pedal triangles similar to a given triangle. It was shown that one of these points was the circumcenter of the given triangle and the others were vertices of the first and second Brocard triangles of the given triangle. The pedal triangles of the inverses of the above points, excepting the circumcenter, were shown to be similar to the original triangle.

4. *On improving the teaching of college mathematics*, by Professor J. H. Zant, Oklahoma A. and M. College.

The task of improving the teaching of college mathematics was attacked from the point of view of the job facing the college teacher of mathematics.

The college teacher of mathematics must (1) have a broad, well-rounded education, (2) must understand young people and know how to direct their learning activities, (3) must be able to select, organize the subject matter of the field of mathematics at many levels and for many purposes, (4) must be able to evaluate the results of his efforts, (5) must be willing and able to engage in some sort of creative activity, and finally, (6) must assume the obligations of an educated citizen in the college community, the state, and the world.

This job calls for preparation in the areas of general education, mathematical education and professional education and a definite effort should be made by the undergraduate and graduate schools and by departments of mathematics to see that prospective college teachers have access to courses which will insure such knowledge and skills. If final answers to these problems exist, they must be worked out by the professors in the departments of mathematics and in the graduate schools where the prospective teachers are being trained and where the beginner starts his work.

5. *Mathematics in the English secondary school and college*, by Professor O. H. Hamilton, Oklahoma A. and M. College.

Professor Hamilton, who has recently returned from an exchange professorship in England, discussed instruction in mathematics in the English secondary school and university. He called attention to the methods by means of which only the best students are selected to attend the university and to the differences between the English and American practices with respect to examinations and the granting of degrees.

J. C. BRIXEY, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1–2, 1952.

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Waynesburg College, Waynesburg, Pennsylvania, May 10, 1952.
ILLINOIS, Western Illinois State College, Macomb, May 9–10, 1952.
INDIANA, Indiana University, Bloomington, May 3, 1952.
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, College of St. Catherine, St. Paul, May 10, 1952.
MISSOURI, Lindenwood College, St. Charles, May 2, 1952.
NEBRASKA
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City, October 31, 1952.
PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.
PHILADELPHIA, University of Delaware, Newark, November 29, 1952.
ROCKY MOUNTAIN, Western State College, Gunnison, Colorado, May 23–24, 1952.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Hobart and William Smith Colleges, Geneva, May 10, 1952.
WISCONSIN, University of Wisconsin in Milwaukee, May 10, 1952.

MACMILLAN

Coming in summer

MERRILL RASSWEILER
IRENE RASSWEILER

FUNDAMENTAL PROCEDURES OF FINANCIAL MATHEMATICS

Here is a thorough treatment of the customary topics of both business and financial mathematics requiring only an arithmetic background. The student receives a good foundation in percentage and its application to business through commissions, taxes, pricing, interest and discount and negotiable instruments. Each topic is followed by a step-by-step procedure to guide the student's thinking and show him how to carry out the calculations.

Just published!

JOHN RANDOLPH

CALCULUS

Maintaining a sound balance between formal problem material and a mathematically rigorous presentation, this new text offers a flexible, well-proportioned modern treatment of the theory and applications of calculus. Enough routine knowledge is given for the needs of the average student. The appendix contains more advanced work for the very good student. Problems are numerous, practical, and well-illustrated. *Published in March—$5.00*

A new text

EARL RAINVILLE

ELEMENTARY DIFFERENTIAL EQUATIONS

All the material of the author's successful *Short Course in Differential Equations* is contained in this book. It offers a sound training in good techniques for obtaining solutions and an explanation of the basic theory behind the techniques. Over 1200 carefully constructed problems with answers are given. *Ready in April*

THE MACMILLAN COMPANY
60 FIFTH AVE., NEW YORK 11, N.Y.

Just published!

THEORY OF MATRICES

By SAM PERLIS

Associate Professor of Mathematics, Purdue University

TABLE OF CONTENTS

1. Introductory concepts
2. Vector spaces
3. Equivalence, rank, and inverses
4. Determinants
5. Congruence and Hermitian congruence
6. Polynomials over a field
7. Matrices with polynomial elements
8. Similarity
9. Characteristic roots
10. Linear transformations

A new text in matrices for graduate students and advanced undergraduates.

Self-contained and simple, yet the text presents a complete theory. Advanced parts of the subject are not avoided, but rather are presented with the care and clarity that make them accessible to workers in fields other than mathematics.

The role of determinants is reduced to a minimum.

Canonical sets are the main theme. The text obtains the well-known canonical sets for equivalence, congruence, Hermitian congruence, similarity, orthogonal similarity, and unitary similarity.

Normal matrices, important in quantum mechanics, are fully discussed.

Applications to differential equations and dynamics are presented.

The aim throughout is to develop ability to reason intelligently with matrices as a tool. Numerical proficiency follows easily when ideas are well understood. For this reason the emphasis throughout the text and its exercises is on clarifying the conceptual content of the subject.

A large collection of problems, providing adequate amounts of numerical work but emphasizing the simple theoretical ideas, is included.

Clothbound—5 5/8" x 8 3/8"—Illustrated—237 pages—$5.50

Another new book in the
ADDISON-WESLEY MATHEMATICS SERIES

ADVANCED CALCULUS

By WILFRED KAPLAN

Published April 1952 *$8.50*

Copies cheerfully sent on approval

ADDISON-WESLEY PRESS, INC. **Cambridge 42, Massachusetts**

Just Published

BRIEF TRIGONOMETRY Revised Edition

Edward A. Cameron, *University of North Carolina*

In revising this popular text a section on vectors has been added, explanations have been amplified, and portions of the book have been completely rewritten. The essentials of plane trigonometry are presented so clearly and concisely that the material can be covered in fewer than thirty assignments. In every way the author has strengthened the teachability of his book. The number of exercises has been greatly increased, with special emphasis on applications. *February 1952, 154 pages, $2.10*

Other Important Texts

BRIEF COURSE IN ANALYTICS Revised Edition

M. A. Hill, Jr., and J. B. Linker, *University of North Carolina*

This book has been rewritten to attain even greater clarity. New additions include: a summary of elementary geometry, algebra, and plane trigonometry; a study of the cycloid and the hypocycloid of four cusps; an enlarged treatment of polar coördinates; and an increased number of examples and exercises. *1951, 224 pages, $2.40*

COLLEGE ALGEBRA Fifth Edition

H. L. Rietz and A. R. Crathorne

Revised by J. William Peters, *University of Illinois*

A more extended review of the topics of secondary school algebra is given in this new edition, including discussions of the number system and the elementary operations of algebra. New topics, such as functional notation, graphs, and determinants have been introduced, and exercise material has been greatly expanded throughout. *1951, 387 pages, $2.95*

MATHEMATICS ESSENTIAL FOR ELEMENTARY STATISTICS Revised Edition

Helen M. Walker, *Teachers College, Columbia University*

This basic text reviews the principles of mathematics valuable to students in a variety of fields. In the revision the scope is enlarged to cover a wider range of topics and explanations and exercises have been expanded. Greater emphasis is placed upon linear interpolation, and sections have been added dealing with permutations and combinations and the binomial and multinomial expansions. There is also a brief treatment of trigonometry. *1951, 382 pages, $3.00*

HENRY HOLT AND COMPANY 383 Madison Avenue, New York 17

SPINOZA DICTIONARY

Edited by Dagobert D. Runes

With a Foreword by Albert Einstein

In this work, Baruch Spinoza, one of the cardinal thinkers of all time, answers the eternal questions of man and his passions, God and nature. In the deepest sense, this dictionary of Spinoza's philosophy is a veritable treasury of sublime wisdom.

Here is what Albert Einstein writes about this book:

"I have read the Spinoza Dictionary with great care. It is, in my opinion, a valuable contribution to philosophical literature.

"If the reader despairs of the business of finding his way through Spinozism, here he will find a reliable guide speaking in Spinoza's own words.

"The grand ideas of Spinoza's *Ethics* are brought out clearly in this book, not less than the magnificent delusions of passion-driven men." $5.00

Expedite Shipment by prepayment

PHILOSOPHICAL LIBRARY

Publishers

15 East 40 St., Desk 300

New York 16, N. Y.

BERTRAND RUSSELL'S DICTIONARY OF MIND MATTER & MORALS

This exhaustive work offers more than 1000 definitions and opinions of the 1950 Nobel Prize winner, arranged as a handy key. Here is Russell's challenging thought on politics, ethics, philosophy of science, epistemology, religion, mathematical philosophy, and on topics crucial to an understanding of international affairs today. Dipped into casually it rewards the browser with stimulating and acute intellectual insights. Read intensively it will be found indispensable to a fuller appreciation of one of the profoundest minds of our age. $5.00

Expedite Shipment by prepayment

PHILOSOPHICAL LIBRARY

Publishers

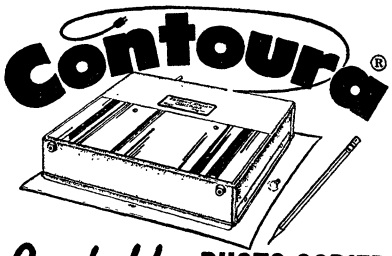
15 East 40 St., Desk 300

New York 16, N. Y.

4 SHORTCUTS FOR EDUCATORS

★

MODELS AS LOW AS $39 Complete



★ *Portable* PHOTO-COPIER

TRULY PORTABLE . . . only 4 lbs., fits in briefcase or desk drawer. Use it anywhere! COPIES ANYTHING written, printed or drawn up to 8½ x 14" . . . including text next to curved binding. Copies only 7¢ each. NEEDS NO DARKROOM . . . use under artificial light. ELIMINATES DRUDGERY and errors of longhand-copying and proof-reading. Affords time and money-savings!

★

WRITE FOR FREE FOLDER TODAY

★

F.G. Ludwig ASSOCIATES

97 Pease Rd. Woodbridge, Conn.

1. CUTS TIME
2. CUTS EFFORT
3. CUTS ERRORS
4. CUTS COSTS



Replace missing or damaged pages from duplicate volumes. Copy passages from rare or single copy editions for reserve book shelf.



Contoura - copy excerpts, or entire pages up to 8½" x 14", from non-circulating books without lugging heavy volumes to photo copy company.



End drudgery of longhand-copying and proof-reading notes. Contoura - copied manuscripts and documents are letter-perfect, need no checking.



Contoura - copy full-page graphs, charts, maps, prints or parts of pages, stamps, signatures, from books that can't be taken from library.

Just Published . . .

CALCULUS

By Atherton Hall Sprague, *Amherst College*

A logically complete course in the calculus, this textbook discusses the subject with exceptional thoroughness. Analytic proofs are accompanied by comprehensive and detailed explanations. Applications to geometry and physics are included, and graded problems and illustrative examples appear throughout. *206 diagrams. 576 pages.*

ANALYTIC GEOMETRY

By Alfred L. Nelson, Karl W. Folley, *and* William M. Borgman, *all of Wayne University*

Planned for use in those courses where preparation for the calculus, rather than the study of geometry, is the paramount objective. Particular attention is given to problems which will be of maximum value to future students of the calculus, the basic sciences, and engineering.

100 illustrations. 4 tables. 215 pages.

COLLEGE ALGEBRA

By Earle B. Miller, *Illinois College; and* Robert M. Thrall, *University of Michigan*

A textbook for the first-year student who wants a grounding in the subject which will fit him for a career where thorough knowledge is indispensable. The exposition avoids the complexities of a too advanced text and the sterility of the oversimplified presentation.

36 illustrations. 7 tables. 493 pages.

New also . . .

THE NATURE OF NUMBER

An Approach to Basic Ideas of Modern Mathematics

By Roy Dubisch, *Fresno State College*

Especially written for those seeking to acquaint themselves with what modern mathematics is about, to gain an insight into its theory, and to familiarize themselves with the types of problem that present-day mathematicians are interested in. Included are problems by which the reader can test his understanding of concepts. *24 illustrations. 159 pages.*

THE RONALD PRESS COMPANY

15 East 26th Street, New York 10

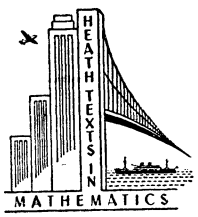
WILLIAM L. HART'S

College Trigonometry

A substantial treatment of plane and spherical trigonometry, of moderate length, incorporating distinctly collegiate viewpoints. Emphasizes analytic trigonometry, oriented for application in later mathematics; presents a mature and well-rounded treatment of numerical plane trigonometry; offers a satisfactory foundation in spherical trigonometry, including a reasonable number of elementary applications. *Text pages: Plane Trigonometry, 151; Complex Numbers and Appendix, 21; Spherical Trigonometry, 35; and Tables, 130. $3.50*

Intermediate Algebra for Colleges

Presents a collegiate substitute for third semester high-school algebra. A frankly elementary treatment of the fundamentals of algebra, couched in language suitable to the maturity of college students. Aims to bring the student to a final stage of mathematical appreciation far above that of the corresponding high-school course. Does not aim to overlap typically collegiate algebra. *276 text pages. $3.00*



DAVID R. CURTISS and ELTON J. MOULTON'S

Essentials of Analytic Geometry

A relatively brief treatment of plane and solid analytic geometry, suitable for semester classes meeting three, four, or five hours weekly. Concise and concentrated, it places main emphasis on those topics considered essential in preparing college students for calculus and engineering courses. A distinctive feature is the inclusion of lists of supplementary exercises designed for the most capable students and introducing ideas of special interest to them. *269 pages. $3.25*

D. C. HEATH AND COMPANY

Sales Offices: New York Chicago San Francisco Atlanta Dallas

Home Office: Boston

Just Published!

PLANE TRIGONOMETRY — *3rd Edition*

by FRED W. SPARKS, Texas Technological College, and PAUL K. REES, Louisiana State University

Important Features of New Edition:

- 1800 new problems, arranged in groups of four. Each problem in a group is on the same level and involves the same principle. Answers are given for three of every four problems.
- Additions to the text matter include: Introductory articles that sum up the content of the chapter; new illustrations; new section on the method used by the Air Force for designating the direction of a line.
- Previous editions of this text were adopted by many schools, including the U. S. Naval Academy, where the midshipmen must have an excellent foundation in triangulation for their advanced courses in navigation and gunnery.

320 pages 6 x 9 in two editions—with and without logarithmic tables

DIFFERENTIAL EQUATIONS— *3rd Edition*

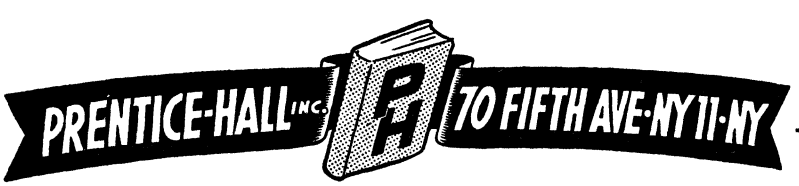
by MAX MORRIS and ORLEY E. BROWN, Case School of Applied Science

Outstanding Features of New Edition:

- More problems, better graded and diversified, especially in the matter of *applied problems*. Theory is stressed only to the extent justified by a background of one year of elementary calculus.
- Exposition simplified and better motivated for student understanding.
- Important topics have been expanded: lineal element, application to electrical circuits, numerical approximation.

260 pages 5⅝ x 8⅜

Send for Your Copies Today





New McGRAW-HILL Books

TRIGONOMETRY

By CECIL T. HOLMES, Bowdoin College. 246 pages (with tables), $3.25

Here is a clearly understandable and thorough text that offers a closer correlation of trignometry with other college mathematics than is done in most books of its kind. This is accomplished by introducing some very elementary notions from analytic geometry and using them consistently in proofs throughout the book. Emphasis throughout is upon understanding, rather than mechanical manipulation.

CALCULUS AND ANALYTIC GEOMETRY

By C. T. HOLMES, Bowdoin College. 416 pages, $4.75

Designed for a combination course in which the concepts and techniques of the calculus are the main objectives. Although calculus is emphasized, the essentials of analytic geometry are presented in sufficient detail for a subsequent major. The concept of integration is introduced early in the text.

LINEAR ALGEBRA AND MATRIX THEORY

By ROBERT R. STOLL, Lehigh University. *International Series in Pure and Applied Mathematics.* In press

A text for college courses in "modern algebra," this book covers the theory of matrices viewed against the background of modern algebra. It presents topics of algebra which are of ever-increasing importance in a wide variety of fields; provides the student of engineering with the necessary algebraic background for studying circuit theory, vibration theory, and dynamics of particles; and introduces students of mathematics to the techniques and point of view of modern algebra via concrete examples from matrix theory.

CALCULUS OF VARIATIONS. With Applications to Physics and Engineering

By ROBERT WEINSTOCK, Stanford University. *International Series in Pure and Applied Mathematics.* In press

Designed to illustrate the applications of the Calculus of Variations in several fields outside the realm of pure mathematics. In addition to a brief treatment of classical elementary problems, sections are included involving calculus of variations in particle dynamics, geometrical optics, quantum mechanics, elasticity, electrostatics, etc.

Send for copies on approval



McGRAW-HILL BOOK COMPANY, INC.

330 WEST 42ND STREET, NEW YORK 18, N. Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 59 NUMBER 6

CONTENTS

The General Chinese Remainder Theorem . . . Oystein Ore 365
The Sums of the Dihedral and Trihedral Angles in a Tetrahedron . . . . . J. W. Gaddum 370
Some Properties of Continuous Functions . . . M. K. Fort, Jr. 372
The Unsatisfactory Story of Curvature . . . . J. L. Coolidge 375
A Canonical Basis for the Ideals of a Polynomial Domain. . . . . G. Szekeres 379
Mathematical Notes . . . . . L. Carlitz, J. G. Campbell, . . . Leonard Tornheim, Richard Bellman, I. Opatowski 386
Classroom Notes. . . . . M. K. Fort, Jr., C. G. Phipps, ... H. J. Hamilton, Editorial, Corrections 392
Elementary Problems and Solutions 407
Advanced Problems and Solutions 412
Recent Publications 421
Clubs and Allied Activities 423
News and Notices 427
The Mathematical Association of America 431
New Members 431
December Meeting of the Maryland-D.C.-Virginia Section . . 434
November Meeting of the Philadelphia Section 436
October Meeting of the Minnesota Section 438
January Meeting of the Northern California Section 440

JUNE-JULY 1952

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	J. W. GIVENS	GEORGE SPRINGER
PATRICK DU VAL	L. M. KELLY	E. P. STARKE
HOWARD EVES	H. D. LARSEN	G. B. THOMAS
F. A. FICKEN	IVAN NIVEN	E. P. VANCE

EDITH R. SCHNECKENBURGER

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of College Mathematics)

OFFICERS OF THE ASSOCIATION

President, SAUNDERS MACLANE, University of Chicago
Honorary President, W. D. CAIRNS, Oberlin College
First Vice-President, F. L. GRIFFIN, Reed College
Second Vice-President, JEWELL H. BUSHEY, Hunter College
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo
Editor, C. B. ALLENDOERFER, University of Washington

Additional Members of the Board of Governors: E. B. ALLEN, C. F. BARR, T. A. BICKERSTAFF, J. C. BRIXLEY, R. H. CAMERON, G. R. CLEMENTS, T. F. COPE, H. S. M. COXETER, W. M. DAVIS, J. M. EARL, G. M. EWING, L. R. FORD, TOMLINSON FORT, J. S. FRAME, A. E. GAULT, R. E. GILMAN, E. H. HANSON, M. R. HESTENES, AUGHTUM S. HOWARD, C. G. JAEGER, B. W. JONES, M. S. KNEBELMAN, R. E. LANGER, D. H. LEHMER, C. C. MACDUFFEE, W. E. MILNE, H. P. PETTIT, J. C. POLLEY, G. E. RAYNOR, E. B. ROESSLER, R. G. SANGER, F. H. STEEN, EARL WALDEN, MARIE J. WEISS, F. B. WILEY

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

THE GENERAL CHINESE REMAINDER THEOREM

OYSTEIN ORE, Yale University

1. Introduction. The Chinese remainder theorem, as one knows, is one of the most useful tools of elementary number theory. It presents a simple method of determining and representing the solution of a system of simultaneous congruences,

$$x \equiv a_i \pmod{m_i} \qquad (i = 1, 2, \cdots, k), \tag{1}$$

provided the moduls m_i are relatively prime in pairs.

I have been unable to find mentioned anywhere in the mathematical literature the fact that it is possible to formulate a general Chinese remainder theorem, which includes the ordinary, but is valid without any restrictions on the moduls. This is the main result, Theorem 1, in the present paper. It has been expressed for rational integers only, but anyone familiar with the concepts of newer algebra will see how it may be formulated for ideals in quite general rings, so that this need not be elaborated. There have been added, however, some remarks on Abelian groups which appear quite naturally in this connection.

2. **The remainder theorem.** Let the system of congruences (1) be given where it is not assumed that the moduls are necessarily relatively prime. We recall that in this case the congruences do not always have a solution: the necessary and sufficient condition that they be solvable is that for all i and j,

$$a_i \equiv a_j \pmod{d_{ij}}, \tag{2}$$

where we have put

$$d_{ij} = m_i \wedge m_j \tag{3}$$

for the greatest common divisor of the two moduls m_i and m_j. When the conditions (2) are fulfilled the solution of (1) is uniquely determined for the least common multiple

$$M = m_1 \vee \cdots \vee m_k \tag{4}$$

as modul.

To formulate our theorem we need some further terminology. For each i we shall write

$$A_i = m_1 \vee \cdots \vee m_{i-1} \vee m_{i+1} \vee \cdots \vee m_k$$

and

$$D_i = m_i \wedge A_i = d_{i,1} \vee \cdots \vee d_{i,i-1} \vee d_{i,i+1} \vee \cdots \vee d_{ik}.$$

The quotient of these two numbers is the integer

$$B_i = A_i/D_i. \tag{5}$$

This expression (5) for B_i may be given various other forms by means of the well known identity

$$a \cdot b = (a \vee b) \cdot (a \wedge b). \tag{6}$$

First we have

$$B_i = \frac{A_i}{m_i \wedge A_i} = \frac{m_i \vee A_i}{m_i} = \frac{M}{m_i}. \tag{7}$$

This may be written

$$B_i = \frac{m_1 \vee m_i}{m_i} \vee \frac{m_2 \vee m_i}{m_i} \vee \cdots,$$

which again by (6) and (3) reduces to

$$B_i = \frac{m_1}{d_{1i}} \vee \frac{m_2}{d_{2i}} \vee \cdots. \tag{8}$$

After these preparations we state the main theorem:

THEOREM 1. *The solution of the simultaneous congruences* (1) *can be presented in the form*

$$x \equiv a_1 c_1 \frac{M}{m_1} + \cdots + a_k c_k \frac{M}{m_k} \pmod{M}, \tag{9}$$

where the c_i form a set of integers satisfying the condition

$$c_1 \frac{M}{m_1} + \cdots + c_k \frac{M}{m_k} = 1. \tag{10}$$

Proof. We must first show that the numbers (7) have no common factor so that the indeterminate equation (10) has a solution set $\{c_i\}$. It suffices to show that for each prime p at least one of the numbers B_i is not divisible by p. Let p^{α_i} be the highest power of p dividing m_i and suppose for instance that α_1 is the greatest among these exponents. Then M is divisible exactly by p^{α_1} and B_1 is not divisible by p.

To prove that the expression x in (9) actually is a solution of (1) let us examine it (mod m_j). When the congruence (2) is multiplied by B_i we obtain

$$B_i a_i \equiv B_i a_j \pmod{d_{ij} \cdot B_i}.$$

But according to (8) the number B_i is divisible by m_j/d_{ij} so that we conclude that

$$B_i a_i \equiv B_i a_j \pmod{m_j}.$$

For each i we multiply this congruence by c_i. When the resulting congruences are added one finds

$$x \equiv a_j(B_1c_1 + \cdots + B_kc_k) \pmod{m_j}$$

or, according to (10),

$$x \equiv a_j \pmod{m_j},$$

as desired.

It may be noticed that in the general solution (9) the multipliers for the residues a_i are independent of the particular set of remainders involved in the system (1).

3. Symmetric functions. It is convenient to introduce operational symbols $\bigvee$ and $\bigwedge$ for taking the l.c.m. and g.c.d. of a set of numbers just as we use the sum sign $\sum$ and the product sign $\prod$. Thus we write

$$\bigvee n_i = n_1 \vee \cdots \vee n_s$$
$$\bigwedge n_i = n_1 \wedge \cdots \wedge n_s$$

for the g.c.d. and l.c.m. of the numbers n_i.

As before let

$$m_1, m_2, \cdots, m_k \tag{11}$$

denote k arbitrary positive integers. From these we form a system of *symmetric functions* by means of g.c.d. and l.c.m. operations, namely

$$M_r = \bigvee m_{i_1} \wedge m_{i_2} \wedge \cdots \wedge m_{i_r}$$
$$N_r = \bigwedge m_{i_1} \vee m_{i_2} \vee \cdots \vee m_{i_r} \qquad (r = 1, 2, \cdots, k) \tag{12}$$

where, for each r, the choice of indices,

$$i_1, i_2, \cdots, i_r,$$

runs through all possible $C_{k,r}$ combinations of the numbers from 1 to k: in particular

$$M_1 = m_1 \vee \cdots \vee m_r,$$
$$N_1 = m_1 \wedge \cdots \wedge m_r.$$

From the distributive law for the g.c.d. and l.c.m. (see f. inst. Ore [3], chap. 5-4) one deduces from (12) that

$$M_r = N_{k-r+1}, \tag{13}$$

and it follows also fairly directly from the definitions (12) that

$$M_1 \geqq M_2 \geqq \cdots \geqq M_k,$$
$$N_k \geqq N_{k-1} \geqq \cdots \geqq N_1, \tag{14}$$

where each term divides the preceding.

When k is an odd number one obtains from (13) the self-dual identity

$$M_{(k+1)/2} = N_{(k+1)/2},$$

as has recently been pointed out by Mitrinovitch [1, 2].

These results can also be obtained quite simply by considering the exponents of the various powers of a prime p which enter into the series of numbers (12). As before let m_i be divisible exactly by p^{α_i}; since our expressions are symmetric in the m_i there is no limitation in assuming the notation such that

$$\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_k.$$

Thus M_i is divisible exactly by p^{α_i}, that is by p to the i-th largest exponent to which it occurs, and N_i is divisible by exactly $p^{\alpha_{i-k+1}}$, that is, p to the $(i-k+1)$-st highest exponent α. Since this holds for every prime p the relations (13) and (14) follow.

An easy consequence of this point of view is the formula

$$m_1 \cdot m_2 \cdots m_k = M_1 \cdot M_2 \cdots M_k, \tag{15}$$

which is a generalization of the simple relation (6). Both sides in (15) contain p to a power with the exponent

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k.$$

4. Abelian groups. The k-tuples

$$(x_i) = (x_1, x_2, \cdots, x_k) \tag{16}$$

with integral x_i form an Abelian group under addition. When we suppose further that each x_i is reduced modul m_i, we obtain the general finite Abelian group A of the type

$$\mu = [m_1, m_2, \cdots, m_k], \tag{17}$$

that is, the group with k basis elements of the respective orders m_i.

This representation of the Abelian group by its type is not unique since there may be several types giving the same group. To obtain a unique type representation one can split each m_i into its prime power factors p^{α_i} and represent the type of A by means of the *invariants*

$$\mu_i = [\cdots, p^{\alpha_i}, \cdots]. \tag{18}$$

A second way of obtaining a unique type representation is by means of the *elementary divisors* where one imposes the restriction on the type that each exponent shall be a divisor of the preceding. To construct this elementary divisor type from the invariant type (18) one can proceed as follows. For each prime p one takes the highest invariant p^{α_i} and then one forms the product of these p^{α_i} over all the primes p in (18). Next one takes the second highest invariants and their product, and so on. These products are evidently the symmetric functions $M_1, M_2, \cdots$ so that we can state:

THEOREM 2. *The elementary divisor type of an Abelian group of type* (17) *is*

$$\mu_e = [M_1, M_2, \cdots, M_k],$$

where the numbers M_j *are the symmetric functions of the numbers* m_i.

From this point of view the formula (15) expresses only the fact that the order of the group is independent of the type representation.

Let us now consider what we may call the *residue k-tuples*

$$(a_i) = (a_1, a_2, \cdots, a_k). \tag{19}$$

These are the elements in A for which the corresponding congruences (1) are solvable, that is, the sets of numbers for which the congruence conditions (2) are fulfilled. It is evident that these residue k-tuples (19) form a subgroup, the *residue group* R of A. But since the congruences (1) are solvable in this case, each a_i may be replaced by the same x so that the residue group R is simply the subgroup of A consisting of all k-tuples

$$(x) = (x, x, \cdots, x)$$

in which all components may be taken to be the same. We notice that the order of R is M_1.

Finally one may say that two k-tuples (x_i) and (y_i) in A are *residue equivalent* if their difference is a residue k-tuple (19). The residue equivalence classes also form a group, the *residue difference group* D isomorphic to the difference group $A-R$ (quotient group A/R). The order of this group is

$$\frac{m_1 \cdots m_k}{M_1} = M_2 \cdots M_k,$$

and we leave it to the reader to verify that its elementary divisors are actually

$$M_2, M_3, \cdots, M_k.$$

Instead of taking the additive group A one could have considered the multiplicative group A' consisting of those k-tuples in (16) in which each component x_i is relatively prime to its corresponding modul. The type of this group is

$$\mu' = [\phi(m_1), \cdots, \phi(m_k)]$$

where, as usual, ϕ denotes Euler's function. Here our symmetric functions $M_j(m_i)$ are replaced by $M_j(\phi(m_i))$. The order of the group may be written in several forms:

$$\begin{aligned} \phi(m_1) \cdots \phi(m_k) &= M_1(\phi(m_i)) \cdots M_k(\phi(m_i)) \\ &= \phi(M_1(m_i)) \cdots \phi(M_k(m_i)). \end{aligned} \tag{20}$$

These identities (20) are extensions of the simple rule

$$\phi(a)\cdot\phi(b) = \phi(a \vee b)\cdot\phi(a \wedge b) = (\phi(a) \vee \phi(b))\cdot(\phi(a) \wedge \phi(b)).$$

Bibliography

1. D. S. Mitrinovitch: Sur une propriété des opérations *max* et *min*, C. R. Acad. Sci., Paris, v. 232, 1951, pp. 286–287.
2. D. S. Mitrinovitch: Sur certaines relations de l'algèbre des ensembles, Ibid., pp. 917–918.
3. Oystein Ore: Number theory and its history, New York, 1948, McGraw-Hill Book Company.

THE SUMS OF THE DIHEDRAL AND TRIHEDRAL ANGLES IN A TETRAHEDRON*

J. W. GADDUM, National Bureau of Standards, Los Angeles, California

There is no theorem on the sum S of the dihedral angles in a tetrahedron analogous to the theorem that the sum of the angles in a triangle is π radians. Furthermore, a little experimentation shows that the sum is not a constant. However, bounds are known. It is well known that the sum of the three dihedral angles around any vertex is between π and 3π, from which it follows immediately that S is between 2π and 6π. These are the best bounds the writer has been able to find in the literature.

The principal result of this note is to show that S is between 2π and 3π, and that these bounds cannot be improved. Since T, the sum of the trihedral angles, is given by $T=2S-4\pi$, this also gives best bounds on T. (One would expect these results to be well known and the writer would appreciate information on this point. In any event, they do not appear to be readily accessible and the elementary proofs given here may be of value.)

We first prove a lemma about spherical triangles.

LEMMA. *If x is a point interior to a spherical triangle abc, then* $ax+bx+cx \leqq ab+bc+ac$.

Proof. Extend ax, bx, cx, intersecting bc, ac, and ab, respectively in p, q, r. Then $cx+ax-ar \leqq cx+xr \leqq br+bc$. Hence $cx+ax \leqq ar+br+bc=ab+bc$. Similarly, $ax+bx \leqq bc+ac$, and $cx+bx \leqq ab+ac$. Adding and dividing by 2, $ax+bx+cx \leqq ab+bc+ac$.

The proof is clearly valid for plane triangles as well.

Proceeding to consider the theorem, given any tetrahedron, let us take an interior point O and drop perpendiculars r_1, r_2, r_3, r_4 to the faces. Now the six dihedral angles of the tetrahedron are supplementary, respectively, to the six

* The preparation of this paper was sponsored (in part) by the Office of the Air Comptroller, USAF.

angles formed by r_1, r_2, r_3, r_4. That is, denoting by R the sum of the angles formed by the r_i, we obtain $S=6\pi-R$.

We are now concerned with finding bounds for R. If we draw a unit sphere about O, r_1, r_2, r_3, r_4 intersect this sphere in points p_1, p_2, p_3, p_4 and R is the sum of the six spherical distances p_ip_j. The fact that O is interior to the tetrahedron is equivalent to the fact that the points p_i are not contained in any hemisphere, or as we may say, are global.

THEOREM. *If R is the sum of the six distances between points of a global quadruple, $3\pi \leqq R$.*

Proof. Denoting the points p_1, p_2, p_3, p_4 and their antipodal points p_1', p_2', p_3', p_4', we wish to show first that p_1' is interior to the triangle $p_2p_3p_4$. (The argument given here was formulated by Professor L. M. Blumenthal.) Since the quadruple is global, p_1' and p_4 are on the same side of the great circle p_2p_3. Similarly p_1' and p_2' are on the same side of p_3p_4, and p_1' and p_3 are on the same side of p_2p_4. Hence p_1' is in the triangle $p_2p_3p_4$.

Now $p_1p_2+p_2p_1'=p_1p_3+p_3p_1'=p_1p_4+p_4p_1'=\pi$. But by the lemma, $p_2p_1'+p_3p_1'+p_4p_1' \leqq p_2p_3+p_3p_4+p_2p_4$. Hence

$$p_1p_2 + p_2p_1' + p_1p_3 + p_3p_1' + p_1p_4 + p_4p_1' = 3\pi \leqq R = p_1p_2 + p_1p_3 + p_1p_4 + p_2p_3 + p_3p_4 + p_2p_4.$$

COROLLARY. $2\pi \leqq S \leqq 3\pi$.

Proof. Since $S=6\pi-R$, the preceding theorem gives $S \leqq 3\pi$. While it is known that $S \geqq 2\pi$, as remarked in the first paragraph, it is interesting to give a proof in the spirit of this note. We have to prove that $R \leqq 4\pi$. Since the perimeter of any triangle (on the unit sphere) is at most 2π, adding the perimeters of the four triangles in any quadruple gives $2R \leqq 8\pi$, and hence $S \geqq 2\pi$.

It is to be remarked that these bounds on S are the best possible. Examples to show this are not difficult to find, but a simple argument, suggested by the referee, is convincing. We simply observe that we can find a tetrahedron arbitrarily close to either of the two limiting forms of zero volume, one of which is a plane quadrangle with its diagonals, the angle sum being 2π, the other of which is a triangle with its three vertices joined to an interior point, with angle sum 3π.

COROLLARY. The best bounds on T are $0 \leqq T \leqq 2\pi$.

This follows from the formula $T=2S-4\pi$.

These methods can be used to characterize tetrahedra whose dihedral angles are less than $\pi/3$ or greater than $2\pi/3$, that is, which have no dihedral angles lying between $\pi/3$ and $2\pi/3$. An examination of cases shows that such a tetrahedron either has two obtuse angles, these being opposite angles, or three obtuse angles, these emanating from one vertex.

SOME PROPERTIES OF CONTINUOUS FUNCTIONS

M. K. FORT, JR., University of Illinois

1. Introduction. A recent paper in this MONTHLY [1] has attempted to base a proof of the Fundamental Theorem of Algebra on Brouwer's Fixed Point Theorem. Since, as is pointed out in [2], this paper contains a basic error, a correct proof of the Fundamental Theorem of Algebra which is based on the Brouwer theorem may be of interest.

In this paper we give a brief exposition of some well known theorems about continuous functions, including the Brouwer theorem for the plane and the Fundamental Theorem of Algebra. The development presented here presupposes a knowledge of only the most elementary properties of continuous functions and complex numbers. A similar expository article has been written by A. W. Tucker [3]. Both [3] and [4] are highly recommended to those who are interested in elementary treatments of some important topological theorems.

2. Existence of continuous logarithms. We let P be the plane of complex numbers, P_0 be the set of all non-zero complex numbers, and S be the set of all $z \epsilon P$ for which $|z| = 1$. If $z \epsilon S$, then we let $A(z)$ be the set of all real numbers θ for which $z = e^{i\theta}$; (*i.e.* $A(z)$ is the set of all arguments of z). By a *disk* we mean the union of a circle and its interior. The word *mapping* is used to denote a continuous function. If f is a mapping of a set X into S, then *f has a continuous logarithm on X* if there exists a real-valued mapping ϕ of X such that $f(z) = e^{i\phi(z)}$ for all $z \epsilon X$. A systematic treatment of logarithms is given by Eilenberg in [4].

We shall assume and make use of the following two elementary properties of complex numbers:

(i) If w_1 and w_2 are points of S, $|w_1 - w_2| < 2$ and $\theta_1 \epsilon A(w_1)$, then there exists a unique $\theta_2 \epsilon A(w_2)$ such that $|\theta_1 - \theta_2| < \pi$.

(ii) If $\theta_1 \epsilon A(w_1)$, $\theta_2 \epsilon A(w_2)$ and $|\theta_1 - \theta_2| < \pi$, then $|\theta_1 - \theta_2| \leqq \pi |w_1 - w_2|$.

Property (ii) merely states that if an arc is contained in a semi-circle, then the length of the arc is less than π times the length of the chord joining the end points of the arc.

THEOREM 1. *If f is a mapping of a disk D into S, then f has a continuous logarithm.*

Proof. The mapping f is uniformly continuous on D, and hence there exists $\delta > 0$ such that if z_1 and z_2 are in D and $|z_1 - z_2| < \delta$ then $|f(z_1) - f(z_2)| < 1/3$. Let q be the center of D and r be the radius of D. We choose a positive integer n such that $r/n < \delta$, and then define $D_k = \{z \mid |z - q| \leqq kr/n\}$ for each integer k, $0 \leqq k \leqq n$. We now define a function ϕ on D by defining ϕ successively on $D_0, \cdots, D_n$. We first define ϕ on D_0 by letting $\phi(q)$ be the $\theta \epsilon A(f(q))$ for which $0 \leqq \theta < 2\pi$. If ϕ has been defined on D_k and $z \epsilon D_{k+1}$, let z' be the point of D_k which is nearest z. Since $|z - z'| < \delta$, $|f(z) - f(z')| < 1/3$; hence by (i) we may define $\phi(z)$ to be the unique member of $A(f(z))$ which differs from $\phi(z')$ by less than π.

Now let S_k be the statement: If z_1 and z_2 are in D_k and $|z_1-z_2|<\delta$, then $|\phi(z_1)-\phi(z_2)|<\pi$. We shall prove successively that $S_0, \cdots, S_n$ are true. S_0 is obviously true since D_0 contains only one point. Now let us assume that we have proved S_k true. Let z_1 and z_2 belong to D_{k+1} and satisfy $|z_1-z_2|<\delta$. We define z_1' and z_2' to be the points of D_k which are the nearest to z_1 and z_2 respectively. It follows from the way in which ϕ is defined on D_{k+1} that $|\phi(z_1)-\phi(z_1')|<\pi$ and $|\phi(z_2)-\phi(z_2')|<\pi$. It is easy to see that $|z_1'-z_2'|<\delta$, and since S_k is true we obtain $|\phi(z_1')-\phi(z_2')|<\pi$. Making use of (ii) and the way in which δ was chosen, we obtain

$$\begin{aligned}|\phi(z_1)-\phi(z_2)| &\leqq |\phi(z_1)-\phi(z_1')|+|\phi(z_1')-\phi(z_2')|+|\phi(z_2'-\phi(z_2)| \\ &\leqq \pi|f(z_1)-f(z_1')|+\pi|f(z_1')-f(z_2')|+\pi|f(z_2')-f(z_2)| \\ &< \pi/3+\pi/3+\pi/3=\pi.\end{aligned}$$

This proves that S_{k+1} is true. Thus we see that S_j is true for $0\leqq j\leqq n$.

From S_n and (ii), we see that if z_1 and z_2 are in D and $|z_1-z_2|<\delta$ then $|\phi(z_1)-\phi(z_2)|\leqq\pi|f(z_1)-f(z_2)|$. The continuity of ϕ now follows easily from the continuity of f. Since $\phi(z)\epsilon A(f(z))$ for each $z\epsilon D$, we have shown that f has a continuous logarithm on D.

Let $I(z)=z$ for each $z\epsilon S$. In contrast to the above theorem we now prove that there are mappings of S into S which do not have continuous logarithms on S.

THEOREM 2. *The identity function I on S does not have a continuous logarithm on S.*

Proof. Suppose that I has a continuous logarithm on S, and that ϕ is a continuous function such that $I(z)=z=e^{i\phi(z)}$ for all $z\epsilon S$. We can define a continuous function g on the closed interval $[0, 2\pi]$ by letting $g(\theta)=\phi(e^{i\theta})-\theta$. For each θ, $g(\theta)$ is an integral multiple of 2π. Since g is continuous, this implies that g is a constant function. This is impossible, however, since $g(0)=\phi(1)$ and $g(2\pi)=\phi(1)-2\pi$.

3. The Brouwer Fixed Point Theorem. In order to prove Brouwer's theorem, we shall need the following theorem which is itself an important theorem of topology.

THEOREM 3. *If D is the disk formed by the union of S with its interior, then there does not exist a mapping f of D into S such that $f(z)=z$ for all $z\epsilon S$.*

Proof. If such a mapping f did exist, then by Theorem 1 there is a mapping ϕ on D such that $f(z)=e^{i\phi(z)}$ for all $z\epsilon D$. By restricting the domain of ϕ, we see that f has a continuous logarithm on S. However, since f is the identity mapping on S, this contradicts Theorem 2.

THEOREM 4 (Brouwer Fixed Point Theorem). *If g is a mapping of a disk D into D, then there exists a point $z\epsilon D$ such that $g(z)=z$.*

Proof. There is no loss in generality in assuming that D is the disk which is the union of S and the interior of S, since all other cases can be reduced to this one by a change of coordinates.

Suppose $g(z) \neq z$ for each $z \in D$. Then, for each $z \in D$, there is a unique point $f(z)$ on S such that z is on the segment joining $f(z)$ to $g(z)$. The function f so defined is continuous, is a mapping of D into S, and $f(z) = z$ for all $z \in S$. We have a contradiction, since Theorem 3 states that such a function does not exist.

4. The existence of continuous nth roots.

THEOREM 5. *If f is a mapping of a disk D into P_0 and n is a positive integer, then there exist n distinct mappings $h_1, \cdots, h_n$ which have the property that $[h_k(z)]^n = f(z)$ for all $z \in D$.*

Proof. We can write $f(z) = r(z)g(z)$ where $r(z) = |f(z)|$ and $g(z) = f(z)/|f(z)|$. Then r is a mapping of D into the set of positive real numbers and g is a mapping of D into S. By Theorem 1, there exists a mapping ϕ such that $g(z) = e^{i\phi(z)}$. We get the desired n-th roots of f by defining

$$h_k(z) = \sqrt[n]{r(z)}\, e^{(\phi(z)+2k\pi)i/n}.$$

5. Fundamental Theorem of Algebra.

THEOREM 6 (Fundamental Theorem of Algebra). *If p is a polynomial of degree $n > 0$, then there exists $z \in P$ such that $p(z) = 0$.*

Proof. Let us assume that the coefficient of z^n in $p(z)$ is $1/2$. We may clearly do this without loss of generality. We now assume that $p(z) \neq 0$ for all $z \in P$ and prove our theorem by establishing a contradiction.

Since $\lim_{z\to\infty} p(z)/z^n = 1/2$, there exists $r > 0$ such that if $|z| \geqq r$, then $p(z)$ and z^n have arguments which differ by less than $\pi/3$ and also $|p(z)| < |z|^n$. Now choose $R > r$ such that $\sqrt[n]{|p(z)|} + |z| < R$ for $|z| \leqq r$. We define D to be the disk with center at 0 and radius R. Since p maps D into P_0, p has a continuous n-th root f on D. We choose f so that $f(r)$ and r have arguments which differ by less than $\pi/3n$. It follows from an easy continuity discussion that $f(z)$ and z have arguments which differ by less than $\pi/3n$ for all z such that $|z| \geqq r$. (Otherwise, for some z such that $|z| \geqq r$, $f(z)$ and z have arguments differing by exactly $\pi/3n$; this implies that $p(z)$ and z^n have arguments differing by exactly $\pi/3$, which is impossible.)

Since $|p(z)| < |z|^n$ for $|z| \geqq r$, we obtain $|f(z)| < |z|$ for $|z| \geqq r$. Thus, if $|z| \geqq r$, z and $f(z)$ both lie in a circular sector of radius $|z|$ and angle $\pi/3n$. It is therefore easy to see that $|z - f(z)| \leqq |z|$ for $|z| \geqq r$. On the other hand, if $|z| \leqq r$, then $|z - f(z)| \leqq |z| + |f(z)| < R$. If follows that if we define $g(z) = z - f(z)$ for $z \in D$, then g is a mapping of D into D. We may now use the Brouwer theorem to find z_0 such that $g(z_0) = z_0$. It follows that $f(z_0) = 0$ and hence $p(z_0) = 0$. This is a contradiction.

References

1. B. H. Arnold, *A topological proof of the fundamental theorem of algebra*, this MONTHLY, vol. 56, 1949, pp. 465–466.
2. B. H. Arnold and Ivan Niven, A correction, this MONTHLY, vol. 58, 1951, p. 104.
3. A. W. Tucker, Some topological properties of the disk and sphere, Proceedings of the First Canadian Mathematical Congress, vol. 1, 1945, pp. 285–309.
4. Samuel Eilenberg, Topology, mimeographed notes for Math. 215, Institute of Mathematics and Mechanics, New York University, New York 3, N. Y.

THE UNSATISFACTORY STORY OF CURVATURE

J. L. COOLIDGE, Harvard University

1. The Preliminaries. The fact that plane loci are sometimes straight and sometimes are not would seem to be about as obvious as any property of such figures; it is therefore very surprising that this was not placed in the foreground by early writers who dealt with the matter. Euclid defines a straight line as one which lies evenly with the points on itself, which meant that the end points would completely cover those which lay in between or completely hide the latter points, and countless variations of this idea were evolved by the early writers of geometry, but the dynamic idea that a straight line was generated by a point which always moved in the same direction was not, as far as I know, stressed by anyone. Of course whoever admits direction as a primary idea of geometry is involving himself in a sea of trouble, and this may be the reason for its avoidance, but the total neglect of anything so fundamental is certainly striking.

The Greek writers were familiar with all sorts of specific curved loci, witness Proclus' famous summary of the history of geometry, and he says that Aristotle recognized three kinds of loci "*C'est aussi la raison pour laquelle il-y-a trois mouvements, l'une en direction de la ligne droite, l'autre circulaire, et le troisième mixte*" [1]. Proclus himself expands this idea in [1] p. 234, "*Tandis que d'autres ont affectuès la section au moyen des quadratices de Hippias et de Nicomède lesquels avaient aussi fait usage de lignes mixtes quadratices.*" The Greeks were familiar with curved loci, but singularly slow in pointing out the distinguishing characteristics.

The fundamental idea lay very close at hand. Some curves were straight, others were curved. Whatever curvature might be, a circle was everywhere equally curved. The greater the curvature, the less the radius, so it would be natural to take as the curvature a quantity inversely porportional to the radius of the circle, and for any curve, the curvature of the circle lying nearest to it. All this is so simple and natural, but it was strangely slow in coming to be realized. I am sure that a complete account can be found; it certainly is not in [2]. I write in the hope that someone will succeed where I have failed to find it.

The question of curvature is intimately connected with that of the centre of curvature, and Apollonius is perfectly aware that from certain points but one normal can be drawn to a conic, and the careful discussion of this question was earnestly pursued, but the writer did not go on to other loci, and it is curious that neither he nor Euclid in discussing optics took the quite obvious steps in this connection. On the contrary, the attention was diverted to the analogous problem of horn angles. A horn or cornicular angle is roughly the figure formed by the circumference of a circle and its tangent; a good discussion by Heath is found in [3]. We learn by Euclid III, 16 that the angle of the semicircle is greater and the remaining angle is less than any acute rectilineal angle. As long as a secant cuts a circumference twice, the angle which it makes with the curve is less than the horn angle, but the latter, though it can be indefinitely increased, by decreasing the radius of the circle, is nevertheless smaller than a right angle. This is very much contrary to the notion that the two have a definite ratio. Vieta in [4] takes up the question of whether a circumference and its tangent really can be said to make an angle, and points out that in any case this must be a different sort of object from other figures which cannot pass from greater to less without passing through equality. Wallis in [5] insists, in a withering attack on Clavius, that a horn angle is not an angle in Euclid's sense as it is not an inclination between two loci. A new word is needed "*Cui respondeo, mihi cum Clavio hactem convenire (et convenisse semper). Quod quem ille vocat Angulum Contactus, nil aliud est quod ego voco Gradum Curvitatis. Sunt utque curvitatum gradum semper proportionales longitudinibus Diametrorum Chordarium arcuum similibus.*"

I will not go further into the complicated question of horn angles but return to the larger question of definition of curvature. The first writer to give a hint of the definition of curvature was the fourteenth century writer Nicolas Oresme, whose work was called to my attention by Carl Boyer. We find Oresme saying in [6] "*Nunc restat de Curvitate dicendum.*" He assumes the existence of something which he calls *Curvitas*, and if we have two curves touching the same line at the same point, and on the same side, the smaller curve will have the greater curvature. He further states that the curvature of a circle is "*Uniformis*" and on p. 219 "*Sit circulus major cuius semidyameter sit AB et circulus minor cuius semidyameter sit AC, tunc si sit semidyameter AB duplo ad semidyanmeter AC, curvitas minoris circuli erit duplo intensior curvitate majoris, et ita proportionibus et curvitatibus.*" We could not have a clearer proof that Oresme conceived the curvature of a circle as inversely proportional to the radius; how did he find this out?

We apparently have to wait nearly three hundred years before finding anything further on the subject of curvature when we turn to the work of Kepler [7]. He undertakes to find the image of a certain brilliant point, the generalized problem of Al Hazen. He shows the standard approach but adds that instead of dealing with the curve itself it would be wiser to deal with its circle of curvature "*At verior ratio jubet invenire circulum, qui continett rationem curvitatis.*" The ra-

tio he connects with a certain circle, the circle of curvature. His editor, Frischauf in a note on p. 403 to [7] adds "*Primum hic occurit circulus quem dicunt osculatorem, quem posteriores mathematici tum add lineas tum ad superficies curvas maximo commodos adhibebunt, cujus inventio Leibnitio huc usque tributa ets.*" We shall presently see that it was certainly a mistake to associate Leibniz with the invention of the circle of curvature, but we have a glimpse of the fact that the problem of the rightful ascription was not too simple.

A great step in advance is due to Huygens. I mention especially his *Horologium oscillatorium sive de motu pendulorum ad horologia aptato demon strationes geometricae* of 1673 [8]. Kepler starts curiously with the involutes of curves. We take a curve all of whose tangents are on one side. To this is attached a flexible string which is pulled taut and then unwound. The curve from which the string springs is called the evolute. He assumes that the string will always remain tangent to this evolute; the locus of a point fixed in the string will be the involute. He begins with a careful proof that the string will always be normal to the involute. Then comes the curious theorem that two curves with tangents on one side which have a common point cannot have the same set of normals. After this we see that if all tangents to one curve are normals to another they will be an involute and evolute as defined above. He next passes to certain specific curves, notably the cycloid, and in theorem XI he undertakes the problem of finding the radius of curvature of any given curve, defined as the distance up the normal from the foot to the point of contact with the evolute. This is defined as the limit of the intersection with an infinitely near normal. In our notation this would involve Δx and $\Delta[y(dy/dx)]$. He says in regard to this "*Illas vero dari in omnibus curvis geometricis.*" He indicates the method of doing this in general, but his ignorance of the calculus prevented him from attaining a satisfactory result in every case.

2. **Sir Isaac Newton.** The first writer to handle the question of curvature of a plane curve in what we should call today a thoroughly satisfactory manner was Sir Isaac Newton, no less. Huygens wrote the *Horologium oscillatorium* in 1673. Newton started thinking about the Calculus in around 1665 but published nothing on the subject till his letter to Collins of 1669; his ideas were well developed in 1671. The first systematic account appeared in [9] with the date of 1736. There is a long appendix due to Colson himself, but the bulk of the work purports to be a direct translation of Newton's own Latin. I shall return later to the possible relation of this to the *Horologium* of Huygens.

I begin by quoting Newton's own words, pp. 59–61 of [9]:

"The same Circle has everywhere the same curvature, and in different Circles it is reciprocally proportional to their Diameters.

"If a Circle touches any Curve on its concave side, in any given point, and if it be of such magnitude that no other tangent Circle be inscribed in the angle of contact of that Point, that Circle will be of the same Curvature as the Curve is of, in the Point of Contact.

"Therefore the Centre of Curvature to any Point of the Curve is the Centre of the Circle equally curved, and thus the Radius or Semi-diameter of Curvature is Part of the Perpendicular to the Curve which is terminated at the Centre.

"And the Proportion of Curvature at different Points will be known from the Proportion of Curvature of aequi-curve Circles, or from the reciprocal Proportion of the Radii of Curvature."

A further explanation comes presently:

"But there are several Symptoms or Properties of this Point C which may be used in its Determination:

1) That it is the Concourse of Perpendiculars to that Arc on each side at an infinitely little distance from DC.
2) If DC be conceived to move while it insists perpendicularly to the Curve, that point of it C (if you accept the motion of approaching to, or receding from the point of insistance C) will be the least moved, but will be its Centre of Motion.
3) If a Circle be described with the Centre C, and the Distance DC, no other Circle can be described that can lie between it and the Circle of Contact.
4) Lastly if the Centre H or h of any other touching Circle approaches by degrees to C the Centre of this, till at last it coincides with it, then any of the points in which the Circle shall cut the Curve, will coincide with the Point of Contact at D.
5) And each of these Properties may supply the means of solving this Problem in different ways. But we shall make choice of the first, as being the most simple."

What all this amounts to is the following. Newton assumes, as does Huygens, that if a point not an inflection is fixed on a curve, and a second point approaches it from either side, the intersection of the normals approaches a definite limiting position. This is the centre of curvature, the centre of a circle having the same curvature, and no other tangent circle can lie between this and the curve. All of these properties can be proved if one of them is assumed. He seeks the intersection of nearby normals, and if we take x as an independent variable so that $\dot{x}=1$ and $z=\dot{y}/\dot{x}$ he proves very simply

$$DC = \frac{(1+z^2)^{3/2}}{\dot{z}}.$$

This is very satisfactory and may be said to close the question of the discovery of curyature, but there remains a troublesome question of priority. Cantor in his *Geschichte der Mathematik*, Vol. 3, p. 17, raises the question whether between 1671 and 1736 Newton did not see Huygens' work of 1673 and find therein an excellent opportunity to make use of his own vastly superior methods. Newton made reference to flexible strings and pendulums and the preoccupation of both mathematicians with the cycloid troubles Cantor. On the other hand, it should

be noticed that this was a time when a good many geometers were occupied with the cycloid where the determination of the radius was particularly easy. Newton pointed out various methods of finding the radius of curvature, and chooses the best. I cannot feel that we are justified in accusing Newton of plagiarism; it was not in his nature.

There remains the question of Leibniz. In [10] he discusses the angles of mutually tangent curves, and the contact with the "*Circulus osculans*" as if this were an already familiar figure. But the date is late, 1686, and he erroneously says that the osculating circle has four coincident intersections, a mistake promptly pointed out by James Bernoulli.

This represents the limit of my most incomplete knowledge. Where did Oresme get the idea that the curvature of a circle was inversely proportional to the radius? Where did Kepler find the circle of curvature? There are various intersecting questions still to be answered. More power to the persevering man who will answer them.

References

1. Les Commentaires sur le livre des Eléments d'Euclide par Paul Ver Ecke, Bruges, 1948.

2. August Haas, Untersuchung einer Darstellung der Geschichte des Krimmungs masses, Tübingen, 1881.

3. The thirteen books of Euclid's Elements by T. L. Heath. Especially vol. II, pp. 39–43.

4. Vieta, Opera, Ch. XIII, p. 386.

5. Wallis, Opera, II, p. 656, Oxford, 1693.

6. Wieleitner, Ueber den Funktionsbegriff und die graphische Darstellung bei Oresme Bibliotheca mathematica, vol. 14, part 1, Ch. 20, p. 217.

7. Kepler, Opera, Frischauf Ed., vol. 2, p. 175, Frankfort, 1869.

8. Huygens, Ouvre completes, vol. 18. The Hague, 1934.

9. The method of fluxions and infinite series with application to the geometry of curved lines by the inventor Sir Isaac Newton. Translated by Colson, 1736.

10. Meditation nova de nature anguli contactus et osculi Acta Eruditorum, 1686, Schriften (2), vol. 3, p. 326.

A CANONICAL BASIS FOR THE IDEALS OF A POLYNOMIAL DOMAIN

G. SZEKERES, University of Adelaide

1. Introduction. This paper is concerned with ideals in the domain of polynomials with integral coefficients. To put the problem in a more general perspective, I begin with compiling some basic algebraic notions and facts. For details the reader may consult Van der Waerden's *Modern Algebra*, Vols. 1 and 2, henceforth quoted as MAI and MAII.*

* Page numbers refer to the English edition, 1949.

An *integral domain* is a commutative ring (MAI, p. 32) with identity element 1, which contains no divisors of zero: $ab=0$ implies either $a=0$ or $b=0$. If R is an integral domain then the polynomials $f(x)=a_0+a_1x+\cdots+a_nx^n$ with coefficients in R also form an integral domain (MAI, p. 47), denoted by $R[x]$. If not stated otherwise, R shall always denote an integral domain.

A *unit* of R is an element e which is a divisor of 1: $ee_1=1$ for a suitable $e_1\epsilon R$. We say, a and a_1 are *associated* if $a_1=ea$, e unit of R. An element p is called *prime* if $p=bc$ implies that either b or c is a unit. In certain integral domains such as the ring of integers, every element can be factorized into the product of prime elements, in essentially one way. That is, if a is neither zero nor a unit and $a=p_1\cdots p_r=q_1\cdots q_s$ where p_μ, q_ν are prime elements and not units, then $r=s$ and p_ν is associated with q_ν, $\nu=1,\cdots,r$, provided that the factors have been arranged in a suitable order. An integral domain with this property is called a *unique factorization domain.*

The following theorem is essentially due to Gauss (MAI, p. 73):

If the unique factorization theorem holds in R then it also holds in $R[x]$. In particular there is a unique factorization of polynomials with integral coefficients and in any number of variables.

For later purposes we note the following theorem which, in a slightly different form, is known as Gauss' Lemma (MAI, p. 71). It is the usual starting point for the proof of Gauss' theorem.

LEMMA. *If R is a unique factorization domain and $af(x)=g(x)h(x)$ where $g(x)$ is a primitive polynomial then $h(x)=ak(x)$.*

A *primitive polynomial* is one whose coefficients have no common divisor except units. Of course it is understood that the polynomials f, g, h, k belong to $R[x]$.

An *ideal* in R is a non-empty subset $M\subset R$ which has the following properties:

1. $a\epsilon M$, $b\epsilon M$ imply $a-b\epsilon M$.
2. $a\epsilon M$, $c\epsilon R$ imply $ac\epsilon M$.

In particular, if $a_1, a_2, \cdots$ are fixed elements of R, then the set of linear combinations $c_1a_1+\cdots+c_ka_k$, $c_\nu\epsilon R$, $\nu=1,\cdots,k$, $k=1, 2, \cdots$, form an ideal which is denoted by $(a_1, a_2, \cdots,)$. The elements a_ν form a *basis* of the ideal. In the particular case when $k=1$, the ideal (a) consists of the multiples of a and is called a *principal ideal.* There are integral domains called *principal ideal domains*, (pr. i.d.), in which every ideal is principal; for example, the ring of integers is a pr.i.d. Also if F is a field, then $F[x]$ is a pr.i.d. (MAI, p. 55).

In a pr.i.d. the prime factorization of an element, if it exists at all, is always unique (MAI, p. 60). The converse is not generally true; *e.g.* if R is the ring of integers then $R[x]$ is a unique factorization domain but not a pr.i.d. Although ideals of $R[x]$ are not necessarily principal, they have at least a finite basis.

That is, if R is a pr.i.d. then every ideal of $R[x]$ has the form

$$(*) \qquad M = (f_1(x), \cdots, f_k(x)).$$

This is a consequence of the following more general theorem of Hilbert (MAII, p. 18):

If in R every ideal has a finite basis then also the ideals of $R[x]$ have a finite basis.

Ideals in polynomial domains have been studied extensively in the past 100 years, mainly from the arithmetical and algebraical point of view. We shall deal here with a more elementary aspect of the problem. Our purpose is to determine, in a purely enumerative manner, all the ideals of $R[x]$ when R is a pr.i.d. At first sight it would seem that Hilbert's basis theorem furnishes a very simple solution of the problem. In fact, all we have to do is to take every finite combination of polynomials and form the ideals (*). But an ideal has usually a great number of representations in the form (*) and the enumeration, although exhaustive, would involve unnecessary repetitions. It is therefore desirable to have a more economical method of enumeration by which every ideal is obtained in one and (possibly) only one way. This we shall achieve by setting up a canonical basis for the ideals of $R[x]$ and deriving certain numerical invariants which characterize the ideals uniquely. To avoid complications, until the last section we shall assume that R is the ring of ordinary integers.

2. If M is a principal ideal $(f(x))$, then we shall call $(f(x))$ or $(-f(x))$ its canonical basis, according as the leading coefficient of $f(x)$ is positive or negative. If $M=(f(x))M_0$ where the leading coefficient of $f(x)$ is positive and M_0 has the canonical basis $(g_1(x), \cdots, g_m(x))$ then the canonical basis for M shall be defined $(f(x)g_1(x), \cdots, f(x)g_m(x))$. Thus it is sufficient to find a canonical basis for every non-principal ideal M whose elements have no common divisor except ± 1. This condition implies that M contains a non-zero constant d. For, let $h(x)=d(a_lx^l+\cdots+a_0)$, $d\neq 0$, $(a_0, \cdots, a_l)=1$, be a polynomial of smallest degree in M and $f(x)$ an arbitrary polynomial of degree $k\geqq l$ in M. By the usual division algorithm we can determine $g(x)$ so that $h_1(x)=da_l^{k-l+1}f(x)-g(x)h(x)$ shall have a degree less than l. But $h_1(x)$ is in M, hence $da_l^{k-l+1}f(x)=g(x)h(x)$ and by Gauss' lemma $f(x)$ is divisible by the primitive polynomial $a_lx^l+\cdots+a_0$. This is true for every $f(x)\epsilon M$, hence $l=0$, $a_0=\pm 1$ and $h(x)=\pm d\epsilon M$.

Since M contains a constant, it contains polynomials of an arbitrary degree k. For every $k\geqq 0$ let a_k be the smallest positive number which is the leading coefficient of a polynomial of degree k in M, and

$$g_k(x) = a_kx^k + \sum_{i=0}^{k-1} a_{ki}x^i, \qquad (k = 0, 1, 2, \cdots),$$

a corresponding sequence of polynomials. We shall call polynomials $g_k(x)\epsilon M$ with this property *minimal*.

By a well-known argument we can show that a_k is a divisor of the leading coefficient of any polynomial of degree k in M. For, let $c_k = qa_k + r$, $0 \leqq r < a_k$, then $f(x) - qg_k(x) = rx^k + \cdots$ is in M; hence by the minimum property of a_k, $r = 0$, $c_k = qa_k$.

It follows that

$$a_k \mid a_{k-1}, \qquad a_{k-1} = q_k a_k \qquad \text{for } k > 0, \tag{1}$$

(2) every $f(x) \epsilon M$ of degree k is already in $M_k = (g_0(x), \cdots, g_k(x))$.

More precisely,

$$f(x) = \sum_{i=0}^{k} b_i g_i(x), \qquad b_k \neq 0 \tag{3}$$

where the b_1 are constants. From Hilbert's basis theorem we conclude:
(4) There is an $m > 0$ such that $M \neq M_{m-1}$, $M = M_m$.

Of course the sequence of minimal polynomials $g_k(x)$ is not uniquely determined by M. In fact, every combination

$$g_k'(x) = g_k(x) + \sum_{i=0}^{k-1} b_i g_i(x) \tag{5}$$

is minimal in M. Nevertheless it follows from (2) that M_k is independent of the choice of the sequence $g_k(x)$, and also m in (4) is an invariant. We shall call m the degree of M.

We can always choose the combination (5) such that the coefficients of $g_k'(x) = a_k x^k + \sum_{i=0}^{k-1} a_{ki}' x^i$ satisfy the inequality

$$0 \leqq a_{ki}' < a_i, \qquad (i = 0, \cdots, k-1).$$

Further, $g_k'(x)$ is uniquely determined by this condition. For, suppose that $g_k''(x) = a_k x^k + \sum_{i=0}^{k-1} a_{ki}'' x^i \epsilon M$ would also satisfy $0 \leqq a_{ki}'' < a_i$, $i = 0, \cdots, k-1$. Then $g_k'(x) - g_k''(x) = \sum_{i=0}^{k-1} (a_{ki}' - a_{ki}'') x^i \epsilon M$ with $|a_{ki}' - a_{ki}''| < a_i$ for $i = 0, \cdots, k-1$, which is only possible (by the definition of a_i) if $a_{ki}' = a_{ki}''$ for $i = 0, \cdots, k-1$; hence $g_k'(x) = g_k''(x)$.

We have now a uniquely determined basis in M which, in a sense, could be termed canonical. There is however the following difficulty: suppose we prescribe arbitrary values for a_k and a_{ki} subject to the condition

$$0 \leqq a_{ki} < a_i, \qquad (k = 0, \cdots, m;\ i = 0, \cdots, k-1). \tag{6}$$

Then we may ask the following question: Is it true that the corresponding polynomials $g_k(x)$ form a canonical basis in the above sense for $M = (g_0(x), \cdots, g_m(x))$? The following example shows that this is not necessarily true; in fact the $g_i(x)$ need not be minimal at all. Take

$$g_0(x) = 4, \qquad g_1(x) = 2x + 2, \qquad g_2(x) = x^2 + x + 1.$$

Then $M=(4,\ 2x+2,\ x^2+x+1)$ contains $2g_2(x)-xg_1(x)=2$, hence $g_0(x)$ is not minimal. We may try to avoid the difficulty by imposing further restrictions upon the coefficients a_{ki}, but the conditions obtained are too complicated to be of any practical value.

3. We shall describe now a different type of normalization for the system $g_k(x)$ which does not have the defect of the trivial normalization (6). Suppose that $g_0(x), \cdots, g_m(x)$ are minimal in M and let $0<k\leqq m$. It follows from (1) that $q_kg_k(x)-xg_{k-1}(x)$ is a polynomial of degree $<k$; hence by (3),

$$q_kg_k(x) - xg_{k-1}(x) = \sum_{i=0}^{k-1} b_{ki}g_i(x). \tag{7}$$

Conversely, if we are given polynomials

$$g_k(x) = a_kx^k + \sum_{i=0}^{k-1} a_{ki}x^i, \qquad (a_k \neq 0,\ k = 0, \cdots, m),$$

which satisfy a recursion of the type (7) then they are minimal in $M=(g_0(x), \cdots, g_m(x))$.

To show this, consider the set of polynomials of the form

$$f(x) = h(x)g_m(x) + \sum_{i=0}^{m-1} b_ig_i(x), \qquad b_i \in R. \tag{8}$$

The multiple by a constant and the sum of two polynomials of the form (8) has again the same form, and also $xg_k(x)$ for $k<m$ by (7); hence the polynomials (8) form an ideal $M'\subseteq M$ which contains every $g_k(x)$, so $M'=M$. It follows that every $f(x)\in M$ can be written in the form (8). Notably if $f(x)$ has the degree $k\leqq m$ then it can be written in the form

$$f(x) = \sum_{i=0}^{k} b_ig_i(x), \qquad b_k \neq 0. \tag{9}$$

Hence the leading coefficient of $f(x)$ is b_ka_k which proves that $g_k(x)$ is minimal in M.

We may assume that

$$0 \leqq b_{ki} < q_k, \qquad (k = 1, \cdots, m;\ i = 0, \cdots, k-1) \text{ in } (7). \tag{10}$$

Otherwise put $b_{ki}=q_kd_{ki}+b_{ki}'$, $0\leqq b_{ki}'<q_k$ and $g_k'(x)=g_k(x)-\sum_{i=0}^{k-1}d_{ki}g_i(x)$ which is obviously minimal in M. We have by (7)

$$q_k g_k'(x) = x g_{k-1}(x) + \sum_{i=0}^{k-1} b_{ki} g_i(x) - \sum_{i=0}^{k-1} q_k d_{ki} g_i(x)$$

$$= x g_{k-1}(x) + \sum_{i=0}^{k-1} b_{ki}' g_i(x) \qquad 0 \leqq b_{ki}' < q_k.$$

Hence, starting from $g_1(x)$ we can exchange the $g_k(x)$ step by step until all of them satisfy (10).

Again b_{ki} and $g_k(x)$ are uniquely fixed by this condition. Suppose that $g_k'(x)$, $k=0, \cdots, m$, are minimal polynomials satisfying (7) and (10) with coefficients b_{ki}'. Obviously $g_0'(x)=g_0(x)$, hence we may assume $g_i'(x)=g_i(x)$ for $i<k$. Then

$$q_k g_k'(x) = x g_{k-1}(x) + \sum_{i=0}^{k-1} b_{ki}' g_i(x), \qquad 0 \leqq b_{ki}' < q_k,$$

and by (9), since the leading coefficients of $g_k(x)$ and $g_k'(x)$ are equal,

$$q_k g_k'(x) = q_k\left(g_k(x) + \sum_{i=0}^{k-1} b_i g_i(x)\right) = x g_{k-1}(x) + \sum_{i=0}^{k-1} b_{ki} g_i(x) + \sum_{i=0}^{k-1} q_k b_i g_i(x),$$

hence $b_{ki}'=b_{ki}+q_k b_i$, $b_{ki}'=b_{ki}$, $g_k'(x)=g_k(x)$.

Finally we show:

$$q_1 q_2 \cdots q_k g_k(x) = f_k(x) g_0(x), \qquad 0 < k \leqq m, \tag{11}$$

where $f_k(x)$ has the degree k and leading coefficient 1. From (7), $q_1 g_1(x) = (x+b_{10})g_0$ which proves (11) for $k=1$. Assuming induction on k, we have from (7)

$$q_1 \cdots q_k g_k(x) = q_1 \cdots q_{k-1}\left(x g_{k-1}(x) + \sum_{i=0}^{k-1} b_{ki} g_i(x)\right)$$

$$= (x + b_{k,k-1}) f_{k-1}(x) g_0 + \sum_{i=0}^{k-2} b_{ki} q_{i+1} \cdots q_{k-1} f_i(x) g_0$$

$$= f_k(x) g_0.$$

It follows immediately from (11) and Gauss' lemma that

$$g_0 \equiv 0 \pmod{q_1 \cdots q_m}, \qquad g_k(x) \equiv 0 \pmod{q_{k+1} \cdots q_m} \qquad \text{for } k > 0.$$

We can write $g_0 = q_0 q_1 \cdots q_m$, $g_k(x) = q_0 q_{k+1} \cdots q_m f_k(x)$ for $k<m$, $g_m(x) = q_0 f_m(x)$, which implies that q_0 is a common divisor of every $g_k(x)$. Consequently $q_0=1$ and

$$g_0 = q_1 \cdots q_m, \qquad g_k(x) = q_{k+1} \cdots q_m f_k(x), \qquad k < m,$$

$$g_m(x) = f_m(x).$$

Summary: We have shown that M possesses a basis $(g_0(x), g_1(x), \cdots, g_m(x))$ with the following properties:

$$g_0 = q_1 \cdots q_m, \tag{12}$$

$$q_k g_k(x) = x g_{k-1}(x) + \sum_{i=0}^{k-1} b_{ki} g_i(x), \tag{13}$$

$$q_k > 0,\ 0 \leqq b_{ki} < q_k, \qquad 0 < k \leqq m,\ 0 \leqq i < k. \tag{14}$$

The integers q_k, b_{ki} are uniquely determined by M and the corresponding $g_k(x)$ are minimal in M. Thus the $g_k(x)$ form a canonical basis of M which does not suffer from the defect of the system in §2. All that remains to be shown is that given an arbitrarily prescribed system of "invariants" (14) we can always determine the $g_k(x)$ from (12) and (13) recursively. Suppose that for $i<k$, $g_i(x)$ has already been determined and turns out to be divisible by $q_{i+1} \cdots q_m$.* Then the right-hand side of (13) is divisible by $q_k q_{k+1} \cdots q_m$ and $g_k(x)$ can be determined from the formula and is divisible by $q_{k+1} \cdots q_m$. Thus there is a one-to-one correspondence between the ideals of $R[x]$ and the systems of integers (14).

4. As a typical application let us determine the number of ideal divisors of degree m of a constant c, that is, the number of different ideals of degree m in $R[x]$ which contain the constant c. Let $q_1 \cdots q_m = d$ be a divisor of c. The number of different systems of invariants belonging to a fixed set of q_i's is, according to (14), $q_1 q_2^2 \cdots q_m^m$. Each of these ideals can be multiplied by a principal ideal (q_0) where $q_0 d$ is a divisor of c. Hence the total number of ideal divisors of degree m of c is

$$N_m(c) = \sum_{q_0 \cdots q_m / c} \prod_{i=0}^{m} q_i^i.$$

If in particular $c = p_1 \cdots p_r$ is square-free, then

$$N_m(c) = \prod_{i=1}^{r} (2 + p_i + p_i^2 + \cdots + p_i^m) = \prod_{i=1}^{r} \left(1 + \frac{p_i^{m+1} - 1}{p_i - 1}\right).$$

5. Turning now to the general case it is clear that if R is an arbitrary pr.i.d., the whole argument of sections 2 and 3 remains valid except that we have to find a substitute for the normalizing conditions (14). Two types of normalizations are required. First, we must find a well defined representative of each class of associated elements of R, and secondly we must have a well defined representative of each class of residues modulo a given (normalized) element of R. In some of the more important pr.i.d.'s this is achieved in a very natural way. For example, if R is the domain of polynomials in one variable and with coefficients in a field F, then each class of associated polynomials has a member whose leading coefficient is 1, and we may regard this as the normalized representative. Also each class of residues modulo a normalized $f(x)$ contains exactly one member whose degree is less than the degree of $f(x)$.

* This is certainly true for g_0 by (12).

Of course the invariants q_k, b_{ki} are now not numbers but elements of R. In the particular case when $R=F[y]$, q_k is a non-zero polynomial in y with leading coefficient 1, and b_{ki} is a polynomial of a lower degree than q_k (*e.g.* 0 if q_k has the degree 0).

If R is not a principal ideal ring, the construction seems to break down completely. It would be particularly desirable to obtain a canonical basis when R is the ring of polynomials with integral coefficients in one or several variables, or the ring of polynomials in more than one variable over a field F. In all these cases there is a unique factorization in R and also the Hilbert basis theorem is valid for the ideals of $R[x]$.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

NOTE ON AN ARITHMETIC FUNCTION

L. CARLITZ, Duke University

1. Put

$$F(a, m) = \sum_{rs=m} \mu(r)a^s,$$

where $\mu(r)$ is the Möbius function (for references see [3, pp. 84–86], [1]). When $a=p^n$, where p is a prime, $F(p^n, m)/m$ is the number of (primary) irreducible polynomials in $GF[p^n, x]$ of degree m; it is also familiar [4, p. 20] that the number of irreducible polynomials of degree m and belonging to the exponent e is $\phi(e)/m$, where $\phi(e)$ denotes the Euler function and m is a *proper* divisor of $p^{ne}-1$; that is,

$$m \mid p^{ne}-1, \qquad m \nmid p^{nr}-1 \qquad \text{for } 1 \leqq r < e.$$

The fact that m is a divisor of $\phi(e)$ appears incidentally; we wish to point out a simple direct proof. More generally we prove that if e is a proper divisor of a^m-1 then $m \mid \phi(e)$.

Indeed it is evident that $a^m \equiv 1 \pmod{e}$. But since e is a proper divisor of a^m-1, it follows readily that m is the exponent to which a belongs (mod e), and therefore m is a divisor of $\phi(e)$.

We remark that in the special case $a=p$, $e=p^m-1$, this result was noted by U. Scarpis [3, p. 139].

The formula

$$F(a, m) = \sum_{d \mid a^m-1}' \phi(d), \tag{1}$$

where the summation is restricted to proper divisors of a^m-1, was proved by Dickson for arbitrary a [2, p. 36]. Thus, the result just proved gives another direct proof of $F(a, m) \equiv 0 \pmod{m}$.

2. It may be of interest to mention a property of $\Phi(M)$, the Euler function for $GF[p^n, x]$, the domain of polynomials in x with coefficients in $GF(p^n)$: $\Phi(M)$ is defined as the number of polynomials in a reduced residue system (mod M), $M \epsilon GF[p^n, x]$. Then we may state the following:

THEOREM. *If E is a proper divisor of A^m-1 then $m \mid \Phi(E)$.*

The proof is exactly like the proof of the analogous property of $\phi(e)$, indeed $A^m \equiv 1 \pmod{E}$ is the exponent to which A belongs (mod E), and therefore m is a divisor of $\Phi(E)$.

Corresponding to (1) we now have

$$F(a, m) = \sum_{D \mid A^m-1}' \Phi(D), \tag{2}$$

where again the summation is restricted to proper divisors, and

$$a = \mid A \mid = p^{n \deg A}.$$

It is easily verified that Dickson's proof of (1) applies to (2) also.

To indicate an occurrence of $\Phi(E)/m$, we take $A = x$ and recall some notions due to Ore [5, pp. 248–250]. For given $M \epsilon GF[p^n, x]$ there exist polynomials $e(x)$ such that

$$e(x) = \sum_{i=0}^{k} a_i x^{p^{ni}} \equiv 0 \pmod{M} \qquad (a_i \epsilon GF(p^n)); \tag{3}$$

if $e(x)$ is a polynomial of the indicated form with minimum k for which (3) holds, then M is said to *belong* to $e(x)$; the polynomial $E = \sum a_i x^i$ *corresponds* to $e(x)$. Also for $x \nmid E$, the exponent of E is the least $m > 0$ such that $E \mid x^m - 1$; Ore calls m the *index* of $e(x)$. Then Ore has proved that the number of irreducible polynomials of degree m and belonging to $e(x)$ is $\Phi(E)/m$, where m is the index of $e(x)$.

References

1. L. Carlitz, An arithmetic function, Bull. Amer. Math. Soc., vol. 43, 1937, pp. 271–276.
2. L. E. Dickson, A generalization of Fermat's theorem, Ann. of Math. (2), vol. 1, 1899–1900, pp. 30–36.
3. L. E. Dickson, History of the Theory of Numbers, vol. 1, Washington, 1919.
4. L. E. Dickson, Linear Groups, Leipzig, 1901.
5. O. Ore, Contributions to the theory of finite fields, Trans. Amer. Math. Soc., vol. 36, 1934, pp. 243–274.

A CRITERION FOR THE POLYNOMIAL SOLUTIONS OF A CERTAIN RICCATI EQUATION

J. G. CAMPBELL, Albany, Kentucky

Let a Riccati equation have coefficients that are polynomials. The degrees of these polynomials restrict the degrees of possible polynomial solutions in a very definite way. We shall prove the following

THEOREM. *Let the Riccati equation*

$$Ay' = B_0 + B_1y + B_2y^2 \tag{1}$$

have polynomial coefficients A, B_0, B_1, B_2 *of degrees* a, b_0, b_1, b_2 *respectively. If* $a \geqq 1$ *then the degree* m *of any polynomial solution of* (1) *is restricted as follows:*

If $a \neq 1+b_1$, *then* m *must be one of the five numbers,* $1+b_0-a$, $a-b_2-1$, b_0-b_1, $\frac{1}{2}(b_0-b_2)$, b_1-b_2;

If $a = 1+b_1$, *then* $b_0-b_1 \leqq m \leqq b_1-b_2$.

There is no implication here that a polynomial solution exists. The theorem states that a polynomial not of one of the degrees stipulated cannot satisfy equation (1).

Proof: The degrees of the four terms in equation (1) are respectively $a+m-1$, b_0, b_1+m, b_2+2m. Two, at least, of these degrees must be equal for the equation to be satisfied by the polynomial y; there must be no single highest degree term. By equating all possible pairs of the four degrees listed, we find that $a=1+b_1$ or m is one of the five numbers listed in the theorem.

If $a=1+b_1$, then the four terms of the equation have degrees b_1+m, b_0, b_1+m, b_2+2m, respectively. Now either $b_1+m<b_0$, or $b_1+m=b_0$, or $b_1+m>b_0$.

If $b_1+m<b_0$, then we must have $b_0=b_2+2m$, so that $m=\frac{1}{2}(b_0-b_2)$.

If $b_1+m=b_0$, then $m=b_0-b_1$.

If $b_1+m>b_0$, then the b_2+2m term cannot stand alone with highest degree, so $b_2+2m \leqq b_1+m$, or $m \leqq b_1-b_2$. Since $\frac{1}{2}(b_0-b_2)$ is the mean of (b_0-b_1) and (b_1-b_2), it lies between them. Thus, if $a=1+b_1$, then

$$b_0 - b_1 \leqq m \leqq b_1 - b_2.$$

This completes the proof of the theorem.

This theorem was suggested by a problem in E. D. Rainville's *Intermediate Differential Equations*, and we borrow an example from that text (p. 42):

$$(9x^4 + 18x^3 + 11x^2 + 2x)y' = -9x^4 - 6x^3 + 4x^2 + (18x^3 + 24x^2 + 4x + 2)y + 3y^2.$$

Here $a=4$, $b_0=4$, $b_1=3$, and $b_2=0$. Thus $a=1+b_1$ and the theorem states that $1 \leqq m \leqq 3$. The above equation has the five polynomial solutions

$$x,\ x^2 + 2x,\ 2x^2 + 7x/3,\ -x^2 - 2x/3 - \tfrac{2}{3},\ 3x^3 + 5x^2 + 3x.$$

These solutions are of degrees 1, 2, 2, 2, 3, so that each value of m admitted by

the inequality is actually present. Thus the inequalities of the theorem cannot be improved for these values of a and the b's.

Rainville [1] proved that the factorization of polynomial differential operators of order two is dependent on the polynomial solutions of a Riccati equation of the type treated in this note.

The author is grateful to the referee for assistance in putting this note into form for publication.

Reference

1. E. D. Rainville, this MONTHLY, vol. 48, 1941, pp. 519–521.

THE SYLVESTER-FRANKE THEOREM

LEONARD TORNHEIM, University of Michigan

G. B. Price* has indicated the existence of interest in a completely simple and direct proof of the Sylvester-Franke theorem. Here is presented a proof based upon elementary transformations.

Let $A=(a_{ij})$ be an n by m matrix. Let $\lambda(p, t)$ $(p=1, \cdots, {}_tC_s)$ range through the ${}_tC_s$ ways of choosing s numbers from the set $1, \cdots, t$. Let $A_{\lambda(p,\ m)\lambda(q,\ n)}$ be the s-rowed minor of A obtained by using the rows determined by $\lambda(p, m)$ and the columns determined by $\lambda(q, n)$. Then the ${}_mC_s$ by ${}_nC_s$ matrix $A^{(s)}=(A_{\lambda(p,\ m)\lambda(q,\ n)})$ is called an s-th compound of A. If $m=n$ we write $\lambda(p)$ for $\lambda(p, m)$.

SYLVESTER-FRANKE THEOREM. *If $A^{(s)}$ is an s-th compound of the n by n matrix A, then*

$$|A^{(s)}| = |A|^e, \quad \textit{where} \quad e = {}_{n-1}C_{s-1}. \tag{1}$$

There are three types E_1, E_2, E_3 of elementary transformations: E_1 is the multiplication of a row or column by a constant k; E_2 is the interchange of two adjacent rows or columns; and E_3 is the addition of k times a row or column to another row or column. If B is transformed into C by E_i, then

$$|C| = \mu_i |B|, \tag{2}$$

where $\mu_1=k$, $\mu_2=-1$, and $\mu_3=+1$.

The proof of the theorem will be based on the following lemma.

LEMMA. *Let B be an arbitrary square matrix such that*

$$|B^{(s)}| = |B|^e. \tag{3}$$

If B is transformed into C by an elementary transformation E, then $|C^{(s)}| = |C|^e$.

We state first that

* Some identities in the theory of determinants, this MONTHLY, vol. 54, 1947, pp. 75–90.

$$|C^{(s)}| = \mu_i^e |B^{(s)}|. \tag{4}$$

We shall prove this in detail only for $i=2$; the other two cases are easier. Suppose that the u-th and $(u+1)$-th rows of B have been interchanged to give a matrix C. The minors of C which do not involve the u-th or $(u+1)$-th rows equal the corresponding minors of B. If both u and $u+1$ appear in $\lambda(p)$, then $C_{\lambda(p)\lambda(q)} = -B_{\lambda(p)\lambda(q)}$. The number of rows of $C^{(k)}$ having such minors is ${}_{n-2}C_{k-2}$. Next suppose u appears in $\lambda(p)$ but $u+1$ does not. Let $\lambda(p')$ be obtained from $\lambda(p)$ by replacing u by $u+1$. Then $B_{\lambda(p)\lambda(q)} = C_{\lambda(p')\lambda(q)}$ and $B_{\lambda(p')\lambda(q)}$, $= C_{\lambda(p)\lambda(q)}$ since the same elements are used and in the same arrangement, because the u-th and $(u+1)$-th rows are adjacent. Thus from this cause ${}_{n-2}C_{k-1}$ pairs of rows have been interchanged in going from $B^{(k)}$ to $C^{(k)}$. Thus the total number of changes in sign in going from $|B^{(k)}|$ to $|C^{(k)}|$ is

$${}_{n-2}C_{k-2} + {}_{n-2}C_{k-1} = {}_{n-1}C_{k-1} = e.$$

From (4), (2), and (3) it now follows that

$$|C^{(s)}| = \mu_i^e |B^{(s)}| = \mu_i^e |B|^e = |C|^e, \tag{5}$$

and the proof of the lemma is complete.

The proof of the Sylvester-Franke Theorem can now be given as follows. It is known that there exists an n by n matrix

$$D = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix} \quad \text{with} \quad 0 \leqq r \leqq n \tag{6}$$

which can be transformed into the given n by n matrix A by a finite sequence of elementary transformations $E^{(1)}, E^{(2)}, \cdots, E^{(v)}$ as follows:

$$E^{(1)}: \quad D \to D_1, \tag{7}$$

$$E^{(2)}: \quad D_1 \to D_2, \tag{8}$$

$$\cdots\cdots\cdots$$

$$E^{(v)}: \quad D_{v-1} \to D_v = A.$$

The matrix D can be calculated by starting with A and reducing it to the form D by a sequence of elementary transformations. Now $|D| = 1$ if $r=n$ and $|D| = 0$ if $r<n$. Also if $r=n$, $D^{(s)} = I_{n^s}$ and $|D^{(s)}| = 1$; otherwise $|D^{(s)}| = 0$. Hence the relation $|D^{(s)}| = |D|^s$ holds in both cases. Then by the lemma, $\left|D_1^{(s)}\right| = |D_1|^s$. Thus, the hypotheses of the lemma are satisfied also in (8), and $\left|D_2^{(s)}\right| = |D_2|^s$. If follows by induction that $|A^{(s)}| = |A|^s$, and the proof is complete.

Similar methods may be used to give the next results.

COROLLARY. *If A is either equivalent, congruent, or similar to B, then $A^{(s)}$ bears the same relation to $B^{(s)}$.*

Thus to determine the elementary divisors of $A^{(s)}$, find B similar to A and in canonical form and find the elementary divisors of $B^{(s)}$.

COROLLARY. *If the rank of an m by n matrix A is r, then $A^{(s)}$ has rank ${}_rC_s$. If $A^{(s)} \neq 0$, then A is equivalent to B if and only if $A^{(s)}$ is equivalent to $B^{(s)}$.*

COROLLARY. *If A is positive definite, then $A^{(s)}$ is also positive definite. If A is symmetric and $A^{(s)}$ has at least two rows, then A is definite if and only if $A^{(s)}$ is.*

A NOTE ON SCALAR FUNCTIONS OF MATRICES

RICHARD BELLMAN, Stanford University

It is well known that the scalar functions of the matrix A which occur as the coefficients of λ in the characteristic equation of A,

$$|A - \lambda I| = 0 = \lambda^n - f_1(A)\lambda^{n-1} + f_2(A)\lambda^{n-2} - \cdots + (-1)^n f_n(A), \tag{1}$$

possess the important commutative property

$$f_k(AB) = f_k(BA). \tag{2}$$

The purpose of this note is to demonstrate a converse result:

THEOREM. *If $\phi(A)$ is a polynomial in the elements a_{ij}, $i, j = 1, 2, \cdots, n$, of A and has the property that $\phi(AB) = \phi(BA)$ for all square matrices A and B of order n, then $\phi(A)$ is a polynomial in $f_1(A), f_2(A), \cdots, f_n(A)$.*

Proof. Let us consider first the case where A has simple characteristic roots. There exists, then, a non-singular matrix T which reduces A to diagonal form,

$$T^{-1}AT = L = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{3}$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are the characteristic roots of A. Thus,

$$\phi(A) = \phi(T^{-1}LT) = \phi(LTT^{-1}) = \phi(L) = \psi(\lambda_1, \lambda_2, \cdots, \lambda_n), \tag{4}$$

where ψ is a polynomial in the λ_i. By choosing T suitably, we may change arbitrarily the order of appearance of the λ_i. From this it follows that ψ is a symmetric function of the λ_i, and hence a polynomial in the elementary symmetric functions of the λ_i, which are precisely the $f_k(A)$.

Since any square matrix may be arbitrarily closely approximated by matrices possessing simple characteristic roots, it follows that the representation obtained above is valid for general square matrices.

LAPLACE TRANSFORM OF $(\text{erf}\sqrt{t})^2$

I. OPATOWSKI, University of Chicago

Let $\text{erf}\, x = (2/\sqrt{\pi})\int_0^x \exp(-x^2)dx$ and

$$I(t, a) = \int_0^t \exp(-x^2)\,\text{erf}\,(ax)dx. \tag{1}$$

In particular $I(t, \infty) = (\sqrt{\pi}/2)\,\text{erf}\, t$, $I(\infty, a) = (1/\sqrt{\pi})\arctan a$, $I(t, 1) = (\sqrt{\pi}/4)(\text{erf}\, t)^2$. The last two relations may be checked by differentiating them with respect to a and t respectively. Using the symbol $L\{F(t)\} = \int_0^\infty \exp(st)F(t)dt$ and expanding erf (ax) in power series we have:

$$\begin{aligned} L\{I(\sqrt{t}, a)\} &= (1/\sqrt{\pi}) \sum_{n=0}^{\infty} [(-)^n a^{2n+1}/n!(2n+1)]L\left\{\int_0^t \exp(-x)x^n dx\right\} \\ &= s^{-1}[\pi(s+1)]^{-1/2} \sum_{n=0}^{\infty} (-)^n(2n+1)^{-1}(a/\sqrt{s+1})^{2n+1} \\ &= s^{-1}[\pi(s+1)]^{-1/2} \arctan (a/\sqrt{s+1}). \end{aligned}$$

The latter series converges only for $-1 < s \geqq a^2 - 1$, but it may be seen directly from equation (1) that the Laplace transform of $I(\sqrt{t}, a)$ exists for any s whose real part is >0. The last equation reduces to a known result* for $a = \infty$:

$$L\{\text{erf}\,\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}.$$

For $a = 1$ the formula gives:

$$L\{(\text{erf}\,\sqrt{t})^2\} = (4\,\text{arccot}\,\sqrt{s+1})/(\pi s\sqrt{s+1}).$$

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

Editorial Note. The following papers are a selection from those submitted to Classroom Notes in response to the editor's invitation appended to the paper by W. R. Ransom, "Bringing in Differentials Earlier," this MONTHLY, vol. 48, p. 336.

DIFFERENTIALS

M. K. FORT, JR., University of Illinois

1. Introduction. The beginning student usually has more trouble understanding differentials than any other topic in Calculus. One reason for this, as

* R. V. Churchill, Modern Operational Mathematics in Engineering, New York, 1944.

has often been noted, is the use of the "dy/dx notation" instead of the "$f'(x)$ notation" for derivatives. However, even the use of the superior "$f'(x)$ notation" does not help the student to understand what the author of his text is trying to get across when he comes to the notorious proof that $dx=\Delta x$. In order to eliminate the confusion concerning differentials, it is sufficient (and probably necessary) to first give an adequate treatment of the concept of function. With regard to functions, the average Calculus text is open to the following criticisms:

(1) "Function" is never actually defined, but instead a definition is given for the logically different concept "the variable y is a function of the variable x."

(2) The symbol "$f(x)$" is used to denote both a function and the value of that function at x.

(3) No distinction is made between a function and an equation which is used to define the function. (*e.g.* Text books frequently speak of "the function $y=x^2+3$" rather than "the function f for which $f(x)=x^2+3$.")

We now outline a treatment of functions and differentials which the author feels is suitable for presentation at the Calculus level.

2. Functions. F is a *function* if F is a rule which associates with each member x of a certain set A of objects a unique member $F(x)$ of a set B of objects. The set A is the *domain* of the function F and the set B is the *range* of F. If x is in the domain of F, then $F(x)$ is the *value of F at x*. The sets A and B are often sets of numbers, but this is not always the case.

Example 1. Let S be the rule which associates with each real number x its square x^2. Then $S(x)=x^2$ for each real number x. The domain of S is the set of all real numbers and the range of S is the set of all non-negative real numbers.

Example 2. Let d be the rule which associates with each point (x, y) of the plane the distance from (x, y) to $(0, 0)$. The domain of d is the set of all ordered number pairs and the range of d is the set of all non-negative real numbers. A function whose domain is a set of ordered number pairs is often called a *function of two variables* and the value of such a function f at (x, y) is denoted by $f(x, y)$ instead of $f((x, y))$. Thus for d we have $d(x, y)=(x^2+y^2)^{1/2}$.

Example 3. Let C be the function which associates with each triangle T its inscribed circle $C(T)$. The domain of C is the set of all triangles and the range of C is the set of all circles.

It is convenient to use the expression "ordinary function" to describe a function whose domain and range are both sets of numbers. As soon as derivatives are defined (carefully avoiding the dy/dx notation) the following example is instructive.

Example 4. Let D associate with each differentiable function f the function f'. Then D is a function whose domain and range are both sets of ordinary functions.

3. Differentials. Let f be a function which has a derivative at a number x. We define the *differential of f at x* to be the function g for which $g(h)=f'(x)\cdot h$ for

each real number h. Thus the differential of f at x is an ordinary function which is linear and whose domain is the set of all real numbers. We shall denote the differential of f at x by df_x. The *differential of* f is a function of two variables whose domain is the set of all pairs (x, h) for which x is a number at which f has a derivative and h is a real number; the differential of f is denoted by df and is defined by the equation $df(x, h) = df_x(h)$. (If the reader chooses, he may replace h by Δx throughout this paper.)

If one defines in the usual way the addition of functions, the multiplication of functions and multiplication of functions by numbers, then it is easy to prove that

$$\begin{aligned} d(f + g) &= df + dg \\ d(f \cdot g) &= df \cdot g + f \cdot dg \\ d(c \cdot f) &= c \cdot df \end{aligned}$$

whenever f and g are functions and c is a number.

We define the identity function I by letting $I(x) = x$ for each real number x. It is easy to prove that $dI_x(h) = h$ for all numbers x and h. Hence, for any function f which has a derivative at x,

$$df_x(h) = f'(x) \cdot dI_x(h)$$

for all h. It follows that the function df_x is equal to the product of the function dI_x by the number $f'(x)$ and thus

$$df_x = f'(x) \cdot dI_x.$$

The above equation is true for each x at which f has a derivative, and hence

(i) $$df = f' \cdot dI.$$

Formula (i) is equivalent to the theorem "$dy = f'(x)dx$" which states that the differential of a function f is equal to the product of the derivative of f and the differential of the identity function.

The differential df is a function of two variables. In order to check his understanding of our notation the reader should prove the following statement. If f'' exists, then the partial derivative of df with respect to the first variable exists and is equal to the differential of f'.

4. Integration. The Editor has requested discussion of the following question:*

(ii) Does the "dx" of differential calculus have the same meaning as the "dx" in the integral $\int_a^b f(x)dx$?

The answer to this question is both yes and no.

If $\int_a^b f(x)dx$ is the Riemann integral (*i.e.* defined as the limit of a finite sum), then there is no connection between the "dx" in the integral and "dx" as used in

* This MONTHLY, vol. 58, 1951, p. 337.

the differential calculus. In this case the "dx" in the integral is used to make the "x" a bound variable. It serves the same purpose that the first "j" does in the expression "$\sum_{j=3}^{5} k^j$." (Notice that if we omitted the j and instead wrote $\sum_{3}^{5} k^j$ then we would be in doubt as to whether $3^j+4^j+5^j$ or $k^3+k^4+k^5$ was intended.) While $\int_a^b f(x)$ would not be a good notation for the integral (since x appears in this expression as a free variable), there is no logical objection to a notation such as $\int_a^b f$. However, it would be difficult to express $\int_2^3 (x^2-5)dx$ in this latter notation since there is no commonly used notation which denotes the function whose value at x is x^2-5. This function can be denoted by an expression such as $\lambda x(x^2-5)$ [see The Calculi of Lambda-Conversion†], and if this notation were used for the function then we could denote our integral by $\int_a^b \lambda x(x^2-5)$ instead of $\int_a^b (x^2-5)dx$. This concludes the discussion of our answer of "no" to (ii).

We now examine the circumstances under which the answer to (ii) is "yes." Let us suppose that f is a continuous function. Then it is possible to prove that there exist functions g such that $g'=f$. It follows from (i) that $dg=f\cdot dI$ for such functions g. We may now define $\int f\cdot dI$ to be the class of all functions g for which $dg=f\cdot dI$. (The "dI" in $f\cdot dI$ is actually superfluous since it is just as reasonable to denote this class of functions by $\int f$.) Finally, we can define $\int_a^b f\cdot dI=g(b)-g(a)$ where g is a member of the class $\int d\cdot dI$. Thus, if integration is introduced via the indefinite integral or primitive method, and "dx" represents the differential of the identity function, then it is possible to interpret the "dx" in $\int_a^b f(x)dx$ as being the same as the "dx" of the differential calculus.

5. Conclusion. The author feels that most of the confusion concerning differentials disappears if the instructor:

(1) uses the word "variable" only to mean a symbol (*i.e.* an ink spot or chalk mark, etc.,) and avoids the usual implication that a variable is a mysterious Dr. Jekyll and Mr. Hyde type of gadget;

(2) carefully distinguishes between a function f and the number $f(x)$ which is the value of f at the number x;

(3) defines the differentials of functions and not the differentials of variables.

THE RELATION OF DIFFERENTIAL AND DELTA INCREMENTS

C. G. PHIPPS, University of Florida

Do mathematicians in general have as much trouble understanding the use and meaning of the differential notation as is indicated by the sporadic references to it which have appeared in this MONTHLY? It is to be hoped not, for there should be no confusion *when the symbolism is properly defined.*

This note is written in response to the Editor's two queries: (1) what is a differential and (2) how is the differential notation related to that for integration? One consistent set of definitions is given here which answers the questions in the order they are proposed.

† Alonzo Church, *The Calculi of Lambda-Conversion*, Princeton University Press, 1941.

Let $y=F(x)$ be a continuous function with continuous derivatives of all orders. The derivative of this function is found by the usual limiting process and the result may be indicated by any one of the usual symbols, say $D_x y$.

Let (x, y) and (x', y') be two different solutions of the above equation. We define the delta-increments as

$$\Delta x = x' - x \quad \text{and} \quad \Delta y = y' - y. \tag{1a}$$

Then we define the differentials of these variables as

$$dx = \Delta x \quad \text{and} \quad dy = (D_x y)\Delta x. \tag{1b}$$

The first definition in (1b) applies specifically to the independent variable and the second to the dependent one. Furthermore, these definitions are consistent for, if we write $y=x$, the second definition yields the same result which is stated in the first.

From the latter two definitions we obtain the usual forms

$$dy = (D_x y)dx \quad \text{and} \quad \frac{dy}{dx} = D_x y. \tag{2a, 2b}$$

The second of these forms supplies the reason for writing the derivative in its usual fractional form. It also shows the change from Δx to dx gives a symmetry to the notation.

The first of these forms has two interpretations:

(i) it is a *rule* for carrying out the operation of "taking the differential"; and

(ii) it is a *formula* for computing the value of the differential of the dependent variable.

The right member of (2a) is a function of two independent quantities, x and dx. If dx is held constant, dy becomes a function of x alone to which we can apply the rule in (i) and define the second differential of y as

$$d(dy) = d^2 y = [(D_x^2 y)dx]dx = (D_x^2 y)dx^2. \tag{3a}$$

Further, we define the higher differentials of y as the result of successive applications of this rule under the same conditions and obtain

$$d^n y = (D_x^n y)dx^n. \tag{3b}$$

These differential equations may then be transformed into the usual fractional notation for derivatives of higher orders.

If the function $F(x)$ is expanded into a Taylor's series, we may write

$$\Delta y = (D_x y)dx + \frac{1}{2!}(D_x^2 y)dx^2 + \cdots + \frac{1}{n!}(D_x^n y)dx^n + \cdots. \tag{4a}$$

By the aid of our definitions this series may be transformed into

$$\Delta y = dy + \frac{1}{2!}d^2y + \frac{1}{3!}d^3y + \cdots + \frac{1}{n!}d^ny + \cdots . \tag{4b}$$

Thus it is seen that our definitions are convenient as well as consistent.

It is well-known that within the interval of convergence the partial sums of the terms of this series may be used as successive approximations to Δy. For instance, the first term alone gives the value Δy would have if its rate of change were constant while x changed from x to x'. The first two terms yield the value Δy would have if this rate of change itself had changed at a constant rate. The first three terms have the constant rate removed one step further. And partial sums with more terms have corresponding interpretations.

If the function $F(x)$ does not possess continuous derivatives beyond the n-th order, we can of course obtain no more than the n-th differential. At the same time, the series approximation would likewise not extend beyond the n-th order.

The extension of these definitions to functions of more than two variables is illustrated by two examples. Let $z=G(x, y)$. As independent variables, we define the differentials of x and y as equal to their delta-increments. Then

$$dz = G_x dx + G_y dy, \tag{5}$$

where the subscript denotes partial differentiation. This form gives both a *rule* for "taking the differential" and a *formula* for computing its value. The definition may be extended to any order. Furthermore, the relation between Δz and its differentials of all orders is given by a form identical with that in (4b).

If $z=G(x, y)$ where $y=F(x)$, the value and form of dz is still given by (5) wherein the value of dy is given by (2a). The second differential, however, is more complicated; it is

$$d^2z = G_{xx}dx^2 + 2G_{xy}dxdy + G_{yy}dy^2 + G_y d^2y. \tag{6}$$

The differentials of higher orders are correspondingly more complicated but can with care be written down. Also, the form in (4b) still holds true.

Turning now to the notation for integration, we find its connection with differential notation dependent upon the meaning attached to the integral.

(a) We may begin by defining the integral sign as the indefinite operator defined in the equation

$$\int dy = y + C.$$

If $y=F(x)$ and $D_x y=f(x)$, we have by substitution that

$$\int f(x)dx = F(x) + C.$$

By this definition, integration is defined (except for the constant C) as the inverse operation of "taking the differential." Hence, the integral sign as an

operator in this sense has no meaning unless it is applied to a perfect differential.

(b) Alternately, we may define Δy exactly by the expression

$$\Delta y = \int_{x_0}^{x_0+\Delta x} f(x)dx,$$

where $y = F(x)$ and $D_x y = f(x)$ as above.

The value of this integral may be approximated by taking $f(x) = f(x_0)$ throughout the interval. This approximate value is $(dy =) f(x_0)\Delta x$. A better approximation may be obtained by dividing the interval into more and smaller parts and using the above approximation for each part. The limit as the number of parts becomes infinite is the exact value of the integral.

This method of approximating Δy does not correspond, except for the first step, with the method which uses either form of (4). Hence the dx in this integral has no relation either to Δx or to the previous form of dx.

The only obvious reason for using the same symbolism in (b) as in (a) is that the definition in (a) may be used to evaluate the limit represented by (b), namely as

$$F(x_0 + \Delta x) - F(x_0).$$

Thus by writing the definite integral in the same form as the indefinite, the known *properties of the latter form* may be used to find the *value of the former*. As with differentials, one interpretation gives the form, the other gives the substance.

TOWARD UNDERSTANDING DIFFERENTIALS

H. J. HAMILTON, Pomona College

1. Introduction. The mathematics teacher hardly needs to excite himself today about identification of differentials with "nascent" or "evanescent" increments or with "vanishingly small" quantities, or even with "infinitesimals." If physics has not entirely abandoned these mystical ideas, at least mathematics has done so. It is now standard mathematical pedagogy to introduce differentials as follows. Let $y = f(x)$, where x is the independent variable. Then we define $dx \equiv \Delta x$ and $dy \equiv f'(x)dx$, where Δx is an arbitrary, non-zero increment in x. A picture goes with these definitions, and we explain to our students that a perfectly good value for dx is the distance in millimeters to the remotest known star; dy/dx is still equal to $f'(x)$.

(Of course, if we are as careful as we should be, we say that dx is a *variable* increment rather than an *arbitrary* increment, for we need to "see" the "dx" in most cases—certainly in integrands, where to give dx an arbitrary value would

is the equation (4); and hence conclude that we have a solution of (1).

In differential geometry expressions such as $ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$ are considered frequently as if ds, du, and dv were well defined quantities. However, a careful inspection of the use of these forms reveals that in every case such a form is really an integrand with the integral sign omitted for convenience. The meanings of the differentials are to be interpreted as if an integral were written explicitly. A similar understanding exists in probability theory when one speaks of the "probability $f(x)dx$" where $f(x)$ is a probability density.

There is a discredited view that differentials are some sort of "infinitely small quantities." Of course, this is nonsense, but there is a germ of truth in it. For differentials do mark the places in mathematical expressions which are held by small, but finite, quantities while some type of limiting process is carried out in which these small quantities tend to zero. To the writer they have no other consistent meaning.

C. B. A.

CORRECTION

Several correspondents have written the editors regarding the paper by J. J. Hart, "A Correction for the Trapezoidal Rule," in this MONTHLY, vol. 59, p. 33. J. P. Ballantine has pointed out that this formula is actually the Euler-Maclaurin expansion (see L. R. Ford, *Differential Equations*, p. 133).

E. G. H. Comfort and Ballantine have noted a computational error in the value of $\int_4^{5.4} \log x\,dx$. The correct value is 1.82784741 instead of 1.82784744. They have also noted that Weddle's rule is more accurate then the proposed rule, in contradiction to the author's statement. Comparative results are:

	Result		*Error*
The Integral	1·82784	74086	
Trapezoidal Rule	1·82765	51387	$-19\ 22699\times10^{-10}$
Simpson's Rule	1·82784	72580	$-\ 1506\times10^{-10}$
Weddle's Rule	1·82784	74073	$-\ 13\times10^{-10}$
Trapezoidal Rule with correction	1·82784	74464	$+\ 378\times10^{-10}$

Hart reports that similar results appear in the M.A. thesis of Ann G. Lutz at the University of Alabama.

CORRECTION

F. ALBERTI, University of Illinois

The following correction should be made in my note "A Theorem on Evolútes" (this MONTHLY, March, 1952, p. 177): In the theorem, replace the phrase "second order contact" by "two point contact," and, at the end of the theorem, add the phrase "provided the circle's right cylinder has three point contact at P."

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, N. Y. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1021. *Proposed by W. R. Ransom, Tufts College*

In the "ride and tie" plan (where the first man rides a horse for a stated time and leaves it for a second man, walking, to ride for a like time while the first man walks on) show that the rate of progress is the harmonic mean between the rates of riding and walking, and find for what fraction of the time the horse can rest.

E 1022. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find all triangles for which the three sides and one of the altitudes are, in some order, measured by four integers in arithmetic progression.

E 1023. *Proposed by D. J. Newman, Harvard University, and H. S. Shapiro, Massachusetts Institute of Technology*

(1) Given any set of points in the plane, not all coincident with the origin, show that there exists a point on the unit circle such that the product of the distances from it to these points is greater than 1.

(2) Given any set of points on the unit circle, not the vertices of a regular polygon, show that there exists a point on the unit circle such that the product of the distances from it to these points is greater than 2.

E 1024. *Proposed by David Gale, Brown University*

Assuming that the four color theorem is true for planar maps containing a finite number of regions, show that it is also true for planar maps containing an infinite number of regions.

E 1025. *Proposed by K. L. Chung, Cornell University*

Show that if

$$a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0$$

and

$$\sum_{j=1}^{k} a_j \leqq \sum_{j=1}^{k} b_j$$

for $k=1, \cdots, n$, then

$$\sum_{j=1}^{n} a_j^2 \leqq \sum_{j=1}^{n} b_j^2,$$

equality holding if and only if $a_j=b_j$, $j=1, \cdots, n$.

SOLUTIONS

E 984 [1952, 252]. *Correction*

In solution I the argument of arc cos should be $x/2$ instead of x.

A Slow Ship Intercepting a Fast Ship

E 991 [1951, 699]. *Proposed by C. S. Ogilvy, Syracuse University*

Two ships leave different points at the same time and steam on straight courses at constant but unequal speeds. Find the condition under which the slower ship can intercept (catch) the faster.

Solution by M. S. Klamkin, Brooklyn Polytechnic Institute. If the slower ship intercepts the faster ship then the ratio of the distances traversed is V_S/V_F, where V_S is the velocity of the slow ship and V_F is the velocity of the fast ship. But the locus of a point such that the ratio of its distances from two fixed points is constant, is a circle. Thus if the path of the faster ship intersects this circle it can be intercepted by the slower one.

Editorial Note. The circle is a circle of Apollonius of the two starting points. If θ is the angle between the faster ship's course and the line joining the two starting points, then it is easily shown that the slower ship can set a course to intercept the faster ship if and only if

$$\theta \leqq \arcsin (V_S/V_F).$$

If we have inequality here, then there are two courses that the slower ship may set. The problem becomes more interesting if the sphericity of the earth is taken into account.

Also solved by J. W. Baldwin, Leon Bankoff, J. H. Braun, R. P. Eisinger, Herbert Emich, Ray Jurgensen, J. D. E. Konhauser, D. C. B. Marsh, Prasert Na Nagara, S. T. Parker, L. A. Ringenberg, Azriel Rosenfeld, and the proposer.

Area-and-Perimeter Bisectors of a Quadrilateral

E 992 [1951, 699]. *Proposed by Kaidy Tan, Anglo-Chinese College, Amoy, China*

Draw a straight line which will bisect both the area and the perimeter of a given quadrilateral.

Solution by Michael Goldberg, Washington, D. C. There is an area bisector for every orientation. Therefore the ratio of the two parts of the perimeter varies from below unity to above unity for a half turn of the area bisector. It follows that there are an odd number of area-and-perimeter bisectors unless there are an infinity of them.

For symmetric quadrilaterals, the axis of symmetry bisects perimeter and area. For an isosceles trapezoid, not only the axis of symmetry, but any line through the midpoint of this axis will bisect the perimeter and the area. For parallelograms, any line through the center will do. The following methods apply to all other cases.

Let the lengths of the sides of the quadrilateral be a, b, c, d. Extend the side b for the distance f and the opposite non-parallel side for the distance e to intersect at P. Let a be the side nearest P. Then if the desired bisector cuts d at the distance x from P, and the side b at the distance y from P, the required halving of the perimeter requires that

$$(x - e) + (y - f) + a = \tfrac{1}{2}(a + b + c + d),$$

or

$$x + y = e + f + \tfrac{1}{2}(b + c + d - a) \equiv k_1. \tag{1}$$

Consider the three triangles which have P as a common vertex and whose opposite sides are a, c, and the desired bisector. The area of the latter triangle is required to be a mean between the areas of the other two triangles. The areas are proportional to the products of the two sides adjoining P. Therefore

$$xy = \tfrac{1}{2}(e + d)(f + b) + \tfrac{1}{2}fe \equiv k_2. \tag{2}$$

Solving equations (1) and (2) for x and y we obtain

$$x,\ y = \tfrac{1}{2}k_1 \pm \tfrac{1}{2}\sqrt{k_1^2 - 4k_2}\,. \tag{3}$$

If the desired bisector cuts two adjacent sides of the quadrilateral, then we have

$$x + y = \tfrac{1}{2}(a + b + c + d) \equiv k_3 \tag{4}$$

and

$$xy = A/\sin\theta \equiv k_4, \tag{5}$$

where A is the area of the quadrilateral and θ is the angle at P. In this case we find

$$x,\ y = \tfrac{1}{2}k_3 \pm \tfrac{1}{2}\sqrt{k_3^2 - 4k_4}\,. \tag{6}$$

Since k_1, k_2, k_3, k_4 are obtainable from the given quadrilateral by euclidean

methods, and only square roots are involved in equations (3) and (6), the desired bisector is obtainable by euclidean constructions.

Also solved by I. A. Dodes, M. S. Klamkin, Sam Kravitz, David Mandelbaum, and B. Martin.

Orthogonal Trajectories of a Family of Cycloids

E 993 [1951, 699]. *Proposed by F. M. Carpenter, Colorado School of Mines*

Find the orthogonal trajectories of the family of cycloids

$$x = a(\theta - \sin\theta),$$
$$y = a(1 - \cos\theta).$$

Solution by E. W. Marchand, Eastman Kodak Company, Rochester, N. Y. Differentiation gives

$$-dx/dy = (\cos\theta - 1)/\sin\theta,$$

and elimination of a gives

$$x/y = (\theta - \sin\theta)/(1 - \cos\theta), \tag{1}$$

which defines θ implicitly as a function of x and y. Hence an orthogonal curve obeys the equation

$$dy/dx = (\cos\theta - 1)/\sin\theta. \tag{2}$$

Differentiation of (1) and elimination of dx between (1) and (2) leads to the separable equation

$$dy/y = [\cot(\theta/2) - 2/\theta]d\theta, \qquad \theta \neq 0,$$

which has the solution

$$y = c(1 - \cos\theta)/\theta^2.$$

Substitution in (1) then yields the solution

$$x = c(\theta - \sin\theta)/\theta^2,$$
$$y = c(1 - \cos\theta)/\theta^2.$$

Also solved by E. S. Keeping, O. E. Stanaitis, and the proposer.

An Inequality Involving Factorials

E 994 [1951, 699]. *Proposed by C. S. Venkataraman, Trivandrum, South India*

Prove that $2!4! \cdots (2n)! > (n+1)!^n$.

I. *Solution by A. E. Livingston, University of Oregon.* The inequality is satisfied for $n=2$. Assume that it is true for $2, \cdots, n-1$. Then, by induction,

$$(2n)!\prod_{k=1}^{n-1}(2k)! > (2n)!(n!)^{n-1} = 2n(2n-1)\cdots(n+1)(n!)^n$$

$$> (n+1)^n(n!)^n = (n+1)!^n.$$

We have equality, instead of inequality, when $n=1$.

II. *Solution by Arthur Sturdley, Massachusetts Institute of Technology.* An easy computation reveals that the sequence $\{a_n\}$, where

$$a_n = \frac{2!4!\cdots(2n)!}{(n+1)!^n},$$

is strictly increasing for $n>0$. Since $a_1=1$, the inequality holds for $n>1$.

Also solved by P. M. Anselone, F. Bagemihl, J. W. Baldwin, Leon Bankoff, J. L. Botsford, J. H. Braun, Aaron Buchman, Kenneth Bush, S. A. Foote, C. V. Fronabarger, Robert Gambill, C. B. Germain, Bernard Greenspan, Emil Grosswald, S. W. Hahn, B. A. Hausmann, Vern Hoggatt, Ray Jurgensen, H. Kaufman, P. G. Kirmser, M. S. Klamkin, Milton Lees, D. B. Marsh, Kovina Milosevich, Rodolfo Morales, Prasert Na Nagara, Robert Oeder, L. A. Ringenberg, Azriel Rosenfeld, J. Shipman, M. R. Spiegel, O. E. Stanaitis, C. A. Swanson, Alfred Sylwester, J. A. Tierney, Peter Treuenfels, F. H. Young, and the proposer. Late solutions by F. F. Dorsey, Herbert Emich, Abraham Franck, Charles Pinzka.

Sum of a Finite Series

E 995 [1951, 700]. *Proposed by L. C. Hsu, National Tsing-Hua University, Peiping, China*

Find the sum of the finite series

$$\sum_{n=1}^{N}(n/2^{2n})\binom{2n}{n}.$$

Solution by M. R. Spiegel, Rensselaer Polytechnic Institute. If we define

$$v_n = \frac{n(n-1)}{3\cdot 2^{2n-1}}\binom{2n}{n},$$

then

$$v_{n+1} - v_n = (n/2^{2n})\binom{2n}{n}.$$

Hence the sum of the series is

$$\frac{N(N+1)}{3\cdot 2^{2N+1}}\binom{2N+2}{N+1}.$$

Also solved by J. W. Baldwin, J. L. Botsford, Kenneth Bush, L. Carlitz, Emil Grosswald, Vern Hoggatt, Philip Kirmser, R. Kissling, M. S. Klamkin, L. A. Ringenberg, O. E. Stanaitis, Chih-yi Wang, and the proposer.

Grosswald pointed out that the sum is asymptotic to

$$\frac{2N+1}{3}\sqrt{\frac{N}{\pi}}.$$

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4470. [1952, 46]. *Correction*

In line 5 read AB, BC, CA instead of BC, CA, AB.

4493. *Proposed by Paul Erdös, University College, London, England*

For each integer n let $\sigma(n)$ be the sum of its divisors. Prove that

$$\sum_{n=1}^{\infty} \frac{\sigma(n)}{n!}$$

is irrational.

4494. *Proposed by D. J. Newman, Harvard University*

Let $a_1 < a_2 < \cdots$ be a sequence of positive integers, and let ξ be the unending decimal fraction formed by juxtaposing the a's (e.g. if the a's are the primes, ξ would be $0.23571113\cdots$). Prove that if $\sum 1/a_n = \infty$, then ξ is irrational.

4495. *Proposed by M. S. Wertheim, Ithaca, New York*

Find all regular functions of a complex variable for which the mean value theorem holds for all a, b, in the form

$$\frac{F(b)-F(a)}{b-a} = F'(c)$$

where c is a point on the straight line segment ab.

4496. *Proposed by H. D. Grossman, New York City*

Let j and k be integral constants, one positive and the other non-negative; and let $r_1, r_2, \cdots, r_j; s_1, s_2, \cdots, s_k$ range over all possible sets of $j+k$ distinct positive integers. Then prove (or disprove) the formula:

$$\sum^{\infty} (1/r_1^4 r_2^4 \cdots r_j^4 s_1^2 s_2^2 \cdots s_k^2) = \frac{(2j+k)!4^j\pi^{4j+2k}}{(2j+1)!k!(4j+2k+1)!}.$$

4497. *Proposed by Jacques Dutka, Rutgers University*

Set down the first n integers, $1, 2, \cdots, n$, in a line and in a second line beneath set down a permutation of them. The number of permutations in the second line which are disjoint with that in the first (no number appears in the same position in both rows) is known to be the integer nearest to $n!/e$. (See, e.g., Whitworth, *Choice and Chance.*)

Suppose now that there are p rows of permutations. How many sets are there in which each row is disjoint with all the others?

SOLUTIONS

Convergence of Series

4415 [1950, 691]. *Proposed by R. P. Boas, Jr. and W. K. Hayman, Brown University*

Find all the values of α and β for which the series

$$\sum_{n=1}^{\infty} n^{\alpha} \sin (n^{\beta})$$

converges.

Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota. Certain special cases are well known. If $\alpha < -1$ the series converges absolutely for any real value of β. If $\beta < 0$, for large n, $\sin n^{\beta} = O(n^{\beta})$, and the series converges for $\alpha+\beta < -1$. For $\beta = 1$, the series converges if $\alpha < 0$.

Hereafter we may assume $-1 \leqq \alpha < 0$ and $\beta > 0$. Put $\nu = -\alpha$, $0 < \nu \leqq 1$ and let $x^{-\nu} \sin x^{\beta}$ be represented by $f(x)$. Using Euler's summation formula in its simplest form (Knopp, *Infinite Series*, 2nd English edition, p. 521) we have

$$\sum_{x=1}^{n} f(x) = \frac{f(1)+f(n)}{2} + \int_1^n \frac{\sin x^{\beta} dx}{x^{\nu}} + \beta \int_1^n \frac{\phi(x) \cos x^{\beta} dx}{x^{\nu-\beta+1}}$$

$$- \nu \int_1^n \frac{\phi(x) \sin x^{\beta} dx}{x^{\nu+1}}, \qquad \phi(x) = -\sum_{k=1}^{\infty} \frac{\sin 2k\pi x}{k\pi}.$$

If we observe that $|\phi(x)| < 1$, it is obvious that the last integral converges absolutely as $n \to +\infty$. Upon substituting $t = x^\beta$, we see easily that the first integral, as $n \to +\infty$, converges if $\nu + \beta > 1$. The question of convergence of the second integral requires further consideration.

Term by term integration of the series $\phi(x)$ is admissible (Bromwich, *Infinite Series*, 1926, ex. 22, p. 144) and the integral may be put in the form

$$\int_1^n \frac{\phi(x)\cos x^\beta dx}{x^{\nu-\beta+1}} = -\sum_{k=1}^{\infty} \frac{1}{2k\pi} \int_1^n \frac{\sin (x^\beta + 2k\pi x)dx}{x^{\nu-\beta+1}} + \sum_{k=1}^{\infty} \frac{1}{2k\pi} \int_1^n \frac{\sin (x^\beta - 2k\pi x)dx}{x^{\nu-\beta+1}}$$

Let us denote the first integral on the right side by G_k and the second by G_{-k}. Integration by parts gives

$$G_k = \frac{-\cos n^\beta}{(\beta n^{\beta-1} + 2k\pi)n^{\nu-\beta+1}} + \frac{\cos 1}{\beta + 2k\pi} - \int_1^n \frac{[\beta\nu x^{\nu-1} + (\nu - \beta + 1)2k\pi x^{\nu-\beta}]\cos (x^\beta + 2k\pi x)dx}{(\beta x^\nu + 2k\pi x^{\nu-\beta+1})^2}.$$

If it is observed that the first term is of the form $O(n^{-\nu})$ and the integrand of the form $O(x^{-\nu-1})$, the convergence of G_k for $\beta > 0$, as $n \to \infty$, is apparent and $G_k = O(1/k)$ is a corollary.

The convergence of G_{-k} may be shown and the corollary $G_{-k} = O(1/k)$ obtained in the very same way if $\beta < 1$. To avoid zero denominators in case of $\beta > 1$ we write

$$G_{-k} = \int_1^{x_1} + \int_{x_1}^n = G'_{-k} + G''_{-k}$$

where the integer x_1 satisfies $(2k\pi)^{1/(\beta-1)} < x_1 < n$. Then, just as in the case of G_k, we have

$$G''_{-k} = \frac{-\cos n^\beta}{(\beta - 2k\pi n^{1-\beta})n^\nu} + \frac{\cos x_1^\beta}{(\beta - 2k\pi x_1^{1-\beta})x_1^\nu} - \int_{x_1}^n \frac{[\beta\nu x^{\nu-1} - (\nu - \beta + 1)2k\pi x^{\nu-\beta}]\cos (x^\beta - 2k\pi x)dx}{(\beta x^\nu - 2k\pi x^{\nu-\beta+1})^2}.$$

Thus the convergence of G''_{-k}, as $n \to \infty$, is established. Since G'_{-k} is finite, the convergence of G_{-k} for $\beta > 0$ and $n \to \infty$ is evident.

From the obvious form

$$G_{-k} = \int_1^n \frac{(2k\pi - \beta x^{\beta-1})\sin (x^\beta - 2k\pi x)dx}{2k\pi x^{\nu-\beta+1}} + \int_1^n \frac{\beta x^{\beta-1}\sin (x^\beta - 2k\pi x)dx}{2k\pi x^{\nu-\beta+1}},$$

upon integrating by parts the first integral on the right we obtain

$$G_{-k} = \frac{1}{2k\pi}\left[\frac{\cos n^\beta}{n^{\nu-\beta+1}} - \cos 1 - (\beta - \nu - 1)\int_1^n \frac{\cos (x^\beta - 2k\pi x)dx}{x^{\nu-\beta+1}} + \beta\int_1^n \frac{\sin (x^\beta - 2k\pi x)\, dx}{x^{\nu-2\beta+2}}\right].$$

Now as $n\to\infty$, the first integral on the right converges if $\nu-\beta+1>0$, and is $O(1)$ as $k\to\infty$. Writing the second integral as

$$\int_1^n x^{2\beta-\nu-2}\sin x^\beta \cos 2k\pi x dx - \int_1^n x^{2\beta-\nu-2}\cos x^\beta \sin 2k\pi x dx,$$

noticing that $\int_1^n x^{2\beta-\nu-2}\sin x^\beta$(or $\cos x^\beta$)dx converges as $n\to\infty$ if $\nu-\beta+1>0$, and using a well known theorem on Fourier coefficients, we obtain easily that the second integral is also $O(1)$. Hence $G_{-k}=O(1/k)$ if $\nu-\beta+1>0$.

Since $\sum O(1/k)/k$ is convergent, the convergence of the integral, as $n\to\infty$, has been established if $\nu-\beta+1>0$. For $\nu-\beta+1=0$ the integral oscillates finitely.

For $\nu-\beta+1<0$ the integral is divergent. In fact a necessary and sufficient condition

$$\left|\sum_{k=x_1}^{m}\frac{G''_{-k}}{k}\right| < \epsilon$$

for the convergence of the integral cannot be satisfied if k and x_1 are properly chosen. To fix the ideas, let k and x_1 be the same as given above in G''_{-k}. If we observe that $2(k+p)\pi x_1^{1-\beta}<1+p/k$, where p is an integer less than k, we see from the above expression for G''_{-k} as $n\to\infty$ that $G''_{-k-p}=O(x_1^{-\nu})$. Since the harmonic series diverges, it is evident that the sum in the above condition can be made larger than an arbitrarily small number ϵ. The reason for the behavior of the series for $\nu-\beta+1<0$ or for $\beta>2$ may be made plausible if we observe that the zeros $x=(2k\pi)^{1/(\beta-1)}$ of the function $N=x^\beta-2k\pi x$ as $k\to\infty$ have a limiting point.

To summarize, in case $-1\leqq\alpha<0$ and $\beta>0$ the series converges if both

(1) $\beta-\alpha>1$, (2) $\alpha+\beta<1$.

If $\beta-\alpha\leqq 1$ and $\beta<1$, or if $\alpha+\beta\geqq 1$ and $\beta>1$, the series diverges.

Integers Equal to the Sum of the Cubes of their Digits

4423 [1951, 42]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine systems of numeration such that there exist pairs of consecutive three-digit integers each of which equals the sum of the cubes of its digits. (For example, 370 and 371 in the decimal scale.)

Solution by P. Na Nagara, College of Agriculture, Bangkok, Thailand. Let the digits of the smaller number be x, y, z in the r-scale. The case $z=r-1\neq y$ must be excluded since

$$x^3 + (y + 1)^3 + 0 = x^3 + y^3 + (r - 1)^3 + 1,$$

or $3y(y+1)=(r-1)^3$, has only the trivial solution $r=1$, as the largest of the three factors in the left-hand member is $\leqq r-1$. Similarly the case $y=z=r-1$ must be excluded.

From $x^3+y^3+(z+1)^3=x^3+y^3+z^3+1$ we get $z=0$. Now by hypothesis the number equals the sum of its digits: $xr^2+yr+0=x^3+y^3+0^3$ or

$$y^3 - yr - x(r - x)(r + x) = 0. \tag{1}$$

Thus y is a factor of $x(r-x)(r+x)$. If $x=ay$, (1) reduces to a Pell equation and for each rational value of a there are infinitely many solutions. In particular, for $a=1$, this equation and its solutions are

$$(2r + 1)^2 - 8x^2 = 1,$$
$$x = y = [(3 + \sqrt{8})^n - (3 - \sqrt{8})^n]/2\sqrt{8},$$
$$r = [(3 + \sqrt{8})^n + (3 - \sqrt{8})^n - 2]/4,$$

with $n>1$, $z=0$. ($r=8$: 660, 661 provides an example.)

The substitutions $ay=r-x$ and $ay=r+x$ lead also to Pell equations from which additional solutions may be obtained. No general formula seems likely. The simplest cases occur with (a) $y=r-x$ which results in $r=3x+1$, with $y=2x+1$, $z=0$, and (b) $2y=r+x$ whence we obtain $r=(3k+1)^2$, $x=(k+1)(3k+1)$, $y=(2k+1)(3k+1)$, $z=0$, k arbitrary.

Also solved (partially) by E. A. Franz, and the Proposer. (For the Proposer's solution see *Mathesis*, t. LIX (1950) pp. 300–303.)

A Functional Equation

4424 [1951, 42]. *Proposed by Rufas Isaacs, the Rand Corporation, Santa Monica, California*

Describe the most general continuous real-valued function $f(x)$ $(-\infty \leqq x \leqq +\infty)$ which satisfies

$$f[1 - f(x)] = 1 - f(x).$$

Solution by N. J. Fine, University of Pennsylvania. More generally, consider a fixed continuous $h(x)$ on the (extended) reals, and the corresponding functional equation

$$f(h[f(x)]) = h[f(x)]. \tag{1}$$

The most general continuous solution f is given by the following construction.

Let R be any closed interval (possibly degenerate or infinite) which is mapped into itself by h, that is, $h(R)\subseteq R$. Let f be any continuous function with range R such that $f(x)=x$ on $h(R)$. Then f satisfies (1) for if x is arbitrary, $f(x)$ is in R, $h[f(x)]$ is in $h(R)$, and (1) follows. Conversely, if f satisfies (1) and R is the range of f, then R is a closed interval. For any x in $h(R)$ there is a y such that $x=h[f(y)]$, and (1) implies $f(x)=x$; hence $h(R)\subseteq R$ and f acts like the identity on $h(R)$.

In the problem as stated, h is the involutory homeomorphism $x\to 1-x$, with fixed point $\frac{1}{2}$. The condition $h(R)\subseteq R$ implies $R\subseteq h^{-1}(R)=h(R)$, that is, $h(R)=R$. Thus R is any closed interval symmetric about $\frac{1}{2}$, and f is any continuous function with range R which leaves each point of R fixed.

Also solved by C. M. Ablow, Richard Arens, R. C. Buck, Aaron Herschfeld, J. B. Kelly, A. K. Kruse, C. E. Langenhop and H. D. Block, G. Lumer, G. Marsaglia, D. R. Morrison, M. J. Norris, C. S. Ogilvy, William Scott, E. B. Shanks, Samuel Skolnik, R. W. Wagner, S. S. Walters, Robert Weinstock, J. G. Wendel, and the Proposer.

Editorial Note. Morrison notes that, if continuity is not demanded of the function $f(x)$ which satisfies $f[1-f(x)]=1-f(x)$, $f(x)$ may be constructed as follows. Let S be any set of real numbers symmetric about $\frac{1}{2}$, and let

$$f(x) = x, \qquad \text{if } x \in S;$$

$$f(x) = y \qquad \text{where } y \text{ is any element of } S, \text{ if } x \notin S.$$

Sum of Two Squares and Sum of Two Triangular Numbers

4425 [1951, 113]. *Proposed by P. A. Piza, San Juan, Puerto Rico*

There are infinitely many triangular numbers (numbers of the form $k(k+1)/2$) which are also perfect squares, *viz.*

$$[(17+12\sqrt{2})^n+(17-12\sqrt{2})^n-2]/32.$$

Find numbers which are at the same time a sum of two squares and a sum of two triangular numbers.

I. *Solution by G. W. Walker, Buffalo, N. Y. and E. P. Starke, Rutgers University.* Let a, b, x, y be any integers satisfying

$$\tfrac{1}{2}x(x+1)+\tfrac{1}{2}y(y+1)=a^2+b^2, \tag{1}$$

which can be written in the form

$$\begin{aligned}(2x-2a+2b+1)(2x+2a-2b+1)-1 \\ =(2a+2b-2y-1)(2a+2b+2y+1)+1=4K.\end{aligned} \tag{2}$$

If the four factors in (2) are identified respectively as u, v, r, s, we have

$$uv = 4K + 1, \qquad rs = 4K - 1, \tag{3}$$

Conversely, if K is an arbitrary integer and integers u, v, r, s are determined which satisfy (3), then

$$\begin{aligned} a &= (r + s - u + v)/8, \qquad x = (u + v - 2)/4, \\ b &= (r + s + u - v)/8, \qquad y = (s - r - 2)/4 \end{aligned} \tag{4}$$

will satisfy (1), as may be verified directly.

It is easy to show that integral values of r, s, u, v satisfying (3) always give integral solutions for a, b, x, y. To get all the basic solutions with no duplications, we may require that $a \geqq b \geqq 0$ and $x \geqq y \geqq 0$; which are equivalent to the following restrictions on the spread of the usable factors

$$\text{(I)} \qquad s + r \geqq v - u \geqq 0, \qquad\qquad \text{(II)} \qquad v + u \geqq s - r \geqq 2.$$

For solutions in positive integers, the first equality sign in (I) and the second in (II) must be eliminated.

II. *Solution by C. M. Sandwick, Sr., Easton High School, Easton, Pennsylvania.* The necessary and sufficient condition that N be a sum of two triangular numbers is that $4N+1$ be a square or a sum of two squares. (The degenerate case in which one of the two triangular numbers is zero occurs if and only if N is itself triangular and $4N+1$ is prime.) This follows at once from the equivalence of the two equations

$$N = \tfrac{1}{2}x(x + 1) + \tfrac{1}{2}y\,(y + 1), \qquad 4N + 1 = (x + y + 1)^2 + (x - y)^2.$$

If now N also is a sum of two squares, a^2+b^2, then it is one of the numbers required by the problem.

The well known criterion that a number be a sum of two squares is that it be of the form $2^\alpha MQ^2$, where $\alpha \geqq 0$, $M>1$; M, Q odd; every prime divisor of M is of the form $4j+1$.

Three special cases are worth notice. (a) If $y=x+1$, then there results a degenerate solution $a=y$, $b=0$. If however $y=p^2+q^2$, then $N=y^2=(2pq)^2 + (p^2-q^2)^2$ and a non-degenerate solution exists.

(b) If X_i are the cited numbers which are both triangular and square, then $N=X_i+X_j$ meets the requirements for any choice of i, j.

(c) Consider $N=\frac{1}{2}x(x+1)$. If $x=s^2$, then $N=[\frac{1}{2}(s-1)s]^2+[\frac{1}{2}s(s+1)]^2$. If $x=\frac{1}{2}r(r+1)$, then $N=\frac{1}{2}r(r+1)+\frac{1}{2}(x-1)x$. Thus: *If x is a square, N is a sum of two squares, and, if x is a triangular number, N is a sum of two triangular numbers.* Therefore, for x equal to any X_i in (b), N is one of the desired numbers.

Also solved by D. H. Browne, L. A. Ringenberg, J. E. Sanders, F. Underwood, and the Proposer.

Editorial Note. For proof of the assertion in the proposal, see 3677 [1935, 573], and E 954 [1951, 568]. Typical of the many special solutions submitted are the following:

a	b	x	y
$n+1$	$2n+3$	n	$3n+4$
$2n+1$	$5n-1$	$3n+1$	$7n-2$
$3m$	m^2-4	m^2-4	m^2+4
a	a^2-n^2	a^2-n^2+n	a^2-n^2-n
$2mn$	n^4+4m^4	$4m^2n^2+n^4-4m^4$	$4m^2n^2-n^4+4m^4$
m^2-n^2	$m^4+6m^2n^2+n^4$	$(m+n)^4-8m^2n^2$	$(m-n)^4-8m^2n^2$

It is obvious that in using any such formulas any negative x may be replaced by $-x-1$, any negative a by $-a$, (similarly for y, b), and a and b may be interchanged, as may x and y.

Lower Bound to the Maximum Modulus

4426 [1951, 113]. *Proposed by D. J. Newman, Harvard University*

Let a, b, c, d, all have modulus unity. Prove that in the unit circle the polynomial

$$P(z) = az^3 + bz^2 + cz + d$$

has a maximum modulus not less than $\sqrt{6}$. What is the result for a polynomial of degree greater than three?

Solution by G. P. Henderson, University of Western Ontario, London, Ontario. We will prove the following generalization of the stated inequality. If

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_m z^m$$

is a polynomial with $a_n \neq 0$, $a_m \neq 0$, $n > m$, then the maximum value of $|P(z)|$ for $|z| \leqq 1$ is not less than

$$\sqrt{\sum_{r=m}^{n} |a_r|^2 + 2|a_m a_n|}.$$

We are going to give z certain values with $|z| = 1$, and for these we may suppose that $m = 0$, since P can be replaced by P/z^m with no change in absolute value. Since $|z| = 1$,

$$|P(z)|^2 = \sum_{r=0}^{n}\sum_{s=0}^{n} a_r \bar{a}_s z^{r-s}.$$

If ω is a primitive nth root of unity,

$$\sum_{t=0}^{n-1} |P(\omega^t z)|^2 = \sum_{r=0}^{n}\sum_{s=0}^{n} a_r \bar{a}_s z^{r-s} \sum_{t=0}^{n-1} \omega^{t(r-s)}.$$

Now the last sum has the value n if $r-s\equiv 0$, (mod n) and vanishes otherwise. We have then

$$(1/n)\sum_{t=0}^{n-1}|P(\omega^t z)|^2=\sum_{r=0}^{n}|a_r|^2+a_0\bar{a}_n z^{-n}+\bar{a}_0 a_n z^n.$$

With $z^n=a_0|a_n|/|a_0|a_n$, the right member becomes

$$\sum_{r=0}^{n}|a_r|^2+2|a_0a_n|.$$

We see that for at least one value of t,

$$|P(\omega^t z)|\geqq\sqrt{\sum_{r=0}^{n}|a_r|^2+2|a_0a_n|},$$

from which the above result follows.

Also solved by M. S. Robertson, G. Szegö, J. L. Ullman, and the Proposer.

Related Absolute Values

4427 [1951, 113]. *Proposed by Paul Erdös, University College, London, England*

Let

$$f_n(x)=\prod_{i=1}^{n}(x-x_i), \qquad -1\leqq x_i\leqq 1.$$

Prove that there cannot exist numbers a, b such that

$$|f_n(a)|\geqq 1, \qquad |f_n(b)|\geqq 1, \qquad -1<a<0<b<1.$$

Solution by Robert Steinberg, University of California, Los Angeles. The proof depends on the following lemma: *If* $-1<-c<0<b<1$, *and* $f(x)=|x+c|^b|x-b|^c$; *then* $f(x)<1$ *for* $-1\leqq x\leqq 1$.

To prove the lemma note that, if $-1\leqq x\leqq -c$, then $f(x)$ has its maximum value at $x=-1$, and this maximum value, $(1-c)^b(1+b)^c$, is less than 1. ($1+b<e^b$ and $1-c<e^{-c}$ imply $(1-c)^b(1+b)^c<e^{-bc}\cdot e^{bc}=1$.)

Similarly $f(x)<1$ if $b\leqq x\leqq 1$.

In the interval $(-c, b)$, differentiation shows that $f(x)=(x+c)^b(b-x)^c$ assumes its maximum value of b^bc^c for $x=b-c$; and this again is less than 1.

Now let a and b be any numbers such that $-1<a<0<b<1$. Then setting $a=-c$, we have

$$|f_n(-c)|^b|f_n(b)|^c=\prod_{i=1}^{n}|c+x_i|^b|b-x_i|^c<1$$

according to the lemma. Since b and c are positive, this implies that at least one of $|f_n(-c)|=|f_n(a)|$ and $|f_n(b)|$ is less than 1.

Also solved by N. J. Fine, L. C. Hsu, and G. W. Walker.

Editorial Note. Walker makes use of a geometrical argument. In Euclidean n-space let the point P have coordinates x_i and consider the 2^n disconnected regions, one in each compartment of coordinates, represented by $|f_n(c)| \geqq 1$ for $c=0$. The inequality $-1 \leqq x_i \leqq 1$ represents all points on and within an n-dimensional hypercube. The effect of varying $c(-1 \leqq c \leqq 1)$ is a rigid translation of the 2^n-sheeted hypersurface along the line joining the origin to $(1, 1, \cdots, 1)$. Of course we get the same effect by translating the hypercube with respect to the hypersurface. It is not difficult to see that there are no points common to the regions bounded by the hypercube and the hypersurface for $0 \leqq c \leqq 1$ which are also common to the hypercube and the hypersurface for $-1 \leqq c \leqq 0$. Thus there exists no set of coordinates $x_i(-1 \leqq x_i \leqq 1)$ for which both $|f_n(a)| \geqq 1$ and $|f_n(b)| \geqq 1$.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Conformal Mapping. By Zeev Nehari. New York, McGraw-Hill, viii+396 pp. $7.50.

The object of this book is to give a student with a minimum amount of background (advanced calculus) an introduction to the Dirichlet problem and the conformal mapping problem for simply and multiply connected domains. The author has accomplished this goal and the book fills a definite need for a textbook on conformal mapping on this level. The book should make an excellent text for advanced undergraduate and beginning graduate work especially for persons interested in applied mathematics. Numerous exercises of above average type are given and these should go a long way toward helping the student understand the theory and application of the theory of conformal mapping. The whole book is written in a rather informal or classroom style and this should appeal to a student.

The first chapter of the book is devoted to the statement of the Dirichlet problem and the introduction of auxiliary functions needed for the statement and solution of the problem. The second, third and part of the fifth chapters give the usual material of a beginning course in complex variables. The fourth chapter gives results on closure properties of families of functions of a complex variable. The fifth chapter gives the solution of the problem of mapping one

simply connected domain conformally upon another simply connected domain neither of which is the entire finite plane. The corresponding problem for the closure of a domain with a simple closed curve for a boundary is also considered. The remainder of the fifth chapter is devoted to the application of these results to the Dirichlet problem and to special mapping problems. The sixth chapter gives the mapping properties of many special classes of functions. The seventh and final chapter considers the conformal mapping problem for multiply connected domains.

On the whole the proofs and explanations given in the book are presented in a neat manner. A few statements in the book need some clarification. As a typical example, in Cauchy's theorem it is stated that two simple curves with the same end points determine a simple closed curve.

E. J. MICKLE
Ohio State University

Finite Matrices. By W. L. Ferrar. Oxford University Press, 1951. vi+182 pp. $4.00.

This is a companion volume to the author's *Algebra* which was published in 1941. Many of the basic theorems concerning matrices are assumed with a foot-note reference to the earlier volume. After an introductory chapter, equivalence over a field and equivalence of λ-matrices over a field are considered, but the field is by definition a subfield of the complex field. This third chapter contains the elementary divisor theory. The term *c*-equivalence is introduced for similarity, and in Chapter IV (entitled *Collineation*) the complex canonical form of a matrix under similarity transformations is obtained. From this the rational canonical form is obtained, valid only for a subfield of the complex field.

Chapter V is devoted to infinite series and functions of matrices, and regarding this material the author claims in his preface that "only patches of it here and there appear to have been published before." It is not clear to the reviewer if Ferrar is familiar with Schwertfeger's monograph *Les Fonctions de Matrices*, Paris, 1938. This is certainly a connected treatise. At any rate Ferrar's Chapter V does not follow Schwertfeger very closely, and contains some interesting new material.

Chapter VI contains a treatment of the congruence of matrices, orthogonal and unitary matrices, quadratic and hermitian forms. The reviewer bridled a bit at the definition of a nilpotent matrix as one whose square is zero, but of course Wedderburn's definition is not patented. In Chapter VII some simple types of matrie equations are considered.

Chapter VIII, entitled *Miscellaneous Notes*, contains an exposition of the resolvent matrix, *i.e.*, the theory of the Frobenius covariants; positive definite quadratic forms whose variables are subject to a linear condition; and the anti-commutative matrices of Eddington and Newman.

Sets of exercises such as are to be found in *Algebra* are absent from the present book, which makes it less convenient for use as a text book.

The printing and paper are excellent, but my copy may have suffered from the Atlantic passage because its board covers are not even developable surfaces.

C. C. MACDUFFEE
University of Wisconsin

NEW BOOKS RECEIVED

Teaching Mathematics in the Secondary School. By L. B. Kinney and C. R. Purdy. New York, Rinehart and Company, Inc., 1952. xvi+381 pages. $5.00.

Mathematics, its Magic and Mastery. Second edition. By Aaron Bakst. New York, D. Van Nostrand Co., Inc., 1952. 14+790 pages. $6.00.

Chemical Calculations. By S. W. Benson. New York, John Wiley and Sons Co., 1952. xi+217 pages. $2.95.

The Nature of Number, An Approach to Basic Ideas of Modern Mathematics. By Roy Dubisch. New York, The Ronald Press Company, 1952, xii+159 pages. $4.00.

Tables of the Exponential Function e^x, Applied Mathematics Series 14. By the National Bureau of Standards Computation Laboratory. Washington, Government Printing Office, 1951. x+540 pages. $3.25.

Colloque de Geometrie Differentielle. (Louvain, 1951) Georges Thone, Liège and Masson et Cie, Paris, 1951, 240 pages. 2450 Fr. francs.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of special features, student papers, bibliographies of program topics, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

EDITORIAL NOTE

All club reports for 1950–51 submitted before January 1, 1952 have now been published. Club reports for 1951–52 will not be compiled.

The publication of club programs was initiated several years ago as a medium for exchanging ideas of program topics and club projects. A considerable fund of suggestions is now available to club sponsors and program chairmen, and the publication of additional club reports would not seem to serve any useful function.

The Editor appeals for material to make this department a source of aid and inspiration to club activities, particularly on the undergraduate level. He would welcome sundry items such as new club projects, bibliographies of club topics, abstracts of student articles, expository articles, mathematical "curiosa," original limericks and poems, plays, radio skits, descriptions of career opportunities, and other items of interest to undergraduate students.

BOOKS CONTAINING MATERIAL FOR UNDERGRADUATE MATHEMATICS CLUB PROGRAMS

C. B. READ, University of Wichita

At a recent national convention there was revealed what one professor called "a woeful ignorance" about source material for undergraduate club programs. Some of those present seemed to be unaware of the existence of certain books which were considered by others as almost classic in the field.

With the idea of obtaining some composite opinion as to the value of certain books, the author volunteered to make a brief survey. With the cooperation of one or two colleagues a tentative list of thirty-one books was prepared. This included a majority of the books known to this group which seemed to have value as source material for undergraduate club programs. This list was sent to a group of nineteen individuals. Individuals were selected as those personally known by the author as having some interest in sponsoring club programs, or whose names had appeared in the club section of this MONTHLY as being connected with club programs.

The individuals were asked to place the number one before the five books considered to be of the greatest value, a number two before the five books considered of next highest value, and a number 3 before the next five books in order of value. Space was provided for adding additional books with their rating.

The response to the check list was greater than might have been anticipated. Seventeen of the nineteen replied. However, before presenting results certain comments should be made. First, several individuals indicated that they were not familiar with all the books on the list and hence that a failure to rate the book does not mean that the book is poor, merely that the individual making the rating was not familiar with the book. Secondly, it was pointed out that there is some excellent material available in articles in papers and various magazines. This fact is clearly recognized, but the purpose of this study was primarily to rate *books* in the field. A third point was raised that it might be very desirable to include more books dealing with fundamental concepts of mathematics. One individual pointed out that the number of such books on the list submitted was small. Another individual raised the point that he did not consider books on puzzles and riddles as a good source for mathematics club programs, rather the program should tend to widen the member's knowledge and interest in mathematics rather than entertain. Still another says, " . . . serious scrutiny would cause one to wonder whether or not one should recommend said books for the consideration of youths, or recreation . . . many of the books give incorrect impressions and most of them cause the reader to work too hard for what he gets." (This individual feels few people are in a position to rate the books and declined to attempt the task.) There is no question that all of these points of view are of distinct merit, but even if opinions differ it is quite likely that a composite opinion is worth more than that of any single individual.

In the light of these introductory remarks the list of books which follows may still be of value in giving sources of material for undergraduate mathematics club programs. The first thought in compiling this list was to include every book mentioned by any individual teacher. However, one of those responding included 38 books not on the original list, 28 of these being in a foreign language. One other list included several books in German, but pointed out that unfortunately many undergraduates find material in a foreign language unavailable. Accordingly with the idea of shortening the list, no book is included unless it was rated in the first or in the second category by at least one or in the third category by more than one individual. In several cases the publisher has more than one office. The address given is that of the office confirming the price and the fact that the book was or was not in print as of November, 1951. It represents a composite opinion giving the number of times a book was listed as of the greatest value, next highest, and third highest (in groups of five).

The numbers in parentheses indicate respectively ratings received in groups of 1, 2, 3 and the weighted totals from an arbitrary weighting of five points for a rating in group 1, 3 points for a rating in group 2, and 1 point for a rating in the third group. It is recognized that there may be disagreement with this scale, but it does offer at least some basis for assigning rank. It should also be repeated that not all individuals rated five books in each of the various classes.

Ball, W. W. Rouse: Mathematical Recreations and Essays. The Macmillan Company, 2459 Prairie Ave., Chicago 16, Illinois. $3.95 (10, 4, 1——63)

Courant, Richard and Robbins, Herbert: What is Mathematics? Oxford University Press, Inc., 114 Fifth Avenue, New York. $4.75 (10, 3, 1——60)

Kasner, Edward and Newman, James: Mathematics and the Imagination. Simon and Schuster, Inc., 1230 Sixth Avenue, New York. $3.50 (6, 6, 0——48)

Steinhaus, H.: Mathematical Snapshots. Oxford University Press, Inc., 114 Fifth Avenue, New York 11, New York. $5.00 (5, 5, 1——41)

Kraitchik, Maurice: Mathematical Recreations. W. W. Norton and Company, Inc., 101 Fifth Avenue, New York 3, New York. Out of Print. (6, 0, 6——36)

Bakst, Aaron: Mathematics, Its Magic and Mastery. D. Van Nostrand Company, Inc., 250 Fourth Avenue, New York 3, New York. New edition in preparation, available approximately February, 1952. (4, 4, 3——35)

Northrop, Eugene P.: Riddles in Mathematics. D. Van Nostrand Company, Inc., 250 Fourth Avenue, New York 3, New York. $3.50 (4, 1, 3——26)

Jones, Samuel I.: Mathematical Nuts. S. I. Jones Company, 1122 Belvidere Drive, Nashville 4, Tennessee. $3.50 (3, 3, 0——24)

Abbott, Edwin A.: Flatland. Barnes and Noble, Inc., 105 Fifth Avenue, New York 3, New York. $2.00 (3, 2, 3——24)

Andrews, William S.: Magic Squares and Cubes. Open Court Publishing Company, Box 402, La Salle, Illinois. Out of Print. (2, 3, 4——23)

Smith, David E. and Ginsburg, J.: Numbers and Numerals. National Council of Teachers of Mathematics, 1201 Sixteenth St., N.W. Washington 6, D. C. $0.75 (3, 2, 2——23)

Licks, H. E.: Recreations in Mathematics. D. Van Nostrand Company, Inc., 250 Fourth Avenue, New York 3, New York. Out of print. (2, 2, 6——22)

Jones, Samuel I.: Mathematical Wrinkles. S. I. Jones Company, 1122 Belvidere Drive, Nashville 4, Tennessee. $3.50 (1, 5, 2——22)

Jones, Samuel I.: Mathematical Clubs and Recreations. S. I. Jones Company, 1122 Belvidere Drive, Nashville 4, Tennessee. $3.00 (2, 2, 5——21)

Rupert, W. W.: Famous Geometric Theorems and Problems. D. C. Heath and Company, 1815 Prairie Ave., Chicago 16, Illinois. Out of Print. (2, 2, 5——21)

Merrill, Helen Abbot: Mathematical Excursions. Bruce Humphries, Inc., 30 Winchester St., Boston, Massachusetts. Out of Print. (2, 3, 1——20)

White, William F.: A Scrapbook of Elementary Mathematics. Open Court Publishing Company, Box 402, La Salle, Illinois. $1.50 (1, 3, 4——18)

Alliston, Norman: Mathematical Snack Bar. Chemical Publishing Company, 212 Fifth Ave., New York 10, New York. Out of Print. (2, 1, 2——15)

Dudeney, Henry E.: The Canterbury Puzzles. Thomas Nelson and Sons, Edinburgh, Scotland. (Not in stock at 385 Madison Ave., New York 17, New York) $2.00 (1, 3, 1——15)

Heath, Royal V.: Mathemagic. Simon and Schuster, Inc., 1230 Sixth Avenue, New York. (Publisher ignores letters as to availability of this book) (0, 3, 3——12)

Lucas, M. Edouard: Recreations Mathematiques. Libraire Gauthier-Villars, Paris. Out of Print. (1, 1, 1——9)

Row, Sundara T.: Geometric Exercises in Paper Folding. Open Court Publishing Co., Box 402, La Salle, Illinois. $1.00 (0, 3, 0——9)

Phillips, Hubert: Caliban's Problem Book. Thomas De La Rue and Co., Ltd., 84/86 Regent Street, London, England. Out of Print. (1, 0, 2——7)

Dudeney, Henry E.: Modern Puzzles and How to Solve them. C. A. Pearson, London, England. Out of Print. (1, 0, 1——6)

Mott-Smith, Geoffrey: Mathematical Puzzles. Doubleday and Company, Inc., Garden City, New York. Out of Print. (0, 2, 0——6)

Yates, Robert C.: Geometrical Tools. Educational Publishers, Inc., 122 N. Seventh St., St. Louis 1, Missouri. $3.50 (1, 0, 1——6)

Collins, A. Frederick: Fun with Figures. Appleton-Century-Crofts, Inc., 35 West 32nd St., New York 1, New York. Out of Print. (0, 1, 2——5)

Finkel, B. F.: Mathematical Solution Book. Kibler and Company, Springfield, Missouri. (Mail sent to publisher returned—unknown) (0, 1, 2——5)

Sainte-Lague: Avec Des Nombres y Des Lignes. Librairie Vuibert, Boulevard Saint-Germain 63, Paris, France. 590 frs. (1, 0, 0——5)

Kramer, Edna E.: The Main Stream of Mathematics. Oxford University Press, Inc., 114 Fifth Avenue, New York 11, New York. $5.00 (1, 0, 0——5)

Dantzig, T.: Number, The Language of Science. The Macmillan Company, 2459 Prairie Avenue, Chicago 16, Illinois. $4.50 (1, 0, 0——5)

Smith, David Eugene: History of Mathematics (2 volumes) Ginn and Company, 2301 Prairie Avenue, Chicago 16, Illinois. $13.00 (1, 0, 0——5)

Merriman, Gaylord M.: To Discover Mathematics. John Wiley and Sons, Inc., 440 Fourth Avenue, New York 16, New York. $3.25 (1, 0, 0——5)

Bell, Eric T.: Men of Mathematics. Simon and Schuster, Inc., 1230 Sixth Ave., New York 20. New York. $6.00 (0, 1, 1——4)

Schubert, H. C. H.: Mathematical Essays and Recreations. Open Court Publishing Co., Box 402, La Salle, Illinois. Out of Print. (0, 0, 3——3)

Degrazia, Joseph: "Math" is Fun. The Gresham Press, Inc., Eleven Broadway, New York 4, New York. $2.75 (0, 1, 0——3)

Dudeney, Henry E.: Amusements in Mathematics. Thomas Nelson and Sons, Edinburgh, Scotland. (Not in stock at 385 Madison Ave., New York 17, New York.) $2.00 (0, 1, 0——3)

Whitworth, William Allen: Choice and Chance. Stechert-Hafner, Inc., 31 East 10th St., New York 3, New York. $4.00 (0, 1, 0——3)

Levens, Alexander S.: Nomography. John Wiley and Sons, Inc., 440 Fourth Ave., New York 16, New York. $3.50 (0, 1, 0——3)

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Dr. F. E. Bothwell of Northwestern University has accepted a position as a consultant with the United States Naval Ordnance Test Station, China Lake, California.

Mrs. Helen E. Brown, formerly a teacher at Fassifern School for Girls, Hendersonville, North Carolina, is teaching now at Martha Berry School for Girls, Mount Berry, Georgia.

Assistant Professor R. E. Carr of Michigan State College has a position as Research Engineer with North American Aviation, Downey, California.

Associate Professor John Cawley of Lafayette College has retired.

Assistant Professor J. O. Chellevold of Lehigh University has been appointed to a professorship at Wartburg College.

Mr. D. F. Coulter, Jr., formerly a mathematician at the United States Mine Countermeasures Station, Panama City, Florida, has been appointed to an instructorship at Grays Harbor College, Aberdeen, Washington.

Mr. J. G. Cox, Alabama Polytechnic Institute, is now Supervisory Analytical Statistician at Eglin Air Force Base, Florida.

Mr. J. W. Crispin of Wayne University has been appointed Research Associate at the Willow Run Research Center, University of Michigan.

Mr. D. O. Davidson has a position as an engineering assistant at Northrop Aviation, Hawthorne, California.

Research Assistant L. E. Davis of Ohio State University has a position as Mathematician at the Naval Research Laboratory, Washington, D. C.

Mr. A. R. DiDonato, previously a student at Duquesne University, has a position at the DuPont Experimental Station, Wilmington, Delaware.

Associate Professor W. C. Doyle of Rockhurst College has been promoted to a professorship.

Mr. Trevor Evans of the University of Wisconsin has been appointed to an assistant professorship at Emory University.

Dr. A. E. Foster of the University of Kentucky has been appointed to an instructorship at Florida State University.

Associate Professor W. W. Gandy of Texas Agricultural and Mechanical College has a position as Senior Aerophysics Engineer with Consolidated-Vultee Aircraft Company, Ft. Worth, Texas.

Mr. S. I. Gass, formerly a mathematician at Aberdeen Bombing Mission, Los Angeles, has a position now as a mathematician in the Planning Research Division, Directorate Program Standards and Cost Control, United States Air Force, Washington, D C.

Associate Professor Abe Gelbart of Syracuse University is on leave of absence and has been appointed Fulbright Lecturer at the Institute of Technology, Trondheim, Norway.

Mr. L. D. Gregory, University of Oklahoma, has a position as Mathematician at Chance Vought Aircraft Company, Dallas, Texas.

Mr. F. J. Hardy, previously a graduate assistant at Duquesne University, has accepted a position as Engineer at Bell Aircraft Corporation, Niagara Falls, New York.

Mr. H. S. Heaps of the University of Toronto has been appointed to an assistant professorship at Nova Scotia Technical College.

Mr. L. L. Helms, formerly a student at Bradley University, is now a research assistant in the Statistical Laboratory, Purdue University.

Mr. David Horwitz, who has been a student at Illinois Institute of Technology, has a position as Assistant Engineer at Armour Research Foundation, Chicago, Illinois.

Mr. H. V. Huneke of the University of Oklahoma has been appointed Lecturer at the University of Wichita.

Mr. Jiro Ishihara, formerly a student at Northwestern University, is now a staff member of the Digital Computer Laboratory, Massachusetts Institute of Technology.

Dr. W. H. Ito of the University of Minnesota has a position as Senior Project Analytical Engineer at Chance Vought Aircraft, Dallas, Texas.

Mrs. Verba W. Iturralde has been teaching at Bowie High School, El Paso, Texas during 1951–1952.

Mrs. Helen S. Keefer, previously a student at Iowa Wesleyan College, has a position as Engineering Aide with the General Electric Company, Schenectady, New York.

Mr. C. E. Kelley who has been associated with the New York Life Insurance Company has been appointed to an instructorship at the University of Missouri.

Associate Professor E. S. Kennedy of American University of Beirut, Lebanon, has been serving as Fulbright Lecturer at Tehran University, Iran.

Mr. M. R. Kenner, formerly a graduate student at the University of Minnesota, has been appointed to an instructorship at Southern Illinois University.

Associate Professor Fred Kiokemeister of Mount Holyoke College is on leave of absence and has been appointed to a visiting professorship at the University of Chicago.

Mr. S. G. Kneale, previously a graduate student at Harvard University, has accepted a position as Mathematician with the Philco Research Division, Philadelphia, Pennsylvania.

Mr. D. H. Kraft is now Second Lieutenant, United States Air Force; he is serving as a mathematician in the Flight Research Laboratory, Wright-Patterson Air Force Base, Dayton, Ohio.

Mr. Sidney Kravitz has accepted a position as Ammunition Design Engineer with the Picatinny Arsenal, Dover, New Jersey.

Mr. E. R. Lancaster of the University of Detroit has a position as Mathematician with International Business Machines Corporation, Poughkeepsie, New York.

Mr. J. N. P. Lawrence, formerly a student at Vanderbilt University, is now a research assistant at Los Alamos Scientific Laboratory, New Mexico.

Assistant Professor E. L. Lehmann of the University of California at Berkeley has been promoted to an associate professorship; he has been on leave of absence during the year 1951–1952 and has been Visiting Associate Professor at Stanford University.

Assistant Instructor F. H. Lloyd of the University of Missouri has been appointed to an instructorship at Westminster College.

Instructor J. H. Manheim of Wilson Junior College, Chicago, Illinois, has accepted a position as Engineer at the Federal Telecommunications Laboratories, Nutley, New Jersey.

Emeritus Professor H. W. March of the University of Wisconsin has been serving as a special consulting mathematician at United States Forest Products Laboratory, Madison, Wisconsin.

Associate Professor P. E. Meadows of Carroll College is now Principal of Sidney High School, Ohio.

Professor J. N. Michie of Texas Technological College has retired with the title of Professor Emeritus.

Dr. T. S. Motzkin of Hebrew University and the National Bureau of Standards has been appointed to a visiting professorship at the University of California, Los Angeles.

Assistant Professor E. N. Nilson, Trinity College, Connecticut, has been promoted to an associate professorship.

Dr. Ilse L. Novak has been appointed a member of the Institute for Advanced Study.

Dr. Rufus Oldenburger, mathematician-engineer of the Woodward Governor Company, Rockford, Illinois, gave a series of lectures in Paris, France, during the month of April on "Mathematical Engineering Analysis"; these lectures were given to l'Institution des Arts et Métiers, Société Francaise des Mecaniciens, Société des Radioelectriciens, and the Société d'Electronique et d'Automatisme.

Assistant Professor Daniel Orloff of Southern Illinois University has accepted a position as Mathematician at Bell Aircraft Corporation, Niagara Falls, New York.

Mr. S. T. Paine of the Rand Corporation has accepted a position as Mathematician with the Department of Meteorology, University of California, Los Angeles.

Mr. A. H. Payne, formerly a graduate student at the University of North

Carolina, is now a mathematician at Ballistics Research Laboratories, Aberdeen Proving Ground, Maryland.

Mr. G. W. Payne, previously a student at St. Mary's University, is now in military service.

Mr. J. M. Pike who has been a graduate student at Kansas State Teachers College is now Liaison Engineer at the United States Naval Air Missile Test Center, Pt. Mugu, California.

Mr. C. F. Pinzka, previously a statistician with E. I. DuPont de Nemours and Company, New Brunswick, New Jersey, has accepted a position with the Educational Testing Service, Princeton, New Jersey.

Mr. Barth Pollak, formerly a student at Illinois Institute of Technology, has a position as Analyst with the Department of Defense, Washington, D. C.

Dr. R. E. Roberson of the Naval Research Laboratory has accepted a position as Research Engineer with North American Aviation, Downey, California.

Assistant Professor Saul Rosen of Drexel Institute of Technology is now Associate Research Engineer with the Burroughs Adding Machine Company, Philadelphia, Pennsylvania.

Mr. Max Rosenberg of the National Bureau of Standards has accepted a position as Mathematician with the Picatinny Arsenal, Dover, New Jersey.

Assistant Instructor J. M. Sandy of the University of Missouri has a position as Mathematician at the United States Naval Ordnance Test Station, Pasadena, California.

Associate Professor C. L. Seebeck of the University of Alabama has been promoted to a professorship.

Mr. D. J. Smith, formerly an Actuarial Trainee at Royal-Liverpool Group of Insurance Companies, New York City, has accepted a position as Engineer with Breeze Corporations, Newark, New Jersey.

Mr. O. S. Spears of the University of Oklahoma is now a member of the Artillery School Board, United States Army, Ft. Sill, Oklahoma.

Mr. R. H. Spohn, previously of the Kellex Corporation, New York City, has accepted a position as Mathematician with the Vitro Corporation of America, West Orange, New Jersey.

Mr. J. R. Swaffield, formerly a student at Boston University, has been appointed to an instructorship at Clark School, Hanover, New Hampshire.

Mr. P. C. Sweetland of Michigan State College has been appointed to an assistant professorship of research at Pennsylvania State College.

Mr. A. D. Talkington of DePauw University is now a mathematician at the National Bureau of Standards, Washington, D. C.

Mr. F. J. Wall who has been a research assistant at the Los Alamos Scientific Laboratory, New Mexico, has accepted a position as Statistician with the Dow Chemical Company, Denver, Colorado.

Dr. L. M. Weiner of the University of Chicago has been appointed to an instructorship at De Paul University for the academic year 1952–1953.

Mr. R. W. Young, Henderson State Teachers College, Arkansas, has been appointed to a professorship at Indiana Technical College.

Associate Professor R. K. Zeigler of the University of California has a position as Statistician with the United States Atomic Commission, Oak Ridge, Tennessee.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred and ten persons have been elected to membership by the Board of Governors on applications duly certified.

W. T. ACHOR, Student, Alabama Polytechnic Institute, Auburn, Ala.

R. E. ALLAN, M. A. (Alabama) Ensign, U. S. Navy, Arlington, Va.

J. L. BAKER, JR., M.S. (Northwestern) Asso. Professor, University of Cincinnati, Ohio

D. O. BANKS, B.S. (Oregon S. C.) Grad. Assistant, Oregon State College, Corvallis, Ore.

C. Y. BARTHOLOMEW, B.S. (Rutgers) Junior Research Engineer, National Union Radio Corporation, Orange, N. J.

J. B. BARTOO, M.S. (Iowa) Instr., State University of Iowa, Iowa City, Ia.

C. D. BELL, M.S. (Michigan) Vice President and Actuary, Pioneer Mutual Life Insurance Company, Fargo, N. D.

E. W. BETH, Ph.D. (Utrecht) Professor of Logic, University of Amsterdam, Netherlands

R. E. BLASCH, Student, Hofstra College, Hempstead, N. Y.

I. E. BLOCK, Ph.D. (Harvard) Mathematical Consultant, Philco Corporation, Philadelphia, Pa.

J. R. BOYD, B.S. (Trinity) Teacher, San Marcos High School, Tex.

GERALD BRIGHAM, Student, University of Rhode Island, Kingston, R. I.

IDA BRISKY, Student, Northeastern University, Boston, Mass.

J. L. BROWN, JR., B.S. (Ohio) Asst. Professor, Pennsylvania State College, Pa.

R. W. BROWN, SR., B.S. (Pacific) Teaching Fellow, Oregon State College, Corvallis, Ore.

LUCY J. BULLOCK, M.A. (Boston U.) Asso. Professor, St. Paul's Polytechnic Institute, Lawrenceville, Va.

MARY J. BURNS, Student, Duquesne University, Pittsburgh, Pa.

MRS. LEILA R. H. BUTLER, B.S. (Savannah) Teacher, Walker High School, Savannah, Ga.

G. C. BYERS, B.A. (Michigan) Hancock, Mich.

L. B. CARPENTER, A.B. (Johns Hopkins) Physicist, Army Chemical Center, Md.

N. H. CHOKSY, B.S. in E.E. (Milwaukee Schl. Engg.) Grad. Student, University of Wisconsin, Madison, Wis.

GERALDINE A. COON, Ph.D. (Rochester) Mathematician, Taylor Instrument Company, Rochester, N. Y.

R. C. COURTER, Student, Columbia University, New York, N. Y.

ANN M. CURRAN, M.A. (Connecticut) Asst. Instructor, University of Connecticut, Storrs, Conn.

D. L. DART, Student, Wayne University, Detroit, Mich.

J. R. DAVIS, JR., Student, University of Kentucky, Lexington, Ky.

V. J. DERBES, M.D. (Tulane) Asso. Professor of Internal Medicine, Tulane University, New Orleans, La.

ANDREW DIPRETE, Student, University of Rhode Island, Kingston, R. I.

REV. M. M. DREILING, M.S. (Catholic) Asso. Professor, St. Joseph's College, Collegeville, Ind.

PATRICK DU VAL, Ph.D. (Cambridge, England) Professor, University of Georgia, Athens, Ga.

R. A. EDMONDSON, M.A. (Cornell) Chairman, Division of Natural Science & Mathematics, Wiley College, Marshall, Tex.

BERNARD EPSTEIN, Ph.D. (Brown) Asst. Professor, University of Pennsylvania, Philadelphia, Pa.

H. L. ESTERMAN, Student, University of Rochester, N. Y.

W. J. FLEMING, JR., B.E. (Yale) Asst. Professor of Engineering, University of Buffalo, N. Y.

R. J. FORTIN, Student, University of Rhode Island, Kingston, R. I.

MRS. RUTH M. FRISCH, A.B. (Brown) Grad. Student, Polytechnic Institute of Brooklyn, N. Y.

J. C. FUNNELL, LL.B. (Northeastern) Retired. 4 Union Park, Boston, Mass.

J. J. GEHRIG, B.S. (Wisconsin) Mathematician, Ballistic Research Laboratories, Aberdeen Proving Ground, Md.

J. M. GIBLIN, M.S. (John Carroll) Mathematician, Army Map Service, Washington, D. C.

BERNICE GOLDBERG, M.A. (Radcliffe) Mathematician, Cambridge Research Center, Mass.

J. K. GOLDHABER, Ph.D. (Wisconsin) Instr., University of Connecticut, Storrs, Conn.

L. W. GREEN, M.A. (Yale) Assistant, Yale University, New Haven, Conn.

R. O. GROSNICK, B.Ed. (S.T.C., Stevens Point, Wis). Instr., University of Wisconsin, Manitowoc, Wis.

J. J. HAILE, B.S. in Ed. (West Virginia S. C.) Instr., Maryland State College, Princess Anne, Md.

E. E. HAMMOND, JR., Sc.M. (Brown) Lieutenant, U. S. Navy, New York, N. Y.

H. W. HANDSFIELD, A.B. (Columbia) Editor, College Department, McGraw-Hill Book Company, New York, N. Y.

W. C. HANSEN, Student, University of Maryland, College Park, Md.

L. R. HARPER, JR., A.B. (Talladega) Grad. Student, University of Chicago, Ill.

C. R. HEIMLICH, Student, Purdue University, Lafayette, Ind.

H. C. HEWES, JR., Student, Northeastern University, Boston, Mass.

E. E. HIGH, M.A. (Atlanta) Instr., West Virginia State College, Institute, W. Va.

C. G. HILL, A.B. (Morehouse) Instr., Voorhees Junior College. Denmark, S. C.

NANCY HINTZE, Student, University of Oklahoma, Norman, Okla.

KATHERINE A. HOLLAND, B.S. (Alabama Poly.) Mathematician, U.S.N.M.C.S., Panama City, Fla.

W. B. HOLMES, B.S. (Oklahoma) Flight Test Analyst, Boeing Airplane Company, Wichita, Kan.

N. Q. HUBBARD, A.M. (Nebraska) Instr., Lincoln Junior College & High School, Kansas City, Mo.

E. H. INSELMANN, M.A. (Columbia) Mathematician, Frankford Arsenal, Philadelphia, Pa.

LEO KATZ, Ph.D. (Michigan) Asso. Professor, Michigan State College, East Lansing, Mich.

D. A. KEARNS, M.A. (Brown) Asst. Professor, Merrimack College, Andover, Mass.

O. W. LAING, M.A. (Southern California) Instr., Pasadena City College, Calif.

P. F. LARDI, Student, Fordham University, New York, N. Y.

MARGARET M. LASALLE, M.S. (Louisiana) Instr., Francis T. Nichols Junior College, Thibodaux, La.

W. E. LAWRENCE, B.S. (Marquette) Grad. Research Assistant, Marquette University, Milwaukee, Wis.

R. B. LEIPNIK, Ph.D. (California) Asst. Professor, University of Washington, Seattle, Wash.

S. R. LENIHAN, B.A. (U.C.L.A.) Teaching Assistant, University of California at Los Angeles, Calif.

BERNARD LEVENSON, M.A. (Columbia) Research Assistant, Columbia University, New York, N. Y.

G. A. LEWIS, M.A. (N.Y.U.) Instr., Storer College, Harpers Ferry, W. Va.

MRS. HELEN W. LINDLEY, M.S. (Mississippi

S. C.) Grad. Assistant, Tulane University, New Orleans, La.

R. F. LOBER, Student, Alfred University, Alfred, N. Y.

R. G. LONG, Student, Reed College, Portland, Ore.

R. D. LOWE, A.B. (Dickinson) Grad. Assistant, Northwestern University, Evanston, Ill.

G. E. MAHONEY, Student, Boston University, Mass.

E. W. MCCLANE, M.S. (DePaul) Asst. Professor, DePaul University, Chicago, Ill.

D. G. MILLER, B.S. (California) Grad. Student, University of Illinois, Urbana, Ill.

R. C. MILLER, JR., M.A. (Redlands) Mathematician, Naval Ordnance Test Station, China Lake, Calif.

T. O. MOORE, M.A. (Arkansas) Instr., University of Missouri, Columbia, Mo.

P. S. NULL, M.S. (West Virginia) Instr., West Virginia University, Morgantown, W. Va.

J. N. PALMIERI, Student, Brown University, Providence, R. I.

E. H. PRIMOFF, Student, University of California at Los Angeles, Calif.

T. J. REGIMBAL, JR., Student, Gonzaga University, Spokane, Wash.

B. V. RITCHIE, M.S. (Purdue) Instr., University of Detroit, Mich.

D. C. ROSE, M.A. (Kentucky) Instr., University of Kentucky, Lexington, Ky.

LAWRENCE ROSENFELD, M.A. (Harvard) Mathematician, Cambridge Research Center, Mass.

DAVID ROTHMAN, Student, University of Wisconsin, Madison, Wis.

D. A. RUX, B.S. (Marquette) Grad. Assistant, Marquette University, Milwaukee, Wis.

V. G. RYAN, M.A. (Villanova) Instr., St. Joseph's College, Philadelphia, Pa.

D. E. SCHAFER, A.B. (Friends) Medical Student, University of Minnesota, Minneapolis, Minn.

MRS. FARIEBEE P. SELF, M.A. (Louisiana) Asst. Professor, Centenary College, Shreveport, La.

MRS. SEDALIA M. SIMS, B.S. (Wiley) Instr., Wiley College, Marshall, Tex.

SISTER M. ELIZABETH, M.A. (DePaul) Mathematics Department, Alverno College, Milwaukee, Wis.

J. W. H. SPENCER, B.S. (George Washington) Chief, Computing Branch, Army Map Service, Washington, D. C.

H. W. SPRING, JR., Student, University of Rhode Island, Kingston, R. I.

W. C. STONE, M.S. (Chicago) Asst. Professor, Union College, Schenectady, N. Y.

W. E. STRIMLING, M.A. (Minnesota) Teaching Assistant, University of Minnesota, Minneapolis, Minn.

W. C. STYSLINGER, JR., Student, Duquesne University, Pittsburgh, Pa.

ENN TATAR, Student, Wartburg College, Waverly, Ia.

J. T. TATE, Ph.D. (Princeton) Higgins Research Associate, Princeton University, N. J.

L. J. TEUTONICO, Student, St. John's University, Brooklyn, N. Y.

C. W. THOMSON, B.S. (S.T.C., St. Cloud, Minn.) Part-time Instructor, University of Colorado, Boulder, Colo.

B. F. TIFFIN, B.Sc. (Loyola, Montreal) Grad. Student, McGill University, Montreal, Quebec

ELMER TOLSTED, Ph.D. (Brown) Asst. Professor, Pomona College, Claremont, Calif.

H. G. TUCKER, M.A. (California) Grad. Student, University of California, Berkeley, Calif.

R. M. VASSAR, Student, College of Idaho, Caldwell, Idaho

SAMUEL VERBLUNSKY, Ph.D. (Cambridge, England) Professor, Queen's University, Belfast, Northern Ireland

J. A. VOLLKOMMER, Student, St. John's University, Brooklyn, N. Y.

A. W. WALLACE, Student, Northeastern University, Boston, Mass.

L. A. WARE, Ph.D. (Iowa) Professor, State University of Iowa, Iowa City, Ia.

ROBERTA A. WILCOX, Student, University of Rhode Island, Kingston, R. I.

R. R. WILLIAMS, JR., M.S. (Michigan) Chairman, Department of Mathematics, Wiley College, Marshall, Tex.

R. F. WILLIS, A.B. (Columbia) Grad. Student, Columbia University, New York, N. Y.

DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the National Bureau of Standards, Washington, D. C., on December 8, 1951. Professor E. R. Sleight, Chairman of the Section, presided at the morning and afternoon sessions.

The meeting was attended by one hundred and twelve persons including the following seventy-five members of the Association:

J. C. Abbott, M. H. Ahrendt, R. P. Bailey, W. E. Barnes, T. J. Benac, E. E. Blanche, J. H. Braun, B. H. Buikstra, W. E. Byrne, Audrey L. Carlan, H. J. Cheston, Jr., Randolph Church, Chester R. Clark, G. R. Clements, J. H. Curtiss, Ruth M. Davis, L. S. Dederick, J. A. Duerksen, William H. Durfee, E. L. Eagle, D. F. Eliezer, P. J. Federico, C. H. Frick, B. C. Getchell, Michael Goldberg, R. A. Good, E. S. Grable, E. C. Gras, R. E. Greenwood, T. N. E. Greville, J. R. Hammond, H. G. Hertz, J. E. Ikenberry, S. B. Jackson, F. E. Johnston, A. E. Landry, Mary A. Lee, D. B. Lloyd, C. J. Maloney, Martin Maltenfort, M. H. Martin, J. M. McLynn, Florence M. Mears, Joseph Milkman, George Millman, W. K. Morrill, C. R. Morris, C. H. Murphy, Jr., W. R. Murray, W. H. Norris, Jr., A. J. Pejsa, G. W. Petrie, III, R. M. Pinkerton, J. W. Popow, C. H. Rawlins, Jr., R. W. Rector, L. V. Robinson, P. C. Rogers, W. G. Saunders, J. B. Scarborough, Paul Shapiro, W. F. Shenton, Sister Gabrielle Marie, E. R. Sleight, J. K. Sterrett, H. C. Stotz, Fedor Theilheimer, O. M. Thomas, P. D. Thomas, J. A. Tierney, Michael Tikson, John Todd, R. E. Walters, M. E. White, and J. W. Wrench.

During the business meeting the Section voted to change the by-laws of the Section making the official name of the Section as follows: The Maryland-District of Columbia-Virginia Section of the Mathematical Association of America. The Section also voted to have a committee appointed to study the suggestions for section activities made at the Minneapolis meeting and report at the spring meeting of the Section. Miss Veryl G. Schult was appointed chairman of this committee. It was announced that Michael Goldberg will act as a member of the executive committee to fill the vacancy made by Professor Klee's leaving the region to go to the Institute for Advanced Study. The members of the nominating committee appointed to submit a slate of new officers at the spring meeting were M. H. Martin, University of Maryland, chairman, W. E. Byrne, Virginia Military Institute, and W. K. Morrill, Johns Hopkins University. It was announced that the invitation to hold the spring meeting at the Virginia Military Institute, Lexington, Virginia, on April 26, 1952 had been accepted by the executive committee. The Section voted unanimously that a letter of thanks be sent to the National Bureau of Standards for their hospitality to the members and guests of the Association during the meeting.

The following papers were presented at the meeting:

1. *The squaring of developable surfaces*, by Mr. Michael Goldberg, Bureau of Ordnance, Navy Department.

The problem of dividing a rectangle into different squares has received considerable attention and many elegant results have been obtained. Mr. Goldberg exhibited some of these results and showed how they may be extended to non-planar surfaces. In particular, the method was applied to

cylinders, Möbius strips and various cones including those which develop into overlapping plane surfaces. Models of these divisions were exhibited.

2. *N-dimensional Hermite polynomials*, by Professor Joseph Milkman, United States Naval Academy.

The definition of N-dimensional Hermite polynomials given by Harold Grad in vol. II, 1949, of *Communications on Pure and Applied Mathematics* is revised to a form suitable for generalization to infinite dimensional spaces and some of their properties including orthogonality developed. A Hermite polynomial of order p is defined to be a polynomial of degree p obtained by the formula:

$$H^p_{i_1 i_2 \cdots i_p} = \frac{(-1)^p}{(2\pi)^p W} \frac{\partial^p W}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_p}}$$

where

$$W = e^{-\pi(x_1^2 + x_2^2 + \cdots + x_N^2)}.$$

Obviously $H^p_{i_1} \cdots {}_{i_p}$ can be expressed as a product of N one dimensional Hermite polynomials.

3. *Partial differential equations and differential operators*, by Dr. L. V. Robinson, Aberdeen Proving Ground.

By making use of the generalized differential operator

$$\lambda = \frac{\partial}{\partial x} + P_1(x, y, z) \frac{\partial}{\partial y} + P_2(x, y, z) \frac{\partial}{\partial z} + \cdots,$$

it was shown how linear partial differential equations of the first and second orders can be solved. Some general structural relations between these differential equations and their solutions were emphasized.

4. *A class of spirals*, by Professor S. B. Jackson, University of Maryland.

In some recent work in gas dynamics, it became important to discuss the geometric behavior as t becomes infinite of curves of the form

$$z = \int_0^t r(u) e^{iu} du$$

where $r(t)$ is a non-negative monotone increasing function, becoming infinite with t. It was shown in this paper that such a curve is an outwinding spiral and that it exhibits one of two characteristics. Either $\lim_{t \to \infty} |z| = \infty$ so that it spirals to infinity in all directions, or the curve lies in a half plane. There is a unique line bounding such a half plane which consists precisely of the limit points of the spiral. Analytic criteria were established to distinguish the two cases.

5. *John Warner's mathematical exercises*, by Professor W. F. Shenton, The American University.

Professor Shenton read and discussed portions of a 32-page pamphlet published in London in 1710. In this pamphlet John Warner, Teacher of Mathematicks, presents the first of a series developing "A New, Plain, and Methodical Institution of Universal Mathematicks Pure and Mixt. Beginning at the very Foundation, and proceeding gradually in the Natural Order those Disciplines ought to be learnt, by such as are minded to go through a Course of this most Useful, Solid Learning. Digested into EXERCISES for the Benefit and Advantage of Learners; which may also contribute somewhat to the Ease and Help of Teachers." Professor Shenton expressed the desire to learn if other numbers of the series had been published.

6. *Use of an electronic analogue machine as a curve fitting device*, by Mr. C. H. Murphy, Jr., Ballistic Research Laboratories, Aberdeen Proving Ground.

An important problem of applied mathematics in the field of aerodynamics is that of determining the coefficients of a differential equation from discrete points of its solution curve. In the field of modern electronic machines precision analogue computers, such as the Reeves Electronic Analogue Computer (REAC), seem to be quite capable of solving the problem. A recent investigation of this use for the REAC was described and graphical results shown. The equation used for the problem was a linear second order complex equation with varying coefficients.

7. *Round table discussion on modern numerical analysis and college mathematics*, by Dr. J. H. Curtiss, Chief, National Bureau of Standards Applied Mathematics Laboratories (Moderator), and Mr. John Todd, Chief, Mr. H. E. Salzer, and Mrs. Ida Rhodes, National Bureau of Standards Computation Laboratory.

Dr. Curtiss began the discussion by stating that with a prospect of fifty new large scale automatic electronic computers to be put into operation within the next five years a need will arise for some two thousand college-trained operators and that from one hundred to two hundred mathematical analysts at the post-Ph.D. level will be needed to insure efficient use of these machines. He further indicated the desirability of mathematics teachers shifting their emphasis from continuous analysis toward those essentially discrete forms of analysis which are used in modern numerical analysis.

Mrs. Rhodes urged that in the undergraduate curriculum more attention be given to errors involved in computing with approximate numbers; calculation using various scales of notation other than the scale with the base ten; theory of polynomial approximation, with special reference to devices for speeding up convergence; modern matrix theory, with special reference to different inversion techniques; asymptotic series; and interpolation, with emphasis on remainder terms.

Mr. Salzer suggested that some of these topics be studied in a one-year course in modern numerical analysis at the junior level (see this MONTHLY, May, 1952) followed by further work at the senior and graduate levels. He further suggested that within the courses now taught in most colleges asymptotic series could be taught while studying integration by parts, difference equations and numerical integration could be taught while studying differential equations, and that field trips and advertising of job openings would stimulate student interest in numerical mathematics.

Mr. Todd discussed the solution of difference equations, showing how some important and interesting results can easily be established which cast light on differential equation theory.

The professional computers urged that only students with the prerequisite qualities of patience, carefulness, and pertinacity be encouraged to take jobs in computing laboratories; but the moderator pointed out that there are opportunities in the research and analytical aspects of computing which do not demand specialized temperamental characteristics. It was suggested from the floor that college teachers could be made to understand the problem best by establishing summer courses or teacher institutes at some of the great computing centers.

C. H. FRICK, *Secretary*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pennsylvania, on Saturday, November 24, 1951. Chairman for this meeting was Professor P. A. Caris, University of Pennsylvania.

There were fifty-eight present, including the following forty-four members:

Emil Artin, Joshua Barlaz, P. A. Caris, Mary L. Constable, E. H. Cutler, D. J. Davis, J. E. Davis, L. J. Deck, F. L. Dennis, Arnold Dresden, N. J. Fine, H. S. Grant, R. C. Haseltine, Katharine E. Hazard, P. J. Kiernan, J. R. Kline, R. A. C. Lane, V. V. Latshaw, F. L. Manning, Clifford Marburger, S. S. McNeary, A. E. Miller, C. A. Nelson, C. O. Oakley, J. C. Oxtoby, G. E. Raynor, Judith A. Richman, J. W. Ridley, R. D. Schafer, Pincus Schub, M. J. Sendrow, C. A. Shook, J. C. Smith, P. I. Speicher, E. P. Starke, R. R. Stoll, A. W. Tucker, R. N. Van Arnam, N. D. Vlachos, G. C. Webber, Anna P. Wheeler, R. C. Yates, H. M. Zerbe, H. J. Zimmerberg.

At the business meeting the following officers were elected for the coming year: Chairman, F. L. Manning, Ursinus College; Secretary, R. D. Schafer, University of Pennsylvania. The Program Committee for the next meeting will be: C. A. Nelson (Chairman), Rutgers University, G. C. Webber, University of Delaware and Albert Wilansky, Lehigh University. The 1952 meeting of the section will be held at the University of Delaware, Newark, Delaware, on November 29.

The program consisted of the following papers:

1. *The stimulation of interest*, by Colonel R. C. Yates, U. S. Military Academy.

Colonel Yates presented items intended to arouse and maintain student interest in mathematics of the first two college years. Such motivations as linkages, cams, paper folding, elementary solutions of maximum-minimum problems, the properties of the crossed parallelogram, caustics, and trisection were used to give point to the general titles: *Integration with experience, Utility, The paradox and the puzzle, Elegance and simplicity, Discovery, Speculation, Creation,* and *The element of surprise.* Devices and models of various sorts were used to illustrate these means of stimulation.

2. *Constructions with ruler and divider*, by Professor Emil Artin, Princeton University.

A few elementary constructions are given that show that the use of a ruler and divider with fixed opening is equivalent to the use of a ruler and a free divider. The answer to the question: "What geometric problems can be solved by the use of these tools?," is the following:

(a) The problem must be possible with ruler and compass.

(b) The algebraic equation describing the geometric problem must have only real roots for all real values of the parameters that enter in the problem.

The proof of this statement makes use of the theory of real fields. The essential theorems concerning real fields are given.

3. *An infinite-product expansion for analytic functions, by* Professor Bernard Epstein, University of Pennsylvania, introduced by the Secretary.

J. F. Ritt obtained in 1930 the theorem: If $f(z)$ is analytic at $z=0$ and $f(0)=1$, then there exists a uniquely determined set of constants $\{a_k\}$ such that

$$f(z) = \prod_1^\infty (1 - a_k z^k)$$

in some circle $|z| < R$. Evidently R cannot exceed the radius of convergence of the Taylor series of $f(z)$ (about $z=0$) but simple examples show that it may be less. Ritt showed that $R > 1/6C^{-1}$, $C = \sup |c_k|^{1/k}$ where $-f'(z)/f(z) = c_1 + c_2 z + c_3 z^2 + \cdots$. It is shown there that $R \geqq C^{-1}$ and that this result is "best possible" in the sense that there exist functions $f(z)$ for which the above product fails to converge for $|z| > C^{-1}$. (On the other hand, there exist functions whose product-expansion converges beyond this "minimum" circle.) The proof consists in obtaining estimates on the moduli

of the a_k in terms of the moduli of the c_k. While the basic idea of the proof is quite simple, some of the necessary estimates are rather intricate. Whether such products possess any important applications remains an open question.

4. *Articulation of secondary and college mathematics*, by Mr. P. J. Kiernan, The Lawrenceville School.

Cooperating members of the secondary schools and colleges are now attempting to make the union between secondary school and college mathematics smooth for the student. Interest in the problem was aroused by a panel meeting in December, 1949, held under the auspices of the New Jersey Committee on Articulation of Colleges and Secondary Schools. Aside from the curriculum, the major point brought up was that of *reading*. While the secondary schools need to do more in the field of reading, they are handicapped by the meager explanations in current texts. This suggests that the college teacher must continue the training of the student in the intensive type of reading that mathematics requires. A sub-committee is trying to determine an essential curriculum core for secondary school mathematics which is necessary for later success in college work. If the number of items in this core can be kept small, it should stimulate schools in New Jersey to offer not only those but also perhaps some more advanced topics.

C. O. OAKLEY, *Secretary*

OCTOBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the Mathematical Association of America was held at the North Dakota Agricultural College in Fargo, North Dakota, on Saturday, October 6, 1951. Sessions were held in the forenoon, at luncheon, and in the afternoon. Professors A. G. Hill, P. A. Rognlie and H. D. Colson, Chairman of the Section, presided at the respective sessions.

Thirty-one persons attended the meeting, including the following twenty members of the Association:

H. M. Anderson, Agnes J. Beckstrom, E. J. Camp, H. D. Colson, R. A. Forseth, G. C. Francis, Ruby M. Grimes, K. L. Hankerson, W. H. McBride, N. S. Mendelsohn, Sigurd Mundhjeld, E. O. Nelson, J. C. Peterson, P. A. Rognlie, S. C. Simonson, F. C. Smith, I. W. Smith, R. C. Staley, O. E. Stanaitis, Matilda B. Thompson.

By invitation of the Executive Committee, Professor W. H. McEwen of the University of Manitoba delivered an address at the morning session entitled *Spectral Theory and Differential Eigenvalue Problems*. Abstract of this address follows:

The fundamental notions of Hilbert Space theory were reviewed and the spectrum of a self-adjoint operator H defined. The properties of the point spectrum (of eigenvalues) and the continuous spectrum were noted, and it was shown in some detail how an arbitrary element f in the subspace determined by the eigensolutions can be expanded by means of a limiting process applied to a contour integral of $(H-\lambda I)^{-1}f$. The appropriateness of this formula for the expansion of elements in the whole space was taken for granted, and it was shown how the formula may be used to solve expansion problems associated with the differential equation $y''+(\lambda-q(x))y=0$ in which $q(x)$ is real valued and continuous in an open integral $a<x<b$, the Hilbert Space being the function class $L_2(a, b)$. By way of illustration the solution of the expansion problem for the case $q(x)\equiv 0$ on $-\infty<x<\infty$ was given.

The following seven short papers were presented:

1. *A note on the teaching of complex numbers*, by Professor E. J. Camp, Macalester College.

The purpose of the paper is to show how to "discover" an adequate definition of addition and multiplication of complex numbers regarded as ordered pairs of real numbers. It is required that the new system should have the ordinary real numbers as a subclass; it should form a field; and the system should be closed with respect to the operation of finding roots.

The definition of addition follows immediately from the corresponding definition for vectors. The definition of multiplication is made first for the real complex numbers $(a_1, 0)$ and $(b_1, 0)$. The definition is then generalized in such a way that the field laws are satisfied, and so that the system is closed with respect to the operation of finding roots. These requirements lead to the usual definition of product, *viz.*

$$(a_1, a_2)(b_1, b_2) = (a_1b_1 - a_2b_2, a_2b_1 + a_1b_2)$$

without the necessity of multiplying (a_1+a_2i) by (b_1+b_2i) as if they were two real binomials and then replacing i^2 by -1.

2. *Annuities certain with variable payments*, by Professor F. C. Smith, College of St. Thomas.

In this paper, the author applies the method of inverse differences to the problem of determining the amount and present value of several types of annuities with variable payments.

3. *On Wren's method of finding the highest common factor of two or more polynomials*, by Professor F. J. Arena, North Dakota Agricultural College, presented by Professor W. L. Woodley who was introduced by Professor A. G. Hill.

Dr. Wren has developed a very ingenious method of finding the highest common factor of two or more polynomials and thus the equal roots of two or more algebraic equations. His method possesses several advantages over other known methods: it is very easy to apply and, in the case of algebraic equations, it tells us what the equal roots are. His method is based on some researches which he published in 1937 in the *Bulletin of the American Mathematical Society*, pages 823–834. The purpose of this paper is to explain this method so that it will be available to a larger number of those who have occasion to use it.

4. *Euler's summation formula and convergence of series*, by Professor O. E. Stanaitis, St. Olaf College.

The formula referred to may be used as a criterion in the theory of convergence and divergence of series. It is remarkable that the formula is applicable in cases where the criteria presented in texts fail. The usefulness of the formula has been illustrated by several examples.

5. *Some problems relating to present pension systems*, by Mr. I. W. Smith, 1126 Thirteenth Street North, Fargo, North Dakota.

The work of mathematicians has developed a substantial basis for pension systems, especially in the field of annuities, probability and statistics. Actuaries have made possible many different plans of pension systems which have functioned on a very sound basis.

The three factors in determining a pension system are interest, amount of contribution to the cost and amount of retirement benefit at a given age.

The disturbing elements to earlier-formed pension plans are present low rates of interest on investments of pension funds, the high index of living cost and taxes required under present war budgets. The employee and employer are affected by the latter two factors in support of a satisfactory retirement income.

Pension systems adopted before the present-day conditions require a revision and adjustment to support an adequate amount of retirement benefit.

6. *Use of finite fields in design of experiments*, by Mr. David Gosslee, North Dakota Agricultural College, introduced by the Secretary.

For some large and complicated experimental designs the task of assigning treatments to the experimental units in proper combinations becomes very tedious. The methods due to R. C. Bose and others can greatly reduce the labor. Methods for obtaining treatment combinations for a 3×3 Latin square and a 3^2 factorial design are discussed. The correspondence between the lines and points of a finite geometrical space and certain experimental elements is shown.

7. *An application of a famous inequality*, by Professor N. S. Mendelsohn, University of Manitoba.

Use is made of the classical inequality which states that the arithmetic mean of n positive quantities is greater than or equal to their geometric mean, in order to obtain a very concise proof of the fact that the sequence $(1+1/n)^n$ has a limit as n approaches infinity.

F. C. SMITH, *Secretary*

JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The fourteenth annual meeting of the Northern California Section of the Mathematical Association of America was held at Stanford University on Saturday, January 26, 1952. Professor C. D. Olds, Chairman of the Section, presided at both the morning and afternoon sessions.

The attendance was one hundred and eighteen including the following fifty-one members of the Association:

H. L. Alder, H. M. Bacon, T. J. Bass, Alice K. Bell, D. C. Benson, M. T. Bird, Z. W. Birnbaum, R. N. Bradt, L. P. Burton, P. F. Byrd, H. M. Cochran, Roy Dubisch, Hazel Eggett, D. J. Ewy, Ruth A. Fish, S. A. Francis, C. M. Fulton, L. C. Graue, C. A. Hayes, J. G. Herriot, Marjorie L. Hoffman, Mary T. Huggins, V. F. Ivanoff, Free Jamison, R. M. Lakness, Charles Loewner, J. C. C. McKinsey, R. B. Merkel, E. D. Miller, H. S. Moredock, F. R. Morris, W. H. Myers, C. D. Olds, H. J. Osner, M. F. Pollack, George Polya, Edris P. Rahn, Raphael M. Robinson, E. B. Roessler, Kathryn B. Rolfe, H. L. Royden, Ruth G. Sumner, Mary V. Sunseri, Irving Sussman, A. V. Sylwester, Gabor Szegö, J. L. Ullman, M. J. Vitousek, L. A. Walker, Elizabeth Winstock, A. R. Williams.

The following officers were elected for the coming year: Chairman, Professor Raphael M. Robinson, University of California; Vice-Chairman, Professor Roy Dubisch, Fresno State College; Secretary-Treasurer, Professor C. D. Olds, San Jose State College.

The following motion prepared by Professor E. B. Roessler and Professor F. R. Morris was approved:

The Joint Committee on Mathematical Education, Northern and Southern California Sections, Mathematical Association of America, shall have membership rules as follows:

(1) The committee shall be composed of the sectional governors, any governor at large who may live within the territory of the two sections, three additional members from the Northern Section, and three additional members from the Southern Section.

(2) The sectional governor of each section shall appoint one member to the committee in July of each year to serve a term of three years.

(3) The committee as a whole shall elect its own chairman for a period of one year; if for some reason the committee has no chairman, the senior governor shall act as chairman until the committee elects otherwise.

(4) The chairman shall not be eligible to serve more than three consecutive years.

The following resolution proposed by Professor H. S. Moredock was approved:

Whereas it has been found that approximately two-thirds of the people teaching arithmetic in the ninth and tenth grades have neither teaching majors nor minors in mathematics and,

Whereas the number of pupils reaching this level and requiring instruction in arithmetic will certainly increase markedly in the immediate future and,

Whereas an alarming shortage in the number of candidates for the general secondary credential with teaching majors or minors in mathematics even now exists,

Therefore, be it resolved that teacher training institutions be urged to insist upon and to provide opportunity to acquire competence in teaching arithmetic on the part of all candidates seeking general secondary credentials and further that the Joint Committee on Mathematical Education of the Northern and Southern California Sections of the Mathematical Association of America be asked to give early and careful consideration to this problem.

By invitation of the Section, Professor M. Schiffer of the Institute of Advanced Study and Stanford University gave an address entitled *Mathematics and Science* during the morning session.

The following papers were presented:

1. *The volume of the sphere and Archimedes' discovery of the integral calculus*, by Professor George Polya, Stanford University.

In his work on *Method* (which was considered as lost till the beginning of this century) Archimedes derives the area of the parabolic segment, the volume of the sphere, the center of gravity of the hemisphere and about a dozen more results on areas, volumes and centers of gravity. He uses a uniform method to solve this variety of problems, which is remarkable in many respects: the method is heuristic, the idea of equilibrium plays an important role in it, and also the idea of coordinate geometry although, of course, the algebraic notation that makes "analytic geometry" was totally unknown to Archimedes. The speaker proposed to enliven the usual treatment of analytic geometry by incorporating the whole work of Archimedes into it—analytic geometry, as usually treated, is badly in need of some enlivening. Using the example of the volume of the sphere, he demonstrated that the essential idea of Archimedes' great discovery appears undistorted, and even more impressive, if the modern notation of analytic geometry is used.

2. *An essentially undecidable axiom system*, by Professor Raphael M. Robinson, University of California, Berkeley.

The axiom system discussed has the primitive concepts 0, S, $+$, $\cdot$, and consists of seven axioms: If $Sa=Sb$, then $a=b$; $0\neq Sb$; if $a\neq 0$, then $a=Sb$ for some b; $a+0=a$; $a+Sb=S(a+b)$; $a\cdot 0=0$; $a\cdot Sb=a\cdot b+a$. As announced at the International Congress of Mathematicians in 1950, this axiom system is essentially undecidable; that is, there is no general method of deciding what statements follow from this axiom system, and the same is true for any consistent extension. The present talk was limited to discussing some elementary consequences of the axioms, and showing that the formula $0+a=a$ does not follow from the axioms.

3. *A simple proof of the binomial theorem*, by Professor C. M. Fulton, University of California, Davis.

This paper has appeared as a classroom note in the April, 1952 issue of this MONTHLY.

4. *The unexpected in mathematics—a museum of pathology*, by Professor J. G. Herriot, Stanford University.

This paper was devoted to a discussion of several examples from various branches of analysis; the examples were chosen to illustrate various types of unexpected behavior. Included were examples of integration of sequences of functions, convergence of double series, and Cantor's set.

5. *Logarithms are exponents*, by Professor M. T. Bird, San Jose State College.

Charles Davies (*Elements of Algebra* on the basis of M. Bourdon, Revised and Re-written, 1877, p. 310) indicated that logarithms may be expressed as simple continued fractions. The author suggested that such a method might be used by algebra teachers to bridge the gap between the definition of a logarithm and the use of a table of logarithms. The method used to find 1/2, 10/21, 21/44, 52/109, and 73/153 (and their decimal approximations) as successive approximations for the common logarithm of 3. It was noted that the theory of continued fractions provides a measure of accuracy for each approximation.

6. *A generalization of the game of Nim*, by Mr. D. C. Benson, Stanford University.

The games of Nim and Kayles are special cases from an extended class of two-party games. For these games one can define winning and losing positions. Further it is possible to define a commutative and associative multiplication of positions and prime positions. Any position can be written as a product of different prime positions, which is called the factorization of the position. If two positions have the same factorization, then they are equivalent. This concept of equivalence possesses the usual reflexive, symmetric, and transitive properties. There is a simple method of determining the factorization of any position. By means of a table of factorizations for the positions consisting of single counters, one can determine whether or not any given position is winning or losing; and given any losing position it is possible to find a good move, *i.e.* a move which transforms the losing position into a winning position.

7. *On the k-ality theory of rings*, by Professor Irving Sussman, University of San Francisco.

The k-ality theory of rings, as developed in a series of papers by A. Foster, was commented upon with relation to its applications to analysis over specialized universal algebras.

The theory consists in supplying the algebra with a convenient number of "coordinate systems" in which to operate by defining over the elements a particular transformation (permutation) group especially chosen for the system. This group then determines a group of mappings within the extensive space of functions of any number of variables, according to the direction:

$$g(x, y, \cdots) \xrightarrow{\rho} g_\rho(x, y, \cdots) = \rho^{-1}[g(\rho(x), \rho(y), \cdots)].$$

A particular example consists of taking the two-group of permutations over any commutative ring R with identity, where $\rho(a) = a^* = 1 - a$, and $a^{**} = a$. Then, for example, the "times" operation yields $a \times b \xrightarrow{\rho} a \times' b = a + b - a \times b$.

If $P(x, y, \cdots, 0, 1, \times', \times, +', +, {}^*)$ is a true proposition (*e.g.* an identity) in the "mixed" system, so is the "dual" proposition $P(x^*, y^*, \cdots, 1, 0, \times, \times', +, +', {}^*)$.

Exploited over an idempotent ring B, using $y^* = 1 + y$, the system $(B, \times, \times')$ subsumes the Boolean algebra B. An extension to p-rings, embracing multi-valued logics, follows.

Such disciplines as measure theories, geometries, topological groups, probability, seem amenable to treatment by the general theory.

MARJORIE L. HOFFMAN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-third Summer Meeting, Michigan State College, East Lansing, Michigan, September 1–2, 1952.

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, College of St. Scholastica, Duluth, October 11, 1952.
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City, October 31, 1952.
PACIFIC NORTHWEST, University of Oregon, Eugene, June 20, 1952.
PHILADELPHIA, University of Delaware, Newark, November 29, 1952.
ROCKY MOUNTAIN
SOUTHEASTERN, Alabama Polytechnic Institute, Auburn, March, 1953.
SOUTHERN CALIFORNIA, Los Angeles City College, March 14, 1953.
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN

THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9 and 10, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. These numbers have been issued to date:

No. 1. *Calculus of Variations* by G. A. Bliss, xiii+189 pages.
No. 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages.
No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.
No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.
No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.
No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.
No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.
No. 8. *Rings and Ideals* by N. H. McCoy, xii +216 pages.
No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.
No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

Off the Press

JOHN F. RANDOLPH's
CALCULUS

Flexibility—offering a happy balance between routine treatment and modern challenging material—is the keynote of this introductory calculus book written by a mathematician noted for his pedagogical skill. A connection between analytic geometry and calculus is made in the first section, and throughout the book a topic is anticipated so that a concept is surreptitiously planted before it is crystallized in a definition or application.

Published in March $5.00

EARL D. RAINVILLE's
ELEMENTARY DIFFERENTIAL EQUATIONS

Intended for students who have completed the standard calculus course, this text includes all of the author's SHORT COURSE IN DIFFERENTIAL EQUATIONS and is designed to help the student obtain a sound knowledge of elementary differential equations and good techniques for solving them. The presentation progresses smoothly from routine work to more and more discussion of principles. This progression coincides very well with the student's increasing ability to comprehend and appreciate more subtle point. *Published in April* $5.00

J. H. MICHELL & M. H. BELZ'
THE ELEMENTS OF MATHEMATICAL ANALYSIS 2/e

While adapted to the particular needs of students of science and engineering, Professors Michell and Belz have prepared a course of mathematical analysis which differs from existing works of this type in that it is rigorous and designed on modern mathematical principles. Little previous knowledge is assumed. *Published in March. $13.60 for the two-volume set.*

The Macmillan Company — *60 Fifth Avenue, New York 11*

Coming in the summer—for your fall classes—

a "how-to-do-it" text requiring only basic arithmetic

MERRILL RASSWEILER & IRENE RASSWEILER's

FUNDAMENTAL PROCEDURES OF FINANCIAL MATHEMATICS

This book contains a thorough treatment of the customary topics of both business and investment mathematics without requiring the student either to know or to learn anything beyond arithmetic. The material is organized to give the student a thorough grounding in percentage and its application to business through commissions, taxes, pricing, interest and discount, and negotiable instruments. Each topic includes discussion followed by a step-by-step procedure to guide the student and to show him how to carry out the calculations. *Ready this summer*

for the basic course in theory of numbers

B. M. STEWART's

THEORY OF NUMBERS

Readily adaptable to either a quarter or semester course, Professor Stewart's book contains 33 chapters—each with a set of exercises—and can be used as a basic text for a course at the junior, senior, and/or graduate level. As a footnote to the title of each chapter, there is a brief discussion as to whether, in the author's opinion, the chapter is a basic one, or one containing material that might be omitted from a short course. *Ready this summer*

The Macmillan Company

60 Fifth Avenue
New York 11

Outstanding McGRAW-HILL Books

GENERAL COLLEGE MATHEMATICS

By W. L. AYRES, C. G. FRY, and H. F. S. JONAH, Purdue University. In press

A new approach to freshman mathematics, designed to help the student in his application of mathematical topics to the biological and social sciences. Keyed primarily for those students who will take only one year of college mathematics, this text examines such subjects as ratios and percentages, linear and quadratic equations, trigonometry, interest and its application to installment buying, laws of growth, statistics, etc.

ADVANCED MATHEMATICS IN PHYSICS AND ENGINEERING

By ARTHUR BRONWELL, Northwestern University. In press

Presents a mathematical foundation in the principal branches of advanced mathematics used in physics and engineering. The fundamental laws of the more important areas are first expressed in very general form. These then become the springboard for the development of vast areas of applications. The text shows how the fundamental formulations simplify to special cases which usually form the starting point in solutions of problems.

INTERMEDIATE ALGEBRA

By PAUL K. REES, Louisiana State University, and FRED W. SPARKS, Texas Technological College. 328 pages, $3.50

An exceptionally clear and teachable presentation of intermediate algebra which brings the student from a limited background in algebra through simultaneous quadratic equations; ratio, proportion, and variation; logarithms; etc. Topics of particular difficulty are considered at length and in detail. 2500 problems, of progressive difficulty, are a particular feature of the text.

ELEMENTS OF STATISTICAL METHOD. New 3rd Edition

By ALBERT E. WAUGH, University of Connecticut. 531 pages, $5.50

A revision of this outstanding book. While changes have been made in the new edition, the text is still designed to introduce the student to statistical concepts and nomenclature and to encourage him to think in statistical terms. Every effort is made to keep the discussion at the beginner's level and to present basic ideas in such a way that the student will find it easy to continue under his own power.

Send for copies on approval



McGRAW-HILL BOOK COMPANY, INC.

330 WEST 42ND STREET, NEW YORK 18, N. Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 59 NUMBER 7

CONTENTS

The Things I Should Have Done, I Did Not Do . R. E. LANGER 443
An Equation in Gaussian Integers . . . . . P. A. SAMET 448
Functions of N Variables in Extended Analytic Geometry . . . . . R. S. UNDERWOOD 453
Mathematical Notes . . . . . P. G. HODGE, JR. 461
Classroom Notes . . . . . J. A. WARD 463
Elementary Problems and Solutions 465
Advanced Problems and Solutions 469
Recent Publications 473
Clubs and Allied Activities 475
News and Notices 478
The Mathematical Association of America 483
New Members 483
February Meeting of the Louisiana-Mississippi Section . . . 487
March Meeting of the Kansas Section 490
March Meeting of the Southeastern Section 491
March Meeting of the Southern California Section 498
April Meeting of the Iowa Section 500
April Meeting of the Kentucky Section 504
April Meeting of the Maryland-D.C.-Virginia Section 507
April Meeting of the Michigan Section 509
April Meeting of the Ohio Section 512
April Meeting of the Southwestern Section 514
April Meeting of the Texas Section 517

AUGUST-SEPTEMBER 1952

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	J. W. GIVENS	GEORGE SPRINGER
PATRICK DU VAL	L. M. KELLY	E. P. STARKE
HOWARD EVES	H. D. LARSEN	G. B. THOMAS
F. A. FICKEN	IVAN NIVEN	E. P. VANCE
	EDITH R. SCHNECKENBURGER	

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of College Mathematics)

OFFICERS OF THE ASSOCIATION

President, SAUNDERS MACLANE, University of Chicago
Honorary President, W. D. CAIRNS, Oberlin College
First Vice-President, F. L. GRIFFIN, Wesleyan University
Second Vice-President, JEWELL H. BUSHEY, Hunter College
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo
Editor, C. B. ALLENDOERFER, University of Washington

Additional Members of the Board of Governors: I. A. BARNETT, C. F. BARR, T. A. BICKERSTAFF, J. C. BRIXEY, B. H. BROWN, R. H. CAMERON, G. R. CLEMENTS, T. F. COPE, H. S. M. COXETER, W. M. DAVIS, J. M. EARL, L. R. FORD, J. S. FRAME, A. E. GAULT, R. F. GRAESSER, E. H. HANSON, M. R. HESTENES, A. S. HOWARD, C. G. JAEGER, R. D. JAMES, B. W. JONES, R. E. LANGER, D. H. LEHMER, F. A. LEWIS, C. C. MACDUFFEE, R. J. MICHEL, W. E. MILNE, C. W. MUNSHOWER, H. P. PETTIT, J. C. POLLEY, G. B. PRICE, G. E. RAYNOR, E. B. ROESSLER, F. H. STEEN, MARIE J. WEISS

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

THE THINGS I SHOULD HAVE DONE, I DID NOT DO*

R. E. LANGER, University of Wisconsin

It is said to have been an edict of Pythagoras, never to let sleep descend until the events of the day had been thrice reviewed. In a humble way I like to think myself a follower of Pythagoras. The edict therefore seemed to suggest topics that I might discuss on this occasion, although, I must say, the analogy between sleep and my retirement from the presidency of the Association does not strike me as particularly apropos.

Projects in which the Association might interest itself are surely appropriate to our concern. If it seems incongruous for me to bring such projects forward at the instant when I have safely stepped from the position that might well seem to have been peculiarly advantageous for action, I must concede the point. In extenuation I shall let Samuel Johnson (The Rambler, April 14, 1750) speak for me. He says: "If the most industrious of men could recall his past moments, it is scarcely to be imagined how few would be marked out by any permanent effect; how small a proportion his action would bear to his possibilities; how many chasms of vacuity he would find even in the most tumultuous hurries of business and the most eager vehemence of pursuit. Modern philosophers say that the hardest of bodies are so porous, that, if all matter were compressed to perfect solidity it might be contained in a cube of a few feet. In like manner, if all the employment of life were crowded into the time which it really occupied, perhaps a few weeks, or days, or hours, would be sufficient for its accomplishment. For such is the inequality of our corporeal to our intellectual faculties, that we contrive in minutes what we execute in years, and the soul often stands an idle spectator of the labour of the hands and the expedition of the feet."

A few years ago there was displayed at the Art Institute of Chicago a painting by a great American artist (Ivan L. Albright). This painting, though very simple, has become widely renowned. It pictured merely the door of a tomb with a wreath that had been deposited by a withdrawing hand. The title of the picture was "The things I should have done, I did not do." You see that I am acknowledging my plagiarism of the title of this address.

Our Association was conceived by its founders as an agency to operate in the collegiate field, and over the years it has continued to profess the concern for matters in this field to be its especial object. This assures me my franchise for what I wish to discuss, and leaves me only to hope that you will find it worth while. That it is original I will not claim. However, in changing times even the raising of oft-discussed questions need not be unprofitable. For under new realignments of prospects and purposes these may well be invested with enhanced importance.

The topics I have chosen are three; first, our elementary college mathematics curriculum under the impact of the widespread trend toward more general

* Presented as a presidential retiring address before the Mathematical Association of America at Minneapolis, September 3, 1951.

education at the expense of specialized subjects; second, our concerns and practices in the training of secondary school teachers; and third, our response to the unprecedented expansion of employment opportunity for mathematically trained personnel.

The great majority of college students are restricted in their choice of mathematics courses to such as we may, perhaps with reservations, refer to as traditional. They are courses in which the essentials of algebra, trigonometry, the calculus, *etc.* are more or less conventionally set forth, in accompaniment with drill that is directed toward technical proficiency. We hear the charge now and then, that these courses are set and dry, that they are withdrawn from matters of immediate vital interest, and that they are centered upon narrow technological objectives, to the extent that they are unrewarding except for a few. In large measure such criticism is certainly mistaken. In the main, these courses are in continual revision. One can claim for them much richness, both in content and method, that is contemporarily wholly to the point. The needs professed by innumerable civilian agencies, and no less by the nation's armed forces, as was emphasized by the recent countrywide examinations, must certainly be considered as eloquent of the profit with which a student may spend time in mathematics courses.

Let us not, however, be smug about this. There is, perhaps, some justice in the criticism of our courses, especially that they are designed for students directed toward technical fields. Can we honestly say that they are not wide of the optimum for students otherwise directed, and especially so for those who can give only a year to mathematical study? The importance of this seems to me to be rising. Over the country there has been, and is, a decisive trend in the colleges toward what is called "more general education." There has been a pretty wide-spread growth of sentiment favoring a broader and less specialized basis of collegiate preparation, in recognition of students' predestined roles as citizens and educated people. There is concern for humanistic and especially for social subjects. Many schools have instituted heavily prescribed courses in so-called "integrated liberal studies."

Now in such courses mathematics invariably finds no place at all, or only a vanishingly insignificant one. Indeed in some places such courses have operated to rule out mathematical study. I will not labor the anomaly of this, at a time like the present, when war and revolutionary scientific and technological advances have made mathematical intelligence an ascendant agent in the molding of our very way of life, and an essential to its preservation. It is, however, the fact, that while newer means of communication are bringing more and more people into daily contact with the quantitative aspects of civilized life, the colleges are pouring forth great majorities of graduates who have only vaguely erroneous or non-existent notions of the nature and significance of our subject. This being so, is it surprising that in the secondary schools mathematics as a curricular subject is, to say the least, on the defensive?

I know, of course, that this problem is not new, and that serious and able

attempts to confront it have been made. If we admit, as I think we must, that our elementary college courses are patterned for technical mastery, must we not also admit that only by an extraordinary coincidence could they at the same time be best adapted for students with liberal objectives? This has engendered a great deal of conviction that something else is needed. In response to it there has been a wealth of publication, much of it distinctly admirable, of professedly non-technical books on the meaning of mathematics, on mathematical surveys, and the like.

I fear the definitive answer to the challenge has not yet been found. I am at least unaware of any general acceptance, either by us or by our critics, of any response that has yet been proposed. We have tried integrations of subject matter—and many regard them as little more than mixtures yielding but little to the purpose. We have tried surveys—and many have found them uninforming and superficial. We have tried new and unorthodox subject matter without discovering many new virtues, and we have tried an emphasis upon an appreciation of mathematics, only to find that appreciation can hardly be called forth by what is unknown and unmastered.

Much time and enthusiasm have gone into these experiments. The returns have generally not been commensurate. All too often the experimental course has been handicapped from the start by its attraction for those students who hoped to find in it the easy course in mathematics.

Has the problem, then, no solution? We must not think that, nor must we cease to apply ourselves to it. College students by thousands are constantly choosing their electives. We cannot be content to have them leave mathematics aside. Rich in its past, dynamic in the present, prodigious for the future, replete with simple and yet profound ideas and methods, surely mathematics can give something to anyone's culture. Does it seem, on the face of it, to be impossible to broach some of this even in a single year's terminal course?

I am sure that the key to the way of doing this is not a golden one that some lucky seeker will accidentally find. It will require a step by step search, and the approach to it will probably be hastened by combined effort. There are, over the country, many small colleges, junior colleges, and teachers colleges, in many of which this problem is more pressing than it is in the universities. It is not a problem for local solution. Therein, it seems to me, lies an opportunity for the Association. To find and enlist the talent that will apply itself, to organize it, to finance it, and in the end to publish its findings, therein may be a chance for great service. Can we, until this has been done, say that we have given the problem the best we are capable of?

I turn to my second topic: our concern with and practice in the training of secondary-school teachers. What a lack of uniformity prevails there! Over the country mathematics teachers are being turned out by institutions ranging from two-year colleges to the universities, and with trainings that may have terminated with the end of the first year of the calculus or with the Ph.D. degree. With such divergence of practice, what standards can there be? By what yardstick

shall an administrator appraise his staff? This is an important matter. It is, however, not the one to which I wish now to speak.

It is my own experience that, on the whole, we treat our teacher training responsibilities pretty casually. The plan of our courses, in college and graduate school, is by and large that of a road which penetrates deeper and deeper into the mathematical domain and eventually leads, through one branch or another, to the frontier. Subconsciously perhaps, but none the less truly, we think of ourselves first as trainers of mathematicians, and content ourselves all too easily with turning out as teachers those we judge to lack the qualifications for the higher estate.

If this is so, can we wonder that our high-school teachers widely fall short in mathematical mastery, even in the branches which concern them most, and that they are themselves so conscious of this that they regard this Association as too "high-brow" for them and consider the MONTHLY as "over their heads." One hears that this is not so in Europe. But we see our mathematics teachers content themselves with their shortcomings. They attend university summer schools in numbers, but to take degrees in Education, not to ameliorate their mathematical ineptnesses. It must be said our teachers are rarely at ease with their subject. Yet they are the ones upon whom we must depend to broach and initially interpret mathematics to the ever on-coming stream of the newer generation.

I believe we are at fault for this condition, and that it roots appreciably in our cavalier attitude toward the job of teacher training. Teaching, at whatever the level, is a dignified profession. And it calls at each level for especial qualifications. Many of our strongest mathematicians would cut but sorry figures before a high-school class. Is it more damning than this that many young men and women who are talented for teaching are less so for the abstractions of higher mathematics?

Do we recognize this in our practice? I think insufficiently so. In the main we send our destined teachers unselectively through mathematicians' courses, and trouble ourselves hardly at all over whether these are so specialized as to offer little in the way of perspective, or anything for future effective teaching. We stand guard that these students pass in partial differentiation and multiple integration, while we concern ourselves little with their arithmetical misconceptions, and condone the meagerness of their geometric mastery or skill. We permit many of them to leave us burdened with a conviction that mathematics is a painful and exacting discipline, a conglomerate of many sparsely related subjects, of which the most are soon and well forgotten. We give them almost no appreciation of the subject's unity, of the sublimity of even its elementary branches, or of the fact that it exists in response to an ineradicably ingrained human need and quest. We show them the trees, and disregard that they never see the forest.

These things have, of course, been said before. It is even a familiar cry that training for secondary-school teaching should feature breadth and comprehension rather than depth and specialization. The MONTHLY contains excellent

papers on this theme. A "Report on the Training of Teachers of Mathematics" by a commission of the Association (The MONTHLY, May, 1935) made an eloquent plea for differentiated teacher training extending even to the Ph.D. degree.

I do not mean to say that no special courses for teachers or differentiation of practice presently exist. The point I wish to make is mainly this: that there is need for more than just local attention to this problem. Therein may be an opportunity for the Association. Divided effort lacks authority and is unequal to the scope of the problem. With coordinated and sustained attention, on the other hand, we may perhaps in time establish norms of good practice which the schools could not afford to ignore, and through which our teachers might develop an *esprit de corps* for mathematics rather than for method.

I come to my third and final topic. During the last decade we have seen a great widening of the field in which mathematically trained persons are sought for. Our graduates at all levels therefore have more diversified opportunities for employment—even for careers. Where, not long ago, students of advanced mathematics had only small choice but to enter the teaching profession, they have now manifold opportunities in utilities and industry, and in the armed and civilian services of the Government. Indeed, this change has been so marked that the teaching profession is threatened with soon finding itself in straights to recruit enough of the best young mathematicians to insure the perpetuation of essential training staffs for the future.

What I have in mind in connection with this is: Granted the change, what responses are we, as teachers, making to it? Diversity of opportunity entrains a diversity of objectives. These, it might be expected, will call for diversified training, unless we are complacently to resign our students to the necessity of sharply reorienting themselves upon leaving us, and of hastening to learn many things we did not teach them. I do not know how widely these matters are calling forth curricular revisions. At the University of Wisconsin we have a program of relatively recent origin which seems pertinent to this. Since I have some knowledge of this I should like to say a little about it.

The usual graduate of an engineering school has neither the mathematical training nor the breadth of scientific experience to qualify him for more than routine engineering practice. He is not equipped to be of much use in what we may style engineering research. The usual mathematics student, on the other hand, is quite ignorant of both the theory and the practice of engineering pursuits. Now much of the enlarged employment opportunity lies precisely in the domain of the applications of mathematics in engineering. We at Wisconsin, therefore, have established a four or five year course, which is administered by a committee of mathematicians and engineers, and which leads to the bachelor's or the master's degree in "Applied Mathematics and Mechanics." The course is largely prescribed, and enrollment in it is restricted to students who maintain a distinctly superior average in their mathematics courses.

The freshman and sophomore years of the course are not especially distinc-

tive. They include a year of the calculus, and along with year courses in physics, chemistry and engineering mechanics, the usual courses in English, social studies, *etc.* In the junior year the technical training is stepped up. This includes courses in advanced calculus, light, electric circuits and electro-magnetic waves, along with engineering courses in the mechanics of materials, electrical machinery, and hydraulics. The courses of the senior year cover mathematical mechanics, vector analysis, Fourier analysis and some complex variable, along with aerodynamics, thermodynamics, and mechanical vibrations. The fifth year, if it is taken, gives considerable opportunity for concentration in one or another engineering field.

My purpose in giving this outline has been mainly to show that teaching adjustments are being made. Probably many others have been made, and are being made, in many places. Now I believe that a great service to mathematical education might be rendered by bringing this whole matter under competent and continued study, by comparing and collating the various adjustments and by publicizing them. Successful innovations could thus be brought to the attention of all. May there not be an opportunity for the Association along this line?

I bring my remarks to a close. In making them I have had no intention to be destructively critical, but only to bring under focus some issues which seem to me to be worth while. The question we must pose ourselves in connection with them is, I believe, as Hamlet has put it:

"Whether 'tis nobler in the mind to suffer
"The slings and arrows of outrageous fortune,
"Or to take arms against a sea of troubles,
"And by opposing end them."

AN EQUATION IN GAUSSIAN INTEGERS

P. A. SAMET, Christ's College, Cambridge, England

This paper is concerned with the equation $ax^2+by^2+cz^2=0$ which was investigated by Legendre when the coefficients are rational integers. He reduced the equation to a "normal form" in which

(i) a, b, c are square-free,

and

(ii) a, b, c are co-prime in pairs.

He found that necessary and sufficient conditions in this case for the equation to be soluble, *i.e.* that a solution in integers x, y, z, not all zero, exists, were that

(a) a, b, c are not all of the same sign,

(b) $-bc, -ca, -ab$ are quadratic residues of a, b, c respectively.

H. J. S. Smith [1] noticed that these conditions carry over to the case when the coefficients and unknowns are Gaussian integers, *i.e.* of the form $U+iV$ where U and V are rational integers. We show now that the proof given in Dickson's book [2] of Legendre's result can be extended to this case.

The equation may be taken to be in the normal form of (i) and (ii). First, we may take a, b, c to be square-free, for if $a=A\alpha^2$, $b=B\beta^2$, $c=C\gamma^2$, where A, B, C are square-free, and the equation $A\xi^2+B\eta^2+C\zeta^2=0$ has a solution (ξ, η, ζ), then $(\beta\gamma\xi, \gamma\alpha\eta, \alpha\beta\zeta)$ is a solution of $ax^2+by^2+cz^2=0$. Secondly, a, b, c can be taken to be prime in pairs. We can take $(a, b, c)=1$ and assume that a and b have a (Gaussian) prime factor p in common; then $cz^2\equiv 0 \pmod{p}$, which gives $z\equiv 0 \pmod{p}$ and therefore $z=pZ$, and our equation becomes $(a/p)x^2+(b/p)y^2+pcZ^2=0$, an equation where the product of coefficients is abc/p and so has a smaller norm than abc, and no two of the coefficients are divisible by p. After a finite number of steps we have coefficients that are co-prime in pairs.

Our result is the following:

THEOREM. *A necessary and sufficient condition for the equation*

$$ax^2 + by^2 + cz^2 = 0 \tag{1}$$

to be soluble in Gaussian integers, not all zero, if (i) *and* (ii) *are satisfied, is*

(iii) *bc, ca, ab are quadratic residues of a, b and c respectively.*

Note. We can dispense with the minus sign of Legendre's condition because $i^2=-1$.

Suppose the equation has a solution x, y, z. We may assume that x, y, z have no common factor and this implies that x, y, z are co-prime in pairs. For if $p \mid x$, $p \mid y$, where p is a Gaussian prime, $cz^2\equiv 0 \pmod{p^2}$, $z^2\equiv 0 \pmod{p}$ and then $p \mid z$ also. Then $ax^2+by^2\equiv 0 \pmod{c}$, which gives ab as a quadratic residue of c, because $(x, c)=(y, c)=1$. The condition is thus necessary.

To prove its sufficiency, we need a preliminary result.

LEMMA. *Conditions* (i) *and* (ii) *exclude the case* $|a|=|b|=|c|>1$.

For suppose $|a|=|b|=|c|>1$; then $N(a)=N(b)=N(c)$ where $N(a)$, the norm of a, is a rational integer. Firstly $N(a)$ cannot be divisible by a prime $p\equiv 3 \pmod{4}$ since then $p \mid a$, $p \mid b$, $p \mid c$ which is false. If $N(a)$ is even, a, b, c have a common factor $1+i$, which is again false. If $N(a)$ has a prime factor $p\equiv 1 \pmod{4}$, then a contains a factor π or $\bar{\pi}$, where π is a Gaussian prime such that $\pi\bar{\pi}=p$. At least one of the pairs a, b; b, c; c, a has either π or $\bar{\pi}$ as a common factor, again contradicting (ii).

We see easily that the case $|a| = |b| = |c| = 1$ always gives us a soluble equation.

By symmetry, we may take $|a| \leqq |b| \leqq |c|$, where now at least one of the equality signs can be omitted. Then $|ab| \leqq |ac| \leqq |bc|$.

We define $|ac| = J$ to be the *index* of equation (1). If we do not make the assumption that $|a| \leqq |b| \leqq |c|$, then the index is the middle one of $|ab|$, $|ac|$, $|bc|$ if they are all different, or the common value of two of them otherwise. The two definitions are equivalent.

The proof will be by induction on J. The values of J^2 form a sequence of positive integers, $J_1, J_2, \cdots, J_n, \cdots$ and we show that if the result holds for $J^2 \leqq J_n$, then it holds for $J^2 \leqq J_{n+1}$.

The case $J = 1$ has been considered. Now assume $J > 1$ and that equation (1) can be solved for all possible values of the index $< J$. By condition (iii), there exists an integer R such that $R^2 \equiv ab \pmod{c}$. By condition (ii), there exists an integer r such that $ar \equiv R \pmod{c}$. We can take r such that $0 < |r| < (1/\sqrt{2})|c|$.

Clearly $ar^2 \equiv b \pmod{c}$ and so

$$ar^2 - b = cQ \tag{2}$$

where Q is an integer and

$$|Q| = \frac{|ar^2 - b|}{|c|} \leqq \frac{|ar^2|}{|c|} + \frac{|b|}{|c|} \leqq \tfrac{1}{2}|ac| + 1 = \tfrac{1}{2}J + 1 < J$$

if $J > 2$.

We must now consider the cases $1 < J \leqq 2$, *i.e.* $J = \sqrt{2}$ and $J = 2$.

If $J = \sqrt{2}$, equation (1) always has a solution. We have $a = \epsilon_1$, $b = \epsilon_2$ or $\epsilon_2(1+i)$, $c = \epsilon_3(1+i)$ where ϵ_1, ϵ_2 and ϵ_3 are units. Further, $b = \epsilon_2(1+i)$ is impossible since b and c are co-prime. In the other case the equation always has a solution, as can be seen by enumerating cases.

If $J = 2$, $|ac| = 2$. Either $|a| = |c| = \sqrt{2}$, whence $|b| = \sqrt{2}$, an impossible case, or $|a| = 1$, $|c| = 2$ and thus c has a factor $(1+i)^2$ and so is not square-free, contradicting (i).

We now consider $J > 2$.

If $Q = 0$, $ar^2 = b$ and $|r| = 1$ since b is square-free. This gives $a = \pm b$ and then we always have a solution of (1).

If $Q \neq 0$, let A be the greatest common divisor of the three terms in (2). Since $A \mid b$, we have $(A, a) = (A, c) = 1$; thus $A \mid Q$ and $A \mid r^2$. Hence $A \mid r$, since A divides b and so is square-free. Put $r = A\alpha$, $b = A\beta$, $Q = AC\gamma^2$, where C is square-free. Substitution in (2) now gives

$$aA\alpha^2 - \beta = cC\gamma^2 \tag{3}$$

where the three terms are relatively prime in pairs. We also put $B = a\beta$.

We shall presently consider the equation

$$AX^2 + BY^2 + CZ^2 = 0 \tag{4}$$

and show now that A, B, C satisfy conditions (i), (ii) and (iii) and that equation (4) has a smaller index than (1). Finally we will show that the solution of (4) leads us to a solution of (1).

None of A, B, C is zero. From $AB = Aa\beta = ab$ we see that A and B are relatively prime and also square-free. From (3), C is prime to $\beta \cdot aA = AB$. Again from (3),

$$cC\beta \text{ is a quadratic residue of } aA,$$
$$aAcC \text{ is a quadratic residue of } \beta,$$
$$AB \text{ is a quadratic residue of } C.$$

Now ac is a quadratic residue of $b = A\beta$, and so of A. Thus $cC\beta$ and ac are quadratic residues of A and so is their product, βaCc^2. Hence $\beta aC = BC$ is a quadratic residue of A.

Since ac and $acAC$ are quadratic residues of β, AC is a quadratic residue of β. Since βcC and βAc are quadratic residues of a, so is $(\beta c)^2 AC$. Further, a and β are relatively prime, and so AC is a quadratic residue of $a\beta = B$.

A, B and C thus satisfy the same conditions as a, b and c.

We next show that the index of (4) is smaller than J if $|b| < |c|$. We have

$$|AC| \leqq |AC\gamma^2| = |Q| < J,$$
$$|AB| = |ab| < J,$$

and the index of (4) is $< J$, by the alternative definition of index.

In case $|b| = |c|$, so that $|AB| = |ab| = J$, we distinguish two cases:

(a) $|BC| < J$. The index of (4) is $< J$.

(b) $|BC| \geqq J$. We have $|AC| < |AB| \leqq |BC|$, and so $|A| \leqq |C| < |B|$. In this case we apply the whole reduction process to the equation $AX^2 + CY^2 + BZ^2 = 0$ (where we note the interchange of B and C) to get $A'X^2 + C'Y^2 + B'Z^2 = 0$ whose index is smaller than J since $|C| < |B|$, analogous to $|b| < |c|$.

If the index of equation (4) is $< J$, a solution X, Y, Z exists by hypothesis. Putting

$$\begin{cases} x = A\alpha X - i\beta Y \\ y = iX + a\alpha Y \\ z = C\gamma Z \end{cases} \tag{5}$$

and using (3) and the definition of B, we have

$$ax^2 + by^2 + cz^2 = cC\gamma^2(AX^2 + BY^2 + CZ^2) = 0. \tag{6}$$

If the index of (4) is $= J$, we find a solution of (4) from that of $A'X^2 + C'Y^2 + B'Z^2 = 0$ by the above substitution, and a second substitution then gives a solution of (1).

Further, the solution (5) of (1) is non-trivial if X, Y and Z are not all zero. For if in (5) $x=y=0$, elimination of X gives $(aA\alpha^2-\beta)Y=0$, where the first factor is $\neq 0$ by (3). Thus $Y=0$ and therefore $X=0$, giving $Z=0$ also, which is false.

Equation (1) thus has a non-trivial integer solution if conditions (i), (ii), (iii), are satisfied. This completes the proof.

I have tried in vain to find real positive numbers l, m, n depending on a, b, c such that a solution of (1) exists with $|x|\leqq l$, $|y|\leqq m$, $|z|\leqq n$, say a minimal solution. It does not seem easy to relate the minimal solution of $AX^2+BY^2+CZ^2=0$ to the minimal solution of $ax^2+by^2+cz^2=0$. Holzer [3] and Mordell [4] have recently studied minimal solutions of $ax^2+by^2-cz^2=0$ with rational integer coefficients, where a, b, c are positive. Holzer found that a solution exists with $|x|\leqq\sqrt{bc}$, $|y|\leqq\sqrt{ca}$, $|z|\leqq\sqrt{ab}$. Mordell found a similar, but slightly weaker, result by a very much simpler method than Holzer's. Neither of their methods seems to apply to the complex case.

Instead of the Gaussian field, we may consider the field $k(\sqrt{-d})$, where d is a positive square-free integer and the field has a Euclidean algorithm. We find on determining R and r that $|r|\leqq\frac{1}{2}\sqrt{d+1}\,|c|$ if $-d\equiv 2, 3 \pmod 4$. This gives $|Q|\leqq\frac{1}{4}(d+1)J+1$ and then this can be $<J$ only if $d=2$. The argument given above will then work if $J>4$, leaving the cases $1<J\leqq 4$ to be considered separately—a tedious though probably not difficult matter. If $-d\equiv 1 \pmod 4$ we find that $|r|\leqq\frac{1}{4}\sqrt{d+4}\,|c|$, $|Q|\leqq\frac{1}{16}(d+4)J+1$. This can be $<J$ for $d=3, 7, 11$, and once more the argument will work if J is sufficiently large, leaving various special cases to be considered separately. The minus sign of Legendre's original conditions would have to be reinstated also, since the equation $x^2=-1$ is no longer soluble.

I wish to thank Professor L. J. Mordell for his helpful criticism and advice in the preparation of this paper and also the Department of Scientific and Industrial Research for a Maintenance Allowance.

References

1. H. J. S. Smith, On the criterion of resolubility in integral numbers of the indeterminate equation $f=ax^2+a'x'^2+a''x''^2+2bx'x''+2b'xx''+2b''x'x=0$, Proc. Roy. Soc., vol. xiii, 1864, pp. 110, 111, Collected works, vol. 1, p. 410.

2. L. E. Dickson, Modern Elementary Theory of Numbers, (Chicago, 1939), p. 155 *et seq.*

3. L. Holzer, Minimal solutions of Diophantine equations, Canadian Journal of Mathematics, II, 2 (1950), pp. 238–244.

4. L. J. Mordell, On the equation $ax^2+by^2-cz^2=0$, Monatshefte für Mathematik, vol. 55, 1951 pp. 323–327.

FUNCTIONS OF N VARIABLES IN EXTENDED ANALYTIC GEOMETRY

R. S. UNDERWOOD, Texas Technological College

Foreword. The general subject here called "extended analytic geometry" is concerned with the effort to obtain, by means of a system of axes through a common origin, actual "drawable" loci of equations in n variables even when n exceeds three. It seemed evident from the beginning that the results, if sufficiently simple, should be helpful in at least such phases of mathematics as the treatment of simultaneous equations. While the possibilities in that field still seem the most promising, other uses have appeared.

Now, in the interval following acceptance of this paper the writer has found what appears to be (so simple and obvious it is in the light of "hindsight") the "best possible" spacing of axes for the purposes discussed above. While it is not the system for which the figures of this paper have been drawn, it may be said that fortunately the techniques developed in this and the foregoing related papers are essentially independent of the angles between the axes. The latter merely determine the particular equations of transformation for the case considered, and affect the basic methods only in the complication or simplification of the algebra. Since the publication date is drawing near it has seemed best to retain in its original form the part of the article dealing with functions of three variables (Sections 1–6). In any case the solids and surfaces obtained when the axes on the plane are 60° apart are much like those found for orthogonal axes, and the full importance of the change to the latter system does not appear until the number of axes on the plane exceeds three.

1. Introduction. A function U of the variables $x_1, x_2, \cdots, x_n$ has of course a simple geometric interpretation, via plane and solid analytic geometry, when $n=1$ or 2. But when $n>2$ the situation in orthodox mathematics has changed drastically. Geometry has absconded or gone abstract, as one wills, and only algebra remains. To the writer this seems especially unfortunate in connection with the interpretation of maxima and minima of functions of several variables. It is the purpose of this paper to suggest a geometric background for the treatment of such problems, with special reference to functions of three and four variables.

2. The three-dimensional system. In plane analytic geometry as extended by the writer's method,* the axes x_1 to x_n lie on a common plane and pass through a common origin. In the solid variant (Paper I, p. 261) an additional axis, called here the U-axis, passes through the origin and is perpendicular to the n-axes plane, with its positive direction upward.

By definition the locus of the equation

$$U = f(x_1, x_2, \cdots, x_n) \tag{1}$$

is the totality of points having coordinates $(x_1, x_2, \cdots, x_n, U)$ which satisfy (1).

It develops that the locus of (1), when it exists, is normally a solid such as, for example, a filled cone; but that exceptionally it may be a surface or some other degenerate form. The solid figure may fill all space, as in the case of most linear

* Two preceding papers on the subject, published in this MONTHLY and hereafter referred to as Papers I and II, are:

I. An Analytic Geometry for N Variables, vol. 52, May, 1945.

II. Some Applications of Extended Analytic Geometry, vol. 56, March, 1949.

functions; but when maxima or minima exist the locus usually has a definite top or bottom. Considering it as a family of surfaces, we may find the equation of the bounding surface, in a very general case, in terms of ordinary rectangular coordinates X, Y, and Z. (As in previous papers, X and Y are the coordinates of the general point on the n-axes plane with respect to a superimposed system of two axes.) Thus the algebraic criteria for maxima and minima of a function of n variables, which become more and more complicated as n increases, are replaced by those for a function of two variables. Moreover, they often need not be applied at all when the shape of the bounding surface is known.

For simplicity we shall confine much of this paper to a discussion of the loci of

$$U = f(x, y, z). \tag{2}$$

The procedure for the general case will be illustrated first as applied to the special equation

$$U = x^2 + y^2 + z^2. \tag{3}$$

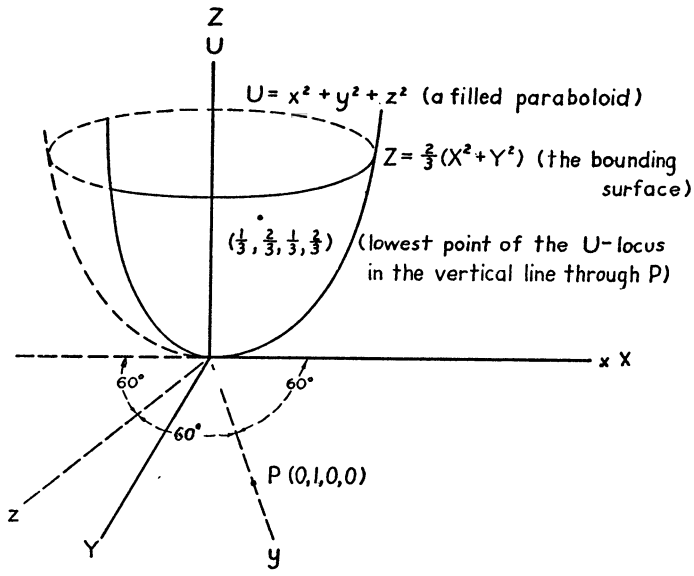


FIG. 1

Consider the point $P(0, 1, 0)$ on the 3-axes plane, which becomes $P(0, 1, 0, 0)$ in the space system (Figure 1). Now the G.C. (generalized coordinates) of P are $(x, 1-x, x)$,§ so that the value of U *over this point* is determined by the equation

§ The G.C. of the point (a, b, c) are $(x, a+b-x, c-a+x)$ by II, (8), p. 160.

$$U = x^2 + (1 - x)^2 + x^2 = 3x^2 - 2x + 1. \tag{4}$$

Evidently the minimum value of U over P is $\frac{2}{3}$, as found by calculus, while there are values of x, and hence coordinates of P, which yield all real values of U exceeding $\frac{2}{3}$.

To get the equation of the bounding surface we may use the general point (X, Y) instead of the specific one selected above. Since, from the fundamental relations [II, (2), p. 159],

$$2X = 2x + y - z, \qquad 2Y = \sqrt{3}(y + z), \tag{5}$$

the G.C. of (X, Y) are

$$(x, Y/\sqrt{3} + X - x, Y/\sqrt{3} - X + x), \tag{6}$$

as obtained by eliminating z and y successively from (5). Substituting in (3) and simplifying, we have

$$U = 3x^2 - 4xX + 2\left(\frac{Y^2}{3} + X^2\right), \tag{7}$$

whence $x = 2X/3$ for a minimal U. When the latter is denoted by Z, we get from (7)

$$Z = \tfrac{2}{3}(X^2 + Y^2). \tag{8}$$

Thus the locus of (3) is the "filled paraboloid" (8).

The step to the general method for a function of three variables is obvious, though of course the way has not been prepared for a rigorous theorem covering all contingencies. In the specific case above the paraboloid (8) is clearly the bottom of the locus, since from (7) $(\partial^2 U/\partial^2 x) > 0$. Of course on occasion the surface corresponding to (8) could be the top of the locus or merely a special surface within it. Usually it is easy to make the decision; but for elusive functions the "posthole method" illustrated above for the point (0, 1, 0) may aid in the pictorial diagnosis. With this understanding we may state the general method.

3. Working rule: If the locus of the equation

$$U = f(x, y, z) \tag{2}$$

has a bounding upper or lower surface, the equation of this surface in terms of rectangular coordinates (X, Y, Z) may normally be found by the procedure hereafter described.

(a) Replace x, y, and z in (2) by t, $K+X-t$, and $K-X+t$ respectively.

(b) Set $\partial U/\partial t = 0$ and solve, thus obtaining t, and hence x, y, and z, in terms of X and K.

(c) Substitute in (2), replacing U by Z and, after simplifying, K by $Y/\sqrt{3}$.

Interesting illustrations are afforded by functions (9) and (11) below.

The locus of

$$U = x^2 + y^2 + z^2 - xy - xz - yz \tag{9}$$

is the filled parabolic cylinder

$$4Z = (Y - X\sqrt{3})^2, \tag{10}$$

which touches the XY plane from above on the line $Y - X\sqrt{3} = 0$ (Figure 2). From (5), the equation of this line in terms of the original variables is $z = x$.

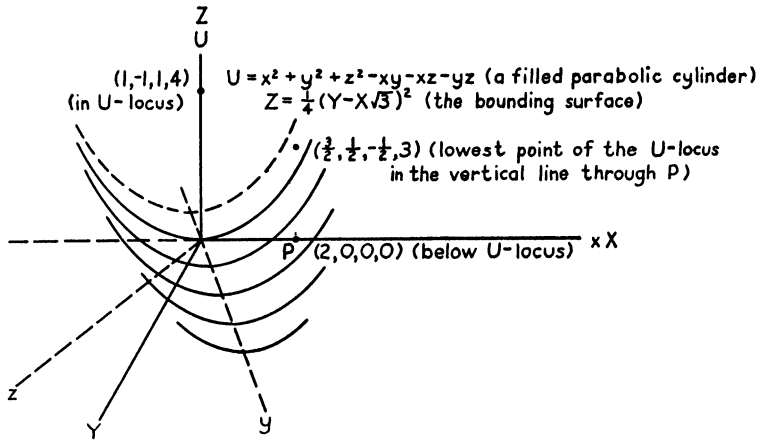


FIG. 2

Thus $U = 0$, its minimum, along this line when $x = z = a$ and $U = (y - a)^2$ with $y = a$; that is, when $x = y = z =$ an arbitrary constant a.

For the equation

$$U = x^2(y + z) \tag{11}$$

the locus is a "double wedge" of which one half fills the region above the XY plane and in front of the XZ plane. Thus half of the plane $Z = 0$ is a minimal surface and half is maximal. In this case the necessary algebraic conditions for extremal values of U yield $x = 0$ and $y + z = 0$; but these equations tell little without the geometric interpretation, since clearly U may have any real value.

4. Loci on the 3-axes plane. An incidental result from the foregoing working rule is a more general method than that previously given for obtaining the locus of

$$f(x, y, z) = 0 \tag{12}$$

on the 3-axes plane. In I, page 257, the equation of the bounding curve about the typical locus area is obtained, when f is a second degree function, by making the substitutions (6) in (12) and then equating to zero the discriminant of the

quadratic in x. But now whenever, with reference to the solid $U=f(x, y, z)$, the equation $Z=F(X, Y)$ of the bounding surface is obtainable by the general rule, we merely set $Z=0$ to find the equation of the curve bounding the locus of (12).

5. Special loci in space. It was stated above that exceptionally the locus of a function of n variables, when $n>2$, may be a surface instead of a solid. Indeed, any equation $Z=F(X, Y)$ of a surface may be changed to the form $Z=f(x_1, \cdots, x_n)$ by substitutions (7), p. 256, in I. We shall now find criteria for the abnormal loci.

DEFINITION: *The locus of*

$$U = f(x, y, z) \tag{2}$$

is degenerate if, for every real and non-isolated point (a, b, c, U_1) *that it contains, the curves* $U_1=f(x, y, c)$, $U_1=f(x, b, z)$, *and* $U_1=f(a, y, z)$ *are identical.*

Note that the locus of (2) will be degenerate if it is non-existent (imaginary) or a surface, curve, or point. Conceivably, though improbably, the conditions of the definition might hold on concentric shells of some highly unusual solid loci.

THEOREM 1. *When*

$$U_y \equiv U_x + U_z \tag{13}$$

the locus of $U=f(x, y, z)$ *is degenerate.*

Proof: Assume that $U_1=f(a, b, c)$, where (a, b, c, U_1) is not an isolated point on the locus. Consider the curve $U_1=f(x, y, c)$ in the plane $Z=U_1$. On this curve it follows from (5) that $dY/dX=(\sqrt{3}dy)/(2dx+dy)$, where dY/dX is the conventional slope of the curve in the plane. The result may be written in the form

$$\left.\frac{dY}{dX}\right]_{z=c} = \left.\frac{\sqrt{3}\,\dfrac{\partial y}{\partial x}}{2+\dfrac{\partial v}{\partial x}}\right]_{z=c}.$$

Similarly,

$$\left.\frac{dY}{dX}\right]_{y=b} = \left.\frac{\sqrt{3}\,\dfrac{\partial z}{\partial x}}{2-\dfrac{\partial z}{\partial x}}\right]_{y=b}.$$

But since $\partial y/\partial x=-(U_x/U_y)$ and $\partial z/\partial x=-(U_x/U_z)$, and since $U_y\equiv U_x+U_z$ by hypothesis, it follows that on the given plane

$$\frac{dY}{dX}\Big]_{y=b} = \frac{dY}{dX}\Big]_{z=c}$$

wherever these quantities are defined. That is, the curves $U_1=f(x, b, z)$ and $U_1=f(x, y, c)$ are identical and coincide (by a similar argument) with the curve $U_1=f(a, y, z)$.

For an illustration, consider the equation

$$U = x^2 + y^2 + z^2 + xy - xz + yz - a^2. \tag{14}$$

Here (13) holds, so that the locus is degenerate. The first step of the working rule yields $U=X^2+Y^2-a^2$ (t drops out), and hence the locus of (14) is the paraboloid

$$Z = X^2 + Y^2 - a^2. \tag{15}$$

In this case the locus of U coincides with that of Z and reaches its minimum $(-a^2)$ when $X=Y=0$. It does not follow, of course, that necessarily $x=y=z=0$ for a minimum U, since the G.C. of the origin are $(x, -x, x)$ and hence U reaches its minimum when $x=-y=z=t$, with t arbitrary.

The converse of Theorem 1 is false, but a partial converse is true, as follows:

THEOREM 2. *If the locus of $U=f(x, y, z)$ is a surface, then $U_y \equiv U_x+U_z$.*

For then

$$U = f(x, y, z) = F(X, Y) = F\left(\frac{2x + y - z}{2}, \frac{\sqrt{3}\,y + \sqrt{3}\,z}{2}\right),$$

whence $U_x=F_X$, $U_y=F_X(\frac{1}{2})+F_Y(\sqrt{3}/2)$, and $U_z=F_X(-\frac{1}{2})+F_Y(\sqrt{3}/2)$. The theorem follows.

6. Special loci on the 3-axes plane. When $U\equiv 0$, the results of Article 5 apply with slight modification for loci on a plane. Theorems 1 and 2 become:

THEOREM 3. *When $f_y\equiv f_x+f_z$, the locus of $f(x, y, z)=0$ is degenerate.*

Note that in this case the definition of the degenerate locus still applies, with $U\equiv U_1\equiv 0$.

THEOREM 4. *If the locus of $f(x, y, z)=0$ is a curve, then $f_y\equiv f_x+f_z$.*

7. Orthogonal axes. When once we realize that the methods thus far developed allow axes to coincide without loss of their usefulness *as separate axes*, the way is opened for a drastic improvement in our system. On the n-axes plane we merely make the positive directions of separate axes 90° apart. Then, if more than four axes are involved, the fifth axis coincides with the first, the sixth with the second, *etc.* In this way the simple lattice points of plane analytic geometry are retained for any number of variables.

The equations of transformation from the two-axes to the n-axes systems [(7), p. 256 in I] now yield the following for the cases of three, four, and five

axes respectively (using x, y, z, u, and v for the axes in that order).

$$X = x - z; \qquad Y = y. \tag{16}$$

$$X = x - z; \qquad Y = y - u. \tag{17}$$

$$X = x - z + v; \qquad Y = y - u. \tag{18}$$

The generalization to the axes x_1, y_1, x_2, y_2, $\cdots$ is obviously

$$X = x_1 - x_2 + x_3 - \cdots; \qquad Y = y_1 - y_2 + y_3 - \cdots. \tag{19}$$

Evidently the G.C. of the point (X, Y) are, in cases (16), (17) and (18), $(x, Y, x-X)$, $(x, y, x-X, y-Y)$, and $(x, y, z, y-Y, X-x+z)$, where the values of the small letters are arbitrary. Thus there are $n-2$ parameters, or degrees of freedom, in the coordinates of the point (X, Y) on the n-axes plane. A function U of n variables becomes, over the point, a function of $n-2$ variables and the constants X and Y. If we equate to zero its partial derivative with respect to each of the variables, solve the $n-2$ equations simultaneously, substitute the results in the expression for U, and replace the latter by Z, we have the equation $Z=F(X, Y)$ of the bounding surface of the solid locus of U, if such exists.

8. Special theorems and functions. With the understanding that henceforth all axes are to be placed orthogonally, we may now combine and restate Theorems 1 and 2 for the case of four variables.

THEOREM 5. *When*

$$U_x + U_z \equiv U_y + U_u \equiv 0 \tag{20}$$

the locus of $U=f(x, y, z, u)$ *is degenerate; while if the locus is a surface, equations* (20) *hold.*

The method is to note that if $U=f(x, y, z, u)=F(X, Y)=F(x-z, y-u)$, then $U_x=U_X(1)$, $U_z=U_X(-1)$, *etc.* For three variables the test is simply: $U_x+U_z\equiv 0$; for five variables, $U_x+U_v-2U_z\equiv U_y+U_u\equiv 0$. The extension to n variables is simple.

To illustrate, the locus of

$$U = x^2 - y^2 + z^2 - u^2 - 2xz + 2yu \tag{21}$$

is degenerate, being the hyperbolic paraboloid $Z=X^2-Y^2$.

The locus of

$$U = \frac{x}{a} + \frac{y}{b} + \frac{z^2}{c^2} + \frac{u^2}{d^2} \tag{22}$$

is all space above and including the plane

$$Z = \frac{X}{a} + \frac{Y}{b} - \frac{c^2}{4a^2} - \frac{d^2}{4b^2}.$$

9. Generalizations of the paraboloid and ellipse. The following rather inclusive theorem will perhaps suggest why the writer believes, in his optimistic moments, that this particular extension of plane and solid analytic geometry is more in the spirit of those subjects, as well as more useful for some purposes, than the algebraic methods which stem from the abstract concept of "n-space," and which touch geometry mainly by definition and analogy.

THEOREM 6. *The locus of the terminating equation*

$$U + 1 = \frac{x_1^2}{a_1^2} + \frac{y_1^2}{b_1^2} + \frac{x_2^2}{a_2^2} + \cdots \tag{23}$$

when the right side consists of three or more terms, is the filled elliptic paraboloid whose lower surface is

$$Z + 1 = \frac{X^2}{a_1^2 + a_2^2 + a_3^2 + \cdots} + \frac{Y^2}{b_1^2 + b_2^2 + b_3^2 + \cdots}. \tag{24}$$

When $U=Z=0$, the plane locus of (23) *becomes the filled ellipse whose boundary is* (24).

The proof is sketched below. We note that, when n is even,

$$X = x_1 - x_2 + x_3 - \cdots + x_{n-1} - x_n. \tag{25}$$

When the value of x_n from (25) is substituted in (23) we find, upon differentiating, that

$$\frac{1}{2} U_{x_i} = \frac{x_i}{a_i^2} + (-1)^{i+1}(x_1 - x_2 + \cdots + x_{n-1} - X). \tag{26}$$

It may be verified that (26) is still true when n is odd. But each of the $n-1$ expressions for the right side of (26) becomes zero when

$$x_i = \frac{(-1)^{i+1}X}{A},$$

where $A = \sum_{i=1}^{n} a_i^2$. And when this value for x_i, together with a similar one for y_i, is substituted in (23) we get (24).

10. Further applications. In the case of two simultaneous equations in two unknowns the algebra of the solution usually takes the dominant role, and the intersections of the curves are merely incidental checks. But when more than two variables are involved, this situation is often reversed. For example, with the geometric background and aids the algebraic processes in the solution of four linear equations in four unknowns are reduced in the general case to the solution of four pairs of equations in two unknowns. Many standard form loci can be derived once for all, and when the loci do not touch there are no real solutions. But elaboration of these ideas cannot be included in this paper.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

ON THE SIGN OF A FUNCTION IN A CLOSED INTERVAL

P. G. HODGE, JR., University of California, Los Angeles

A problem which arises frequently in applied mathematics is to show that a given function of one variable is always positive in a given interval. If the function is entirely known, this may be done, of course, by showing that it is positive at the endpoints and has no roots in the interval. However, this is not always feasible in the case where the function contains a parameter. In the present note a method is presented which may ease the computations considerably in certain cases. The related problems of showing that the function is always non-negative, always negative, or always non-positive, may also be solved by obvious adaptations of the method.

Consider a twice continuously differentiable function $F(x)$ defined in the interval $a \leqq x \leqq b$, where $F(a) > 0$, $F(b) > 0$. If $F(x)$ is non-positive at some interior point of $[a, b]$, it must have a relative minimum which is equal to or less than zero. Let $F(x_0)$ be such a minimum. Then the following relations are all valid at x_0:

$$F(x_0) \leqq 0, \qquad F'(x_0) = 0, \qquad F''(x_0) \geqq 0. \tag{1}$$

Now let $q(x)$ be non-negative in $[a, b]$, and let $p(x)$ be arbitrary. Define the *test function*:

$$T(x) = F(x) + p(x)F'(x) - q(x)F''(x). \tag{2}$$

The substitution of (1) into (2) shows that $T(x_0) \leqq 0$. Therefore, if it can be shown that the test function is positive for all $a < x < b$ it will follow that the hypothesis is false and therefore $F(x) > 0$ for all $a \leqq x \leqq b$. The test is a weak one, in that the converse is not necessarily true. If $T(x) \leqq 0$ for some x, no conclusion can be drawn as to the sign of $F(x)$

As a first example, consider the function

$$F(x) = (x - \alpha)e^x + e^{-\beta x} \tag{3}$$

in the interval $0 \leqq x \leqq 1$. Then

$$F(0) = 1 - \alpha, \qquad F(1) = (1 - \alpha)e + e^{-\beta},$$

which are both positive provided $\alpha < 1$. The test function for this case is

$$\begin{aligned} T(x) = {} & x(1 + p - q)e^x + [(1 - \alpha)p - (2 - \alpha)q - \alpha]e^x \\ & + (1 - \beta p - \beta^2 q)e^{-\beta x}. \end{aligned} \tag{4}$$

Constants p and q can be determined so that the coefficients of the last two terms

vanish:

(5) $p = (\alpha\beta^2 + 2 - \alpha)/\Delta, q = (1 - \alpha - \alpha\beta)/\Delta,$ where $\Delta = (1 - \alpha)\beta^2 + (2 - \alpha)\beta.$

The corresponding test function is

$$T(x) = x(1 + \beta)^2 e^x/\Delta. \tag{6}$$

For the method to yield results, $T(x)$ must be positive and $q(x)$ non-negative. It follows from (5) and (6) that $\Delta > 0$ and $1 - \alpha - \alpha\beta \geqq 0$. Sufficient conditions for the function (3) to be always positive in $[0, 1]$ are thus seen to be

$$\alpha < 1, \qquad \beta > 0, \qquad \alpha(1 + \beta) \leqq 1. \tag{7}$$

The method is particularly valuable when $F(x)$ is a polynomial. In this case, take

$$p(x) = Ax + B, \qquad q(x) = (Cx + D)^2. \tag{8}$$

If $F(x)$ is a polynomial of degree n, the test function will also be a polynomial of degree n. In general, the four constants A, B, C, D may then be determined so that the coefficients of the four highest terms vanish, so that $T(x)$ becomes of degree $n-4$. In practice, this will be somewhat laborious, and it may be more economical to set some of the constants equal to zero and reduce the polynomial by a smaller degree.

As an example, consider the fourth degree polynomial

$$P_4(x) = x^4 - 3\lambda x^3 + 4\lambda x^2 - 2\lambda^2 x + 1 \tag{9}$$

in the interval $0 \leqq x \leqq \infty$, where $0 \leqq \lambda \leqq 1$. Take $q(x) = 0$, $p(x) = Ax + B$, so that

$$\begin{aligned} T(x) = {} & (1 + 4A)x^4 + (-3\lambda - 9\lambda A + 4B)x^3 + (4\lambda + 8\lambda A - 9\lambda B)x^2 \\ & + (-2\lambda^2 - 2\lambda^2 A + 8\lambda B)x + (1 - 2B\lambda^2). \end{aligned} \tag{10}$$

To reduce $T(x)$ to a quadratic, set $A = -1/4$, $B = 3\lambda/16$. Then

$$T(x) = (1/16)[(32\lambda - 27\lambda^2)x^2 + (16 - 6\lambda^3)]. \tag{11}$$

Since this is greater than zero for all $0 \leqq \lambda \leqq 1$ and for all x, the given polynomial is positive for all positive x.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

AN EXTENSION OF PLANE ANALYTIC GEOMETRY

J. A. WARD, University of Kentucky

An interesting sidelight may be shown to plane analytic geometry by letting one of the variables, say y, assume imaginary values. To this end replace the Y-axis by the complex UV-plane perpendicular to the X-axis to give three mutually perpendicular axes with the XU-plane identified as the conventional XY-plane. Then $x=a$, $y=b+ci$ may be written $x=a$, $u=b$, $v=c$ and is represented as the point (a, b, c) in space, and conversely. The "semi-complex" graph of a real function, $f(x, y)=0$, will consist of the locus of all points (a, b, c) such that $x=a$, $y=b+ci$ satisfy the equation.*

In $x^2-y^2=a^2$, $y=\pm\sqrt{x^2-a^2}$ is real for $x^2\geqq a^2$. Therefore $u=\sqrt{x^2-a^2}$, $v=0$ is the usual equilateral hyperbola in the XU-plane. This is the real portion of the curve. For $x^2<a^2$, $y=\pm i\sqrt{a^2-x^2}$ is pure imaginary and $v=\pm\sqrt{a^2-x^2}$, $u=0$, is a circle in the XV-plane, which is the imaginary portion. (See figure 1.) Thus the "semi-complex" graph of $x^2-y^2=a^2$ is a curve in two perpendicular planes. Notice that the circle in the XV-plane connects the (real) vertices of the hyperbola so that the curve is now all in one piece.

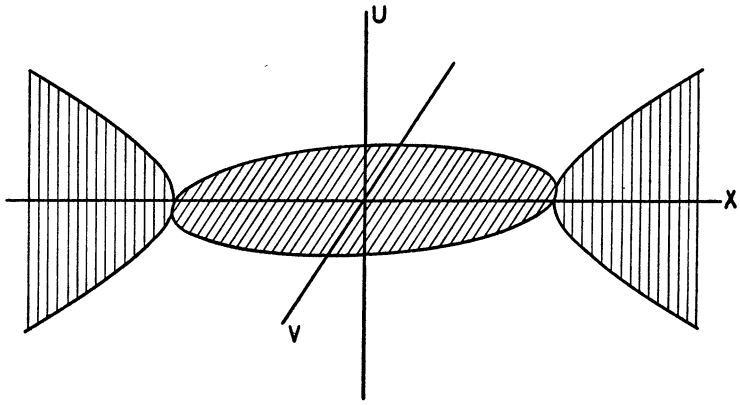


FIG. 1

Moreover if $y^2=f(x)$ is any real function of x, $u=\pm\sqrt{f(x)}$, $v=0$ will give the real portion and $v=\pm\sqrt{-f(x)}$, $u=0$ will give the imaginary portion of the "semi-complex" graph. The non-excluded portions of the real and imaginary parts of the planes complement each other to give a single connected whole, so that the real and imaginary portions of the graph intersect at right angles on the

* See also, A. J. Kempner, Bulletin A. M. S., vol. 41, 1935, pp. 809–843.

X-axis at the boundaries of the excluded portions of the real (or imaginary) planes.

Certain "mysteries" of other curves become clear when studied from this standpoint. The real curve of $y^2=(x-1)^2(x-2)$ has an isolated point at (1, 0). The reason for this is obvious in the "semi-complex" graph (figure 2), for we see the isolated real point is located where the loop in the imaginary part closes and is no longer isolated.

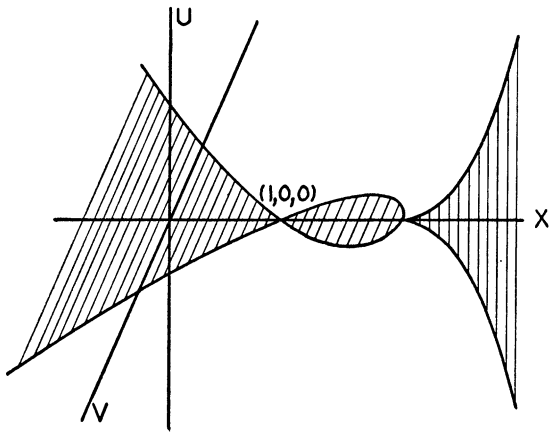


FIG. 2

"Semi-complex" graphs are not always as simple as those given above. If in the real equation $f(x, y)=0$ we let $y=u+iv$ (u and v real) we get $g(x, u, v)+iv\cdot h(x, u, v)=0$. Hence the "semi-complex" graph is the intersection of the real surfaces $g(x, u, v)=0$ and $v\cdot h(x, u, v)=0$. Since the latter function factors, the graph in space consists of two parts: $g(x, u, 0)=0$ (which is the usual graph $f(x, y)=0$ with y real) and intersection of the surfaces $g(x, u, v)=0$ and $h(x, u, v)=0$ determined by the imaginary values of y.

For $y^2+(ax+b)y+f(x)=0$, $h(x, u, v)=0$ becomes the plane $2u+ax+b=0$. Thus the semi-complex graph of any integral quadratic function of y lies in two perpendicular planes.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1026. *Proposed by Michael Golomb, Purdue University*

Orders have been given in a certain country that from now on all scientific writing shall be done in a secret transcription, consisting of a permutation of the alphabet of the country. A foreigner, unacquainted with the language and its alphabet is given the privilege of seeing the transcribed form of any five letters he may designate. He wishes to learn the common form of the word *atom*, which he is told has five letters. How can he manage this, if reference to the inverse transcription is ruled out?

E 1027. *Proposed by Ruth Clark and Robert Oeder, Los Alamos Scientific Laboratory*

Find the total area enclosed by the set of circles formed as follows: Construct the inscribed circle of a triangle ABC and draw the tangent to this circle which is parallel to a selected side of the triangle; construct the inscribed circle of the new triangle so cut off and draw the tangent to this circle which is parallel to the selected side of triangle ABC; repeat the process indefinitely. Also show that the set of maximum area is obtained if the lines are drawn parallel to the shortest side of triangle ABC.

E 1028. *Proposed by Vern Hoggatt, Oregon State College*

Do there exist Pythagorean triangles whose sides are Fibonacci numbers?

E 1029. *Proposed by A. E. Currier, United States Naval Academy*

Sum the series $1+1/3+2!/(3)(5)+3!/(3)(5)(7)+\cdots$.

E 1030. *Proposed by Charles Salkind, Polytechnic Institute of Brooklyn*

Consider the polygon formed by the internal trisectors of the angles of a given n-gon, intersecting in neighboring pairs.

(1) Prove that a necessary and sufficient condition that the trisector polygon be regular is that the parent n-gon be regular when $n \geqq 4$.

(2) Prove that the area ratio between the parent and trisector polygons is always irrational.

SOLUTIONS

An Inequality for the Triangle

E 996 [1952, 41]. *Proposed by Josef Andersson, Vaxholm, Sweden*

If a, b, c and r, r_a, r_b, r_c denote the measures of the sides and of the radii of the inscribed and escribed circles of a triangle, show that

$$abc/r \geqq a^3/r_a + b^3/r_b + c^3/r_c.$$

Solution by C. V. Fronabarger, Southwest Missouri State College. If the desired relation is true, then, clearing of fractions and making use of the relations $rr_ar_br_c=r^2s^2$, $r_ar_b=s(s-c)$, $r_br_c=s(s-a)$, $r_cr_a=s(s-b)$, where s is the semi-perimeter, we get, after division by rs,

$$abcs \geqq a^3(s-a) + b^3(s-b) + c^3(s-c),$$

which may be written as

$$\phi \equiv a^2(a-b)(a-c) + b^2(b-c)(b-a) + c^2(c-a)(c-b) \geqq 0.$$

But, obviously, $\phi=0$ when $a=b=c$ and $\phi>0$ when a, b, c are not all equal. Hence, since the steps may be reversed, we may conclude that the desired relation is true.

Also solved by Ruth Clark, Tadashi Kishi and Robert Oeder (jointly), M. S. Klamkin, L. A. Ringenberg, Azriel Rosenfeld, Roscoe Woods, E. J. Zirkel, and the proposer.

Some Inequalities Related to Fermat's Last Theorem

E 997 [1952, 41]. *Proposed by David Ellis, University of Florida*

Let x, y, z, n be positive integers such that $n>1$ and $x^n+y^n=z^n$. Show that

$$z^{n-1} < \min(x^n, y^n) < \max(x^n, y^n) < z^n, \tag{1}$$

$$1/z < \min((x/y)^n, (y/x)^n) < \max((x/y)^n, (y/x)^n) < z. \tag{2}$$

Solution by R. Kissling, University of California. Since $2^{1/n}$, $n>1$, is irrational, we can suppose that $z>x>y\geqq 1$. Then

$$\begin{aligned} y^n &= z^n - x^n \geqq z^n - (z-1)^n \\ &= z^{n-1} + z^{n-2}(z-1) + \cdots + (z-1)^{n-1} > z^{n-1}. \end{aligned}$$

Thus $\min(x^n, y^n) > z^{n-1}$. That $\max(x^n, y^n) < z^n$ is obvious. This disposes of (1).

Again,

$$y^n = z^n - x^n \geqq (x+1)^n - x^n > nx^{n-1},$$

whence $(y/x)^n > n/x$, and consequently $(x/y)^n < x/n$. Thus we have

$$n/x < \min((x/y)^n, (y/x)^n) < \max((x/y)^n, (y/x)^n) < x/n,$$

which implies (2) but is considerably sharper.

Also solved by Herbert Emich, Vern Hoggatt, M. S. Klamkin, Kovina Milosevich, Azriel Rosenfeld, W. L. Strother, and the proposer.

Maximum Area in a Corner

E 998 [1952, 41]. *Proposed by Leo Moser, University of Alberta*

Show that the largest area A that can be enclosed by two fixed lines, inclined at an angle $\theta \leqq \pi$, and n straight line segments of unit length is given by

$$A = (n/4) \cot (\theta/2n).$$

Solution by Vern Hoggatt, Oregon State College. We attain the desired result through the following sequence of theorems.

THEOREM 1. *Among all the curves of given length joining two given points A and B, the one which, together with the chord AB, bounds the maximum area is a circular arc.*

See, *e.g.*, Bolza, *Lectures in the Calculus of Variations*, ex. 13, p. 210.

THEOREM 2. *The triangle AOB having the maximum area subject to the conditions* $AB = m$ *and* $\sphericalangle AOB = \alpha$ *is the one in which* $OA = OB$.

For of all triangles having a common base and equal opposite angles, the isosceles triangle has the maximum altitude.

THEOREM 3. *Among all the curves of given length s whose endpoints A and B lie on two half-lines* OL_1 *and* OL_2, *forming an angle* $\theta \leqq \pi$, *the one which, together with the half-lines, bounds the maximum area* (assuming such a maximizing curve exists) *is a circular arc having its center at O.*

Suppose there exists a maximizing curve C of length s. Then C must be a circular arc, since otherwise the area between C and the chord AB could be increased by theorem 1, and the triangle AOB must be isosceles, since otherwise the area of triangle AOB could be increased by theorem 2. Thus C is a circular arc with its center on the bisector of angle AOB. We now determine the radius r so that the area will be a maximum. If ϕ is the central angle of the arc, the area of the figure is found to be

$$r^2(\phi - \sin \phi)/2 + (r \sin \phi/2)^2 \cot \theta/2.$$

Setting $r = s/\phi$, differentiating with respect to ϕ, setting the derivative equal to zero, and solving for $\cot \theta/2$, we find

$$\cot \theta/2 = \frac{\phi(1 + \cos \phi) - 2 \sin \phi}{\phi \sin \phi - 2(1 - \cos \phi)} = \frac{\sin \phi}{1 - \cos \phi} = \cot \phi/2.$$

Thus, for maximum area, $\phi = \theta$, or $r = OA$.

THEOREM 4. *The maximum area A that can be enclosed by two fixed half-lines, inclined at an angle* $\theta \leqq \pi$, *and n straight line segments of unit length is that given*

by the situation where the n unit line segments occur as bases of n equal isosceles triangles having a common vertex at the intersection of the two fixed half-lines.

Draw the circular arc C, having the intersection of the half-lines as center, and passing through the extremities of the n unit segments when arranged as described above. Then, if u denotes the area of a circular segment bounded by C and one of the unit segments, and if r denotes the radius of C, we have

$$A + nu = r^2\theta/2.$$

Suppose, now, that the area A' of some other polygon bounded by the two half-lines and n unit segments exceeds the area A. On the outward side of each of the n unit segments place one of the circular segments of area u considered above. We now have a figure bounded by the two half-lines and n circular arcs and having an area

$$A' + nu > r^2\theta/2.$$

Since we now have a contradiction of theorem 3, our present theorem is established.

THEOREM 5. *The maximum area A that can be enclosed by two fixed half-lines, inclined at an angle $\theta \leqq \pi$, and n straight line segments of unit length is given by*

$$A = (n/4) \cot (\theta/2n).$$

By theorem 4 the maximum area consists of n equal isosceles triangles with unit bases and with vertex angles equal to θ/n. The altitude of each triangle is $\frac{1}{2} \cot (\theta/2n)$, and the area of each triangle is $\frac{1}{4} \cot (\theta/2n)$, whence the total maximum area is $(n/4) \cot (\theta/2n)$.

Also solved by Julian Braun, E. I. Gale, R. D. Oeder and Tadashi Kishi (jointly), and C. S. Ogilvy. Most of these solutions assumed theorem 4 above.

As the proposer pointed out, if $\theta = \pi/m$, where m is a positive integer, the result follows easily from a well known extremal property of the regular polygon of $2mn$ sides. The particular case $\theta = \pi$, $n = 3$, is the familiar 'gutter' problem found in many calculus texts.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4498. *Proposed by D. J. Newman, Harvard University, and H. S. Shapiro, Massachusetts Institute of Technology*

Prove that every complex number is a zero of some entire function whose power series has rational coefficients.

4499. *Proposed by P. G. Federbush, Student, Rutgers University*

The coefficients of the congruence

$$a_0x^n + a_1x^{n-1} + \cdots + a_n \equiv 0 \pmod{p}$$

are subject to the restriction $0 \leqq a_j \leqq p-1$, $a_0 \not\equiv 0$, but otherwise are taken at random. p is a prime. What is the probability that the congruence has no solution?

4500. *Proposed by J. R. Musselman, Western Reserve University*

Given three points A_i $(i=1, 2, 3)$ and a line L cut by the lines A_jA_k making angles α_i in the positive sense. Show that the lines drawn through A_i making angles $\pi-\alpha_i$, in the positive sense, with L are concurrent at a point P on the circumcircle of $A_1A_2A_3$. Further, if the altitudes from A_i make angles β_i with L, then the lines through A_i making angles $\pi-\beta_i$ with L are concurrent on the circumcircle of $A_1A_2A_3$ at a point Q diametrically opposite to P.

4501. *Proposed by C. D. Olds, San Jose State College, California*

Show that

$$\log (1.1010010001 \cdots) = \frac{1}{11} + \frac{1}{2}\,\frac{1}{101} + \frac{1}{3}\,\frac{1}{1001} + \cdots .$$

4502. *Proposed by Karl Goldberg, Brooklyn, New York*

Prove that two groups A and B are isomorphic if and only if there exist operations "$\cdot$" and "$*$" defined over the elements of the groups as follows:

$$a \cdot b \in A \qquad a \cdot (b*a') = (a \cdot b)a'$$

$$b*a \in B \qquad b*(a \cdot b') = (b*a)b'$$

where $a, a' \in A$; $b, b' \in B$.

SOLUTIONS

4421 [1951, 42]. *Correction*

In the May issue, p. 338, in the first equation in solution I, replace the summation sign by product sign.

An Application of Wolstenholme's Theorem

4428 [1951, 194]. *Proposed by C. D. Olds, San Jose State College, California*

If p is any prime number >3, prove that

$$_{kp-1}C_{p-1} \equiv {}_{jp-1}C_{p-1} \pmod{p^3},$$

where j, k are positive integers and ${}_nC_r$ is the number of combinations of n different things taken r together.

Solution by Ernest Trost, Technikum Winterthur, Zürich, Switzerland. From Fermat's theorem follows

$$\begin{aligned} x^{p-1} - 1 &\equiv (x-1)(x-2)\cdots(x-p+1) \\ &= x^{p-1} - s_1x^{p-2} + \cdots - s_{p-2}x + s_{p-1} \pmod{p}. \end{aligned}$$

Hence we have $s_i \equiv 0 \pmod{p}$, $(i=1, 2, \cdots, p-2)$, $s_{p-1}=(p-1)!$ and for $x=p>3$

$$(p-1)! \equiv s_{p-3}p^2 - s_{p-2}p + s_{p-1} \pmod{p^3},$$

whence

$$s_{p-2} \equiv ps_{p-3} \equiv 0 \pmod{p^2}.$$

Now, if $p>3$, we obtain, upon putting $x=kp$,

$$\prod_{i=1}^{p-1}(kp-i) \equiv s_{p-3}k^2p^2 - s_{p-2}kp + s_{p-1} \equiv (p-1)! \pmod{p^3}.$$

Therefore, upon dividing by $(p-1)!$, we have

$$_{kp-1}C_{p-1} \equiv 1 \pmod{p^3}, \quad k = 1, 2, \cdots,$$

which implies the desired result.

Also solved by A. N. Aheart, D. H. Browne, Leonard Carlitz, Margaret W. Maxfield, Leo Moser, J. H. Wahab, and R. E. Wild.

Editorial Note. Several solvers refer to Wolstenholme's and other relevant theorems. See Hardy and Wright, *An Introduction to the Theory of Numbers*, pp. 85–88. Carlitz finds a proof by J. W. L. Glaisher in the *Quarterly Journal of Mathematics*, vol. 31 (1900), pp. 20–21, 325, where also additional results are obtained, including the following: If $p \geqq 5$, and if B_{p-3} denotes a Bernoulli number in the even suffix notation, then

$$_{kp-1}C_{p-1} \equiv 1 - \tfrac{1}{3}(k^2 - k)p^3B_{p-3} \pmod{p^4}.$$

$$\sum{}' \frac{1}{N^2R} = 2.$$

Also solved by T. M. Apostol, D. H. Browne, O. E. Stanaitis, J. Vales and A. Petracca, and the Proposer.

Editorial Note. Vales and Petracca obtain the following generalizations. Put

$$S(r, n) = \sum_{i=1}^{n} \frac{1}{i^r}, \qquad K(r, n) = \frac{1}{n + r - 1} \sum_{i=1}^{n} K(r - 1, i),$$

with $K(1, n) = 1/n$. Then

$$\sum_{n=1}^{\infty} \frac{1}{n^{h+2}} = \sum_{n=1}^{\infty} \sum_{i=1}^{h} \frac{S(i, n)}{(n + 1)^{h+2-i}}, \qquad \sum_{n=1}^{\infty} \frac{1}{n^{h+1}} = \sum_{n=1}^{\infty} \frac{K(h, n)}{h + n - 1}.$$

Browne finds

$$\sum_{n=1}^{\infty} \frac{S_n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^4} + \sum_{n=1}^{\infty} \frac{S_n}{(n + 1)^3} = 5 \sum_{n=1}^{\infty} \frac{S_n}{(n + 1)^3},$$

but further extensions do not assume so neat a form.

A Circle Tangent to the Nine-Point Circle

4432 [1951, 195]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

With the midpoints of the sides of a triangle as centers, circles are described passing through the feet of the corresponding altitudes. Show that the circle orthogonal to these three circles is tangent to the nine-point circle of the given triangle at a point such that its distance from the foot of one altitude is equal to the sum of the distances from the feet of the other two altitudes.

Solution by O. J. Ramler, Catholic University, Washington, D. C. Let A, B, C be the midpoints of the sides of the triangle $A'B'C'$, and let the feet of the altitudes of $A'B'C'$ be D, E, F. Draw the circles (A), (B), (C) with centers A, B, C and radii AD, BE, CF respectively. Draw the common chords of the circumcircle (O) of ABC and each of the circles (A), (B), (C). Let X, Y, Z be the vertices of the triangle formed by these common chords. Then the incircle of XYZ is the circle mutually orthogonal to circles (A), (B), (C).

Let the common chord of circles (A) and (O) meet (O) again in D'. Then angle $AD'D$ = angle $ADD' = B - C$, since AD is parallel to BC. Hence YZ is antiparallel to BC with respect to AB and AC. Similarly XZ, the common chord of (B) and (O), is antiparallel to AC with respect to AB and CB. Hence ZX and ZY form an isosceles triangle with AB. The radical axis of circles (A) and (B) therefore bisects angle Z of triangle XYZ. From this it follows readily that the radical center of circles (A), (B), (C) is the incenter of triangle XYZ.

Let the incircle (I) of triangle XYZ touch the sides YZ, ZX, XY at L, M, N respectively. With respect to circle (A) the side YZ inverts into circle (O). Since

the incircle (I) is tangent to YZ at L, (I) must touch (O) where AL meets (O); call this point T. Point T is unique, that is, T is the point where M inverts with respect to circle (B), and N inverts with respect to (C). Moreover, since L and T are inverse with respect to (A), (I) and (A) are orthogonal. Similarly (I) is orthogonal to (B) and (C). Circles (I) and (O) are therefore tangent at T, and T is the center of similitude of (I) and (O). Hence triangles ABC and LMN are homothetic with respect to T so that

$$TL/TA = TM/TB = TN/TC = r/R$$

where r, R are the radii of (I) and (O). From $TL/TA = r/R$ and $AL \cdot AT = AD^2$ we obtain $TL \cdot AL = (r/R)AD^2$. Similarly we have $TN \cdot CN = (r/R)CF^2$ and thence $(TL \cdot AL)/(TN \cdot CN) = AD^2/CF^2$. From $TL/TN = AL/CN$ and $TL/TA = TN/TC$ it follows that

$$AT \cdot CF = TC \cdot AD. \tag{1}$$

Now $ACET$, $ACFT$ and $ACDT$ are cyclic quadrangles. Ptolemy's theorem provides the equations

$$ET \cdot AC = AT \cdot CE + AE \cdot TC, \tag{2}$$

$$FT \cdot AC = AT \cdot CF + AF \cdot TC, \tag{3}$$

$$DT \cdot AC = AT \cdot CD - AD \cdot TC. \tag{4}$$

Adding (3) and (4) and subtracting (2) we have

$$AC(DT + FT - ET) = AT(CD + CF - CE) + TC(AF - AD - AE).$$

Since $AD \| BC$ and $BE \| AC$, it follows that $CD = CE = AB$, and $AF = AE = BC$. Hence

$$AC(DT + FT - ET) = AT \cdot CF - TC \cdot AD = 0$$

from (1). Since $AC \neq 0$, it follows that $TD + TF = TE$.

Also solved by W. E. Byrne and Joseph Langr.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Meaningful Mathematics, A survey course for college students. By H. S. Kaltenborn. New York, Prentice-Hall, Inc., 1951. 15+397 pages. $4.75.

This is a text for a one year "cultural" type of survey course for students whose mathematical background may be limited. It aims to help the student gain insight into underlying ideas and processes as well as appreciation of the

role of mathematics in civilization.

A wide range of topics is treated: the nature and value of mathematics; number and computation; function and graph; algebra; numerical and analytic trigonometry; analytic geometry; differential and integral calculus. A chapter on geometry includes discussions of the nature of the postulational approach and of the relation of Euclidean to non-Euclidean geometries. A chapter on consumer mathematics features a more realistic discussion of installment buying than is usually found.

The chapters on calculus neglect almost all derivations, the connection of the indefinite with the definite integral being asserted without even being made plausible, yet these chapters treat a wide variety of applications: extrema; related rates; differentials; area; volume; work; motion; exponential growth; integration of series; *etc.* Throughout the book, the student is frequently offered a taste of a new mathematical morsel, which is snatched away before a good bite can be had—perhaps a necessary caution in that it might prove indigestible to a stomach unused to such fare.

Statements or discussions which may mislead the reader occur on pages 160, 169, 175, 305–306, 351. The explanation of a variation of "Nim" on pages 22–23 is faulty. The proof on page 298 can be simplified. The description of typical trigonometric problems on pages 195–197 might well be supplemented by scale construction suggestions.

Instructors who have tired of survey texts which conceal standard topics under fancy names and which strive to capture student interest by sparkling diction and bright jests, may welcome this book. It retains customary terminologies, and uses familiar types of examples and illustrations. On the whole, the presentation is competent, the writing simple and lucid, the format attractive. There is an answer section, and brief log-trig tables. Many of the exercises request brief written or oral reports on special mathematical topics, case histories, recreations, *etc.* Such sidelights are also stressed in the text proper.

It is difficult to evaluate the gain to the student from his skimming the surface of so much mathematical material. Perhaps he may occasionally delve into the richer strata with the help of the oral reports, which Professor Kaltenborn feels form a vital part of the course.

R. L. SWAIN
State University of New York
New Paltz, N. Y.

NEW BOOKS RECEIVED

Industrial Process Control by Statistical Methods. By J. D. Heide. New York, McGraw-Hill Book Company, 1952. ix+297 pages. $6.00.

College Algebra. First Edition. By R. R. Middlemiss. New York, McGraw-Hill Book Company, 1952. xix+344 pages. $3.50.

The Elements of Mathematical Analysis. Second Edition. Volumes I and II. By J. H. Michell and M. H. Belz. New York, The Macmillan Company, 1952.

24+516 pages and 12+569 pages. $13.60 the set.

Brief Trigonometry. Revised Edition. By E. A. Cameron. New York, Henry Holt and Company, 1952. 6+151 pages. $2.10.

Aufgabensammlung zu den gewöhnlichen und partiellen Differentialgleichungen. By Dr. Guido Hoheisel. Berlin, Germany, 1952, Walter de Gruyter and Company. 136 pages. DM 2,40.

Dictionary of Mind, Matter and Morals. By Bertrand Russell. New York, Philosophical Library, 1952. xiv+290 pages. $5.00.

Einführung in die freie Geometrie ebener Kurven. By Dr. Louis Locher-Ernst. Basel, Switzerland, Verlag Birkhäuser, 1952. 87 pages. Schw. Fr. 12.50.

Smithsonian Logarithmic Tables. By G. W. Spenceley, R. M. Spenceley, and E. R. Epperson. Washington, D. C., Smithsonian Institution. xii+402 pages, 1952. $4.50.

Econometrics. By Gerhard Tintner. New York, John Wiley and Sons, 1952. xiii+370 pages. $5.75.

Intermediate College Mechanics, A Vectorial Treatment. By D. E. Christie. New York. McGraw-Hill Book Company, 1952. xvi+454 pages. $7.00.

Elementary Differential Equations. By E. D. Rainville. New York, The Macmillan Company. xii+392 pages. 1952. $5.00.

Calculus. By A. H. Sprague. New York, The Ronald Press Company, 1952. xi+576 pages. $6.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of special features, student papers, bibliographies of program topics, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB TOPICS

Bibliographies have been published for a considerable number of topics frequently used for club programs. A summary list may be of value to program directors and others.

The bibliographies which follow were published in this MONTHLY, the numbers referring to volume and page.

American mathematicians. **47**, 107.
Apportionment of representatives. **47**, 484; **49**, 115.
Arithmetical prodigies. **25**, 91.
The binary scale of notation, etc. **25**, 139.
Calculating machines. **43**, 99; **46**, 233; **47**, 106.
The cattle problem of Archimedes. **25**, 411.
The Chinese suan p'an. **27**, 180.
Codes and ciphers. **26**, 409.

Constructions with a double-edged ruler. 25, 358.
Constructions with compasses alone. 47, 107.
La courbe du diable. 33, 273.
Development of present day numerals. 43, 99.
Euler integrals and Euler's spiral, etc. 25, 276.
A Fibonacci series. 25, 235.
Fiedler's cyclography. 32, 517.
Finite geometries. 28, 85.
First printed mathematical books. 47, 108.
Forecasting the population of the U. S. 47, 484.
The four-color problem. 43, 181.
Functional equations. 32, 428.
Geometrography, etc. 25, 37.
Geometry of four dimensions. 25, 316.
Golden section. 25, 232.
Historical items pertaining to the calculus. 46, 233.
History of algebra. 46, 234; *47*, 107.
History of American mathematicians. 46, 233; *47*, 107.
A home made mathematics exhibit. 40, 555.
The logarithmic spiral. 25, 189.
Magic squares and circles. 47, 106.
Maps and map projection. 46, 650.
Mathematics and art. 47, 108.
Mathematics and defense. 47, 484.
Mathematics and music. 47, 108.
Mathematics in the ancient world. 46, 234.
Mathematics in certain countries. 47, 107.
Nomographs. 47, 106.
Notes on famous mathematicians. 47, 109.
The number π. 26, 209.
Numerals and number systems. 46, 234.
The oldest mathematical work extant. 25, 36.
Origin of various mathematical terms and symbols. 46, 233; *47*, 107.
Paper folding. 25, 95.
The pasturage problem of Sir Isaac Newton. 33, 155.
The polar planimeter. 46, 45.
Proportional representation and preferential voting. 47, 484.
Ptolemy's theorem and formulae of trigonometry. 25, 94.
Scholars in other fields—interest in mathematics. 47, 108.
Symbolic logic. 46, 289.
Theorem of Bang. 33, 224.
Women as mathematicians and astronomers. 25, 136.

The following bibliographies appeared in *The Pentagon*.

The abacus. 10, 93.
Amicable numbers, 8, 85.
Apportionment in Congress. 7, 85.
The bee as a mathematician. 6, 19.
Calculating machines. 6, 18.
Calendar problems. 7, 30.
The cattle problem of Archimedes. 5, 67.
Codes and ciphers. 7, 81.

The construction of sundials. 8, 19.
Constructions with limited means. 7, 29.
La Courbe du diable. 10, 94.
Duplication of the cube. 10, 31.
Fermat's last theorem. 9, 33.
Fibonacci series. 8, 20.
The "fifteen" puzzle. 9, 104.
The four-color problem. 8, 86.
The golden section. 11, 103.
History of mathematics in the United States. 10, 33.
Linkages. 7, 84.
The magic number nine. 11, 32.
Magic squares. 10, 28.
Mathematics and music. 7, 31.
Mathematical prodigies. 6, 17.
Nim. 11, 101.
Non-euclidean geometry. 9, 33.
Paper folding. 5, 68.
Perfect numbers. 11, 102.
The planimeter. 6, 69.
Plays. 11, 32.
Proofs of the Pythagorean theorem. 7, 28.
Ptolemy's theorem. 8, 85.
Rational-sided triangles. 9, 102.
Repeating decimal fractions. 10, 95.
Scales of notation. 6, 67.
Solutions of the quadratic equation. 6, 66.
Squaring the circle. 9, 105.
The story of pi. 11, 34.
Trisection of an angle. 8, 21.
Women as mathematicians. 5, 67.

Several comprehensive bibliographies have been prepared by William L. Schaaf and published in *The Mathematics Teacher*. The following selections contain material for clubs.

The correlation of mathematics and science. 44, 340. (1. Curriculum integration in mathematics and science. 2. Biology and mathematics. 3. Chemistry and mathematics. 4. Physics and mathematics. 5. Mathematics and aviation. 6. Mathematics and navigation. 7. Miscellaneous applications of mathematics to science.)

Guidance: the case for mathematics. 44, 130. (1. Why learn mathematics? 2. Guidance in mathematics. 3. Functional competence in mathematics. 4. Mathematics for industry, trade, and business. 5. Mathematics for professional careers. 6. Cultural values of mathematics.)

Laboratory mathematics. 44, 422. (1. Laboratory methods and equipment. 2. Laboratory devices for plane geometry. 3. Laboratory devices for solid geometry. 4. Models for polyhedrons. 5. Laboratory devices for algebra. 6. Laboratory devices for trigonometry. 7. Models for conic sections. 8. Paper folding: dissection of figures. 9. Linkages. 10. Mechanical construction of curves.)

Logarithms and exponentials. 45, 361. (1. History of logarithms. 2. Theory of logarithms. 3. Logarithms and exponential functions. 4. Teaching logarithms.)

Mathematics and art. 43, 423. (1. Art. aesthetics, architecture. 2. Ornament and design; pattern; geometric drawing. 3. Dynamic symmetry. 4. The golden section.)

Mathematics and engineering. 44, 54. (1. Mathematics and engineering education. 2. Engineering and mathematics. 3. Texts and reference works on mathematics for engineers.)

Mathematics and modern science. 43, 294.

Mathematical plays and programs. 44, 526. (1. Plays, skits, and pageants. 2. Programs for assemblies and mathematics clubs.)

Modern calculating machines. 45, 110. (1. General articles. 2. Technical books. 3. Digital machines; electronic calculators. 4. Differential analyzers. 5. Harmonic analyzers and synthesizers. 6. Network analyzers. 7. Algebraic equation solvers.)

The numerical solution of equations. 44, 204. (1. General approximation methods. 2. On the location of roots. 3. Miscellaneous methods of solving cubic and quartic equations.)

The theorem of Pythagoras. 44, 585. (1. Proofs and discussions. 2. Pythagorean numbers: integral and rational right triangles. 3. Special cases of rational right triangles; Heronian triangles. 4. Pythagorean theorem in design. 5. Books and monographs.)

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items should be submitted at least two months before publication can take place.

A REPORT OF THE GENERAL ASSEMBLY OF THE INTERNATIONAL MATHEMATICAL UNION

The first General Assembly of the new International Mathematical Union met in Rome, Italy, on March 6–8, 1952. Eighteen member countries were represented by delegates and, in addition, there were observers present from Spain and Poland. The members of the United States Delegation were as follows: Professor M. H. Stone, University of Chicago, Chairman of the Delegation; Professor Einar Hille, Yale University; Professor J. R. Kline, University of Pennsylvania; Professor Saunders MacLane, University of Chicago; and Professor G. T. Whyburn, University of Virginia.

There was a lively and profitable discussion of the future activities of the Union; for example, a committee was set up to study the possibility of publishing a world directory of mathematicians. Other committees are to deal with methods of decreasing the cost of mathematical publication and with methods of facilitating the international exchange of persons. Of greatest interest to the Association is the reestablishment of the International Mathematical Instruction Committee (IMUK). This committee, charged with the study on an international basis of the important and difficult problems of improving mathematical instruction at all levels, is to consist of Professors Behnke (Germany), Châtelet (France), Fehr (Switzerland), Kurepa (Yugoslavia) and Jeffery (Canada). It is hoped that auxiliary national committees will be set up to assist in this work.

The business of the General Assembly proceeded in the most friendly fashion, though it was delayed somewhat by the necessity of saying everything both in French and in English. The chairmanship of the meeting was rotated from

country to country. Three new countries—Pakistan, Spain and Yugoslavia—were admitted to membership, and it was the hope of all that still other countries will apply soon for membership. It was decided that both English and French shall be equally official languages for the work of the Union.

Officers of the Union were elected as follows: President, M. H. Stone (U.S.A.); Vice-presidents, E. Borel (France) and E. Kamke (Germany); Secretary, E. Bompiani (Italy); Members of the Executive Committee: W. V. D. Hodge (England), S. Iyanaga (Japan) and B. Jessen (Denmark).

As is often the case with such meetings, one of the most important features was the chance of talking personally with colleagues from other countries. These opportunities were extended by an excursion to Tivoli on March 9th and a two day session for mathematical papers on March 10–11. There were ten papers, from as many countries, full of interest. For my own part, the high point was the penetrating new results of H. Cartan on Hilbert's theorem on chains of syzygies.

SAUNDERS MACLANE

PERSONAL ITEMS

Professor J. Hobart Bushey of Hunter College was the delegate of the Association at the Convocation which was held at the United States Military Academy on May 20, 1952 in celebration of the sesquicentennial year of the Academy.

Professor Wayne Dancer of the University of Toledo represented the Association at the inauguration of President Ralph MacDonald of Bowling Green State University on May 10, 1952.

Professor L. M. Graves of the University of Chicago was the representative of the Association at the annual meeting of the American Council on Education in Chicago on May 2–3, 1952.

Professor T. L. Reynolds, Millsaps College, was the delegate of the Association at the program commemorating the seventy-fifth anniversary of Jackson College on May 1–2, 1952.

Professor V. F. Cowling of the University of Kentucky has been awarded a Ford Foundation Faculty Fellowship for the year 1952–1953; he is on leave of absence from the University of Kentucky and is located at Yale University.

Professor F. L. Griffin has been honored by the establishment of the F. L. Griffin Fund for the Improvement of Teaching by the alumni of Reed College; Professor Griffin has retired from his position as Professor of Mathematics at Reed College after forty-one years of service.

Professor J. S. Miller of Dillard University has been awarded a Ford Foundation Fellowship for 1952–1953.

Professor George Polya of Stanford University has been elected an honorary member of the Board of the Mathematical Society of France.

Professor Herbert Robbins, University of North Carolina, has been awarded a Guggenheim Fellowship and will spend the year 1952–1953 as a member of the

Institute for Advanced Study.

Kent State University announces the following promotions: Associate Professor Emerson Jenkins to a professorship; Assistant Professor Marvin Johnson to an associate professorship.

Louisburg College announces the following: Professor T. C. Amick, head of the Department of Mathematics, will retire at the end of the fall semester of 1952; Associate Professor Elizabeth Johnson will succeed Professor Amick as Head of the Department.

New Mexico College of Agriculture and Mechanic Arts makes the following announcements: Assistant Professor Max Kramer has been promoted to an associate professorship; Dr. O. B. Ader, formerly of Mississippi Southern College, has been appointed to an instructorship.

Skidmore College announces: Associate Professor Mary E. Williams is on leave of absence during the first semester of 1952–1953 and is studying at Cambridge University; Professor Emeritus C. H. Forsyth of Dartmouth College is teaching at the college during her absence.

Swarthmore College announces the retirement of Professor Arnold Dresden and Professor R. W. Marriott; Professor Dresden has been a member of the faculty for twenty-five years, Professor Marriott has taught at the college for forty-six years.

Western Michigan College reports the promotion of Assistant Professors H. H. Hannon, J. K. Peterson, and C. S. Sams to associate professorships.

Dr. Jerome Blackman has been appointed to an instructorship at Syracuse University.

Dr. Archie Blake of the Mechanical Research Corporation, Chicago, Illinois, has accepted a position as Head of the International Business Machines Computing Section of the Cornell Aeronautical Laboratory, Buffalo, New York.

Instructor R. G. Bradshaw of Clarkson College has been promoted to an assistant professorship.

Dr. H. F. Bright, formerly of the University of Texas, has accepted a position as Chief, Technical Services, Human Resources Research Office, George Washington University.

Mr. H. H. Brown, previously of the Armour Research Foundation, Illinois Institute of Technology, has accepted a position as Senior Research Engineer with the Franklin Institute Laboratories for Research and Development, Philadelphia, Pennsylvania.

Associate Professor Kenneth E. Brown of the University of Tennessee has been appointed Specialist for Mathematics in the United States Office of Education.

Professor Emeritus W. B. Carver of Cornell University was Visiting Professor at Southern Illinois University during the 1952 Summer Session.

Mr. S. H. Chasen who has been a graduate student at Emory University has a position as Mathematician at the United States Naval Air Testing Station, Patuxent River, Maryland.

Mr. M. L. Coffman, East Texas State Teachers College, has been appointed to an instructorship in the Department of Physics of Texas Agricultural and Mechanical College.

Dr. G. F. Cramer of the Defense Department has accepted a position as Engineer with R. C. A. Victor, Camden, New Jersey.

Assistant Professor Albert Edrei of the University of Colorado has been appointed to an associate professorship at Syracuse University.

Mr. Rudolf Feige, formerly of the University of Cincinnati, has a position as Electrical Engineer with the American Can Company, Cincinnati, Ohio.

Mr. A. R. Friedenheit, previously a student at Washington and Jefferson College, is now a stress analyst with North American Aviation, Columbus, Ohio.

Associate Professor Fritz Herzog of Michigan State College has been promoted to a professorship.

Dr. T. J. Jaramillo who has been Chairman of the Department of Mathematics and Professor of Mathematics at Far Eastern University, Manila, Philippine Islands, for the last four years has returned to the Engineering Mechanics Division of Armour Research Foundation, Illinois Institute of Technology, as Senior Scientist.

Professor E. P. Lane of the University of Chicago has retired.

Dr. R. A. Leibler of the Navy Department has accepted a position as Mathematician with the Sandia Corporation, Albuquerque, New Mexico.

Mr. H. G. Mazurkiewicz, previously of the Willow Run Research Center, University of Michigan, is now a group engineer, electronics, with the Glenn L. Martin Company, Baltimore, Maryland.

Mr. Jack Minker who has been a dynamics engineer at Bell Aircraft Corporation, Niagara Falls, New York, is now an engineer in the Advanced Development Section, Radio Corporation of America, Camden, New Jersey.

Miss Florence Moline of Union College, Lincoln, Nebraska, has been promoted to an assistant professorship.

Mr. Robert E. Montgomery is a mathematical research analyst with the Sandia Corporation, Albuquerque, New Mexico; he is also a part-time faculty member of the Department of Mathematics and Astronomy of the University of New Mexico.

Dr. W. K. Moore of the University of Kansas has been appointed to an assistant professorship at Albion College.

Mr. C. R. Morris, formerly of the United States Navy Hydrographic Office, Suitland, Maryland, has accepted a position with the Geotechnical Corporation, Dallas, Texas.

Dr. Paul Nesbeda of Catholic University of America has a position as Mathematician with the Radio Corporation of America, Camden, New Jersey.

Dr. John Reckzeh has been appointed to an associate professorship at State Teachers College, Jersey City, New Jersey.

Dr. F. Virginia Rohde of the University of Florida has been promoted to an assistant professorship.

Associate Professor M. F. Rosskopf of Syracuse University has been appointed to an associate professorship at Teachers College, Columbia University.

Assistant Professor Edith R. Schneckenburger of the University of Buffalo has been promoted to an associate professorship.

Mr. Cecil Schwartz, formerly an engineer with the Armour Research Foundation, Illinois Institute of Technology, is now an engineer with the Glenn L. Martin Company, Baltimore, Maryland.

Mr. W. L. Shepherd of the University of Oregon has been appointed to an instructorship at Texas Western College of the University of Texas.

Assistant Professor J. C. Smith of Lafayette College has been promoted to an associate professorship.

Dr. Roscoe Stinetorf has been appointed Chairman of the Department of Mathematics of Wagner College.

Associate Professor R. R. Stoll of Lehigh University has been appointed to a professorship at Oberlin College; during the past summer Professor Stoll served as a consultant to the National Science Foundation.

Mr. N. D. Vlachos, formerly of the Kellett Aircraft Corporation, Camden, New Jersey, has accepted a position as Research Associate with the Hercules Powder Company, Allegany Ballistics Laboratory, Cumberland, Maryland.

Mr. R. F. Wheeling of Brown University is now a technologist with the Socony-Vacuum Oil Company, Paulsboro, New Jersey.

Professor A. E. Whitford of Alfred University has retired with the title of Professor Emeritus; Professor Whitford has been engaged in college teaching for fifty-one years.

Mr. W. F. Winn of the Armed Services Electro Standards Agency, Fort Monmouth, New Jersey, has been promoted to the position of Electronic Engineer.

Associate Professor Emeritus Joseph Barnett, Oklahoma Agricultural and Mechanical College, died on April 12, 1952. He had been a member of the Association for twenty-three years.

Mr. R. D. Bartz, an undergraduate student at the University of Wisconsin, died on February 14, 1951.

Professor M. C. Foster of Wesleyan University, Middletown, Connecticut, died on April 9, 1952.

Professor Marie Litzinger, chairman of the Department of Mathematics of Mount Holyoke College, died on April 7, 1952.

Mr. J. J. Newman, a student at Columbia University, died on September 10, 1951.

Professor Emeritus I. C. Nichols of Louisiana State University died on March 13, 1952. He was a charter member of the Association.

Mr. R. R. Shumway, a retired professor of the University of Minnesota, died on April 19, 1952. He was a charter member of the Association.

Professor Emeritus E. W. D. Taylor of Pomona College died on April 7, 1952.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1952 by a mail vote of the membership of the Association in the Sections indicated:

Kansas	G. B. Price, University of Kansas
Missouri	R. J. Michel, Southeast Missouri State Teachers College
Ohio	I. A. Barnett, University of Cincinnati
Pacific Northwest	R. D. James, University of British Columbia
Southeastern	F. A. Lewis, University of Alabama
Southwestern	R. F. Graesser, University of Arizona
Upper New York State	C. W. Munshower, Colgate University
New England Region	B. H. Brown, Dartmouth College

As usual, great interest has been shown in each Section in the selection of a Governor to represent the members of that Section. The close contests in many Sections seemed to indicate that the members felt that the losing candidates as well as the successful ones were worthy of membership on the Board of Governors. Of the Sections voting this year the highest percentage of votes cast was 53% in the Kansas Section.

H. M. GEHMAN, *Secretary-Treasurer*

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 130 persons have been elected to membership by the Board of Governors on applications duly certified.

M. J. AEGERTER, M.S. (Wisconsin) Grad. Student, University of Wisconsin, Madison, Wis.

M. A. A. AL-BASSAM, Ph.D. (Texas) Asst. Professor, Huston-Tillotson College, Austin, Tex.

A. G. ANDERSSON, Student, University of Alabama, University, Ala.

JOSEPH ANDRUSHKIW, Ph.D. (Ukrainian Free U.) Asso. Professor, Seton Hall University, South Orange, N. J.

P. L. ARMSTRONG, A.M. (Southwestern Presby. U.) Asst. Professor, Clemson College, S. C.

M. J. ARTHURS, Student, Harvard University, Cambridge, Mass.

G. B. AX, M.A. (Columbia) Asst. Professor, Virginia Military Institute, Lexington, Va.

J. L. BARNES, Ph.D. (Princeton) Professor of Engineering, University of California at Los Angeles, Calif.

MRS. MABEL S. BARNES, Ph.D. (Ohio State) Lecturer, Occidental College, Los Angeles, Calif.

PETER BEDROSIAN, Student, McMaster University, Hamilton, Ontario, Canada

R. J. BEEBER, M.S. (Catholic U.) Grad. Stu-

dent, Columbia University, New York, N. Y.

M. R. BOTTACCINI, B.A. (Southern California) Head, Torpedo Dynamics Section, Naval Ordnance Test Station, Pasadena, Calif.

EVELYN BOYD, Ph.D. (Yale) Asso. Professor, Fisk University, Nashville, Tenn.

B. J. BOYER, Student, Purdue University, Lafayette, Ind.

R. B. BREDEMEIER, B.A. (Walla Walla) Grad. Assistant, Oregon State College, Corvallis, Ore.

G. E. BREDON, Student, Stanford University, Calif.

W. R. BRITTENHAM, M.S. (Wisconsin) Grad. Student, University of Wisconsin, Madison, Wis.

J. C. CALDWELL, Ed.M. Instr., Edward Waters College, Jacksonville, Fla.

H. E. CAMPBELL, Ph.D. (Wisconsin) Asst. Professor, Emory University, Ga.

RUTH D. CLARK, M.A. (Wisconsin) Staff Member, Los Alamos Scientific Laboratory, N. M.

J. T. CLAUSEN, JR., B.S. (Beloit) Chief, Methods and Data Analysis, White Sands Proving Ground, Las Cruces, N. M.

C. L. COPE, M.A. (Georgia) Asst. Professor, University of Georgia, Atlanta, Ga.

WALTER CORNETZ, B.A. (N.Y.U.) Communications Engineer, Grumman Aircraft Engineering Corporation, Long Island, N. Y.

G. E. DECKER, M.A. (N.Y.S.C. for Teachers, Albany) Teacher, Lewiston High School, N. Y.

AUGUST DECKERT, B.A. (Washington S.C.) Mathematician, White Sands Proving Grounds, Las Cruces, N. M.

J. J. DENNIS, Ph.D. (Northwestern) Chairman, Department of Mathematics, Clark College, Atlanta, Ga.

A. W. DICKINSON, Student, University of Massachusetts, Amherst, Mass.

RAYMOND DOBY, M.S. (N.Y.U.) Development Engineer, Sylvania Electric Products, Kew Gardens, N. Y.

ALBERT EDREI, Dr.Sci.Math. (Zürich) Asst. Professor, University of Colorado, Boulder, Colo.

HARVEY EISENBERG, B.A. (Brooklyn) Mathematician, Ft. Monmouth, N. J.

J. M. ELKIN, M.A. (Columbia) Chief Statistician, Railroad Retirement Board, Chicago, Ill.

H. MARGARET ELLIOTT, Ph.D. (Radcliffe) Asst. Professor, Washington University, St. Louis, Mo.

G. V. EMERSON, Student, Hobart College, Geneva, N. Y.

A. M. FLEISHMAN, B.A. (Bridgewater) Mathematician, Naval Proving Ground, Dahlgren, Va.

S. A. FOOTE, Student, University of Alabama, University, Ala.

J. W. FRANKOWSKY, M.S. (N.Y.U.) Acting Chairman, Department of Mathematics, Lincoln University, Pa.

G. V. FRANKS, Student, Holy Cross College, Worcester, Mass.

A. R. FRIEDENHEIT, Student, Washington and Jefferson College, Washington, Pa.

P. W. GILBERT, Ph.D. (Duke) Asst. Professor, Syracuse University, N. Y.

T. L. GLAHN, B.S. (Oregon S.C.) Grad. Assistant, Oregon State College, Corvallis, Ore.

REV. H. H. GOTTBRATH, Sc.M. (DePaul) Instr., Bellarmine College, Louisville, Ky.

C. C. GRANDY, Student, Colorado Agricultural and Mechanical College, Ft. Collins, Colo.

BERNARD GREENSPAN, M.A. (Brooklyn) Asst. Professor, Drew University, Madison, N. J.

J. D. HAGGARD, Ed.D. (Missouri) Asst. Professor, Kansas State Teachers College, Pittsburg, Kans.

LOIS S. HALE, Student, University of Georgia, Athens, Ga.

J. L. HAMMERSMITH, B.S. (Michigan) Mathematician, Naval Research Laboratory, Washington, D. C.

L. F. HAYWARD, Student, Southwestern College, Winfield, Kan.

H. S. HEAPS, M.A. (Toronto) Asst. Professor of Civil Engineering, Nova Scotia Technical College, Halifax, Nova Scotia, Canada

P. K. HENNEY, Student, Ball State Teachers College, Muncie, Ind.

R. E. HILL, Student, University of California, Berkeley, Calif.

M. H. HOEHN, M.A. (California) Instr., University of Idaho, Moscow, Idaho

J. H. HORNBACK, Ph.D. (Illinois) Asst. Professor, University of Alabama, University, Ala.

H. F. HUNTER, B.S. (Northwestern) IBM CPC Operator, Northrop Aircraft Company, Hawthorne, Calif.

R. T. IVES, Student, Haverford College, Pa.

R. E. JACKSON, Student, University of California at Los Angeles, Calif.

L. W. JACOBS, III, Student, Central College, Fayette, Mo.

E. H. JAKUBOWSKI, B.A. (American International) Physicist, U. S. N. Mine Countermeasures Station, Panama City, Fla.

H. H. JOHNSON, B.A. (San Jose S.C.) Grad. Assistant, Oregon State College, Corvallis, Ore.

W. W. JONES, M.A. (Cornell) Asso. Professor, Kentucky State College, Frankfort, Ky.

R. B. KELMAN, Student, University of California, Berkeley, Calif.

R. R. KEMP, Student, McMaster University, Hamilton, Ontario, Canada

F. B. KEY, M.A. (Duke) Asst. Professor, University of Richmond, Va.

R. B. KILTIE, Student, Columbia University, New York, N. Y.

JACK KLUGERMAN, Student, Brooklyn College, N. Y.

KARL KOZARSKY, M.S. (Michigan) Mathematician, U. S. Naval Proving Ground, Dahlgren, Va.

SANTOSH KUMAR, Student, Hindu College, Amritsar, India

C. H. LEE, Mathematical Statistician, Bureau of the Census, Washington, D. C.

ROY LEE, M.S. (Kansas S.T.C.) Professor of Chemistry, Barber-Scotia College, Concord, N. C.

R. A. LEIBLER, Ph.D. (Illinois) Mathematician, Sandia Corporation, Albuquerque, N. M.

STANISLAW LEJA, M.A. (Lwow) Dyer, Spinnerin Yarn Company, South Hackensack, N. J.

GILBERT LEMIEUX, C.E. (Ecole Poly. de Montreal) Asst. Principal Engineer, Highways Department of Quebec, Canada

C. J. LINDSAY, B.A. (Buffalo) Grad. Student, University of Buffalo, N. Y.

L. A. MAGNANTI, Student, Syracuse University, N. Y.

JOSEPH MAHLER, M.S. (Harvard) Project Engineer, Flight Research Laboratory, Dayton, Ohio

S. W. MALINOWSKI, Student, Rensselaer Polytechnic Institute, Troy, N. Y.

CARL MARKELLO, Student, University of Buffalo, N. Y.

ELIZABETH A. MARTZ, Student, University of Cincinnati, Ohio

R. D. MAYER, Student, Purdue University, Lafayette, Ind.

G. J. MAZZARA, Student, Hofstra College, Hempstead, N. Y.

J. H. MEANS, M.S. (Iowa) Asso. Professor, Huston-Tillotson College, Austin, Tex.

PINCHAS MENDELSON, Student, University of Pennsylvania, Philadelphia, Pa.

R. E. MONTGOMERY, M.S. (Illinois) Mathematical Research Analyst, Sandia Corporation, Albuquerque, N. M.

L. J. MONTZINGO, JR., M.A. (Buffalo) Instr., Roberts Wesleyan College, North Chili, N. Y.

B. J. MORSE, M.A. (Columbia) Grad. Student, Columbia University, New York, N. Y.

SHARON F. MURNICK, Student, Hofstra College, Hempstead, N. Y.

D. V. NEWTON, B.S. (Whitworth) Teaching Fellow, University of Washington, Seattle, Wash.

R. C. O'NEAL, B.S. in Ed. (Missouri) Teacher, Quincy High School, Quincy, Ill.

S. T. PAINE, B.A. (U.C.L.A.) Mathematician, Department of Meteorology, University of California at Los Angeles, Calif.

F. J. PALAS, Ed.M. (Oklahoma) Grad. Student, University of Oklahoma, Norman, Okla.

E. C. PALMER, B.S. Cashier, Palmer Electric Company, Seattle, Wash.

J. P. PIERCE, M.E.E. (Brooklyn Poly.) Asso. Professor, U. S. Naval Postgraduate School, Monterey, Calif.

D. H. Pilgrim, B.A. (Morningside) Grad. Student, State University of Iowa, Iowa City, Iowa

R. S. RAVEN, M.A. (Stanford) Senior Engineer, Westinghouse Electric Corporation, Friendship Airport, Baltimore, Md.

K. I. RAYMOND, Ed.M. (St. Lawrence) Com-

puter, Bell Aircraft Corporation, Niagara Falls, N. Y.

F. C. REED, B.S. (Redlands) Mathematician, Flight Determination Laboratory, White Sands Proving Ground, Las Cruces, N. M.

W. F. REYNOLDS, M.A. (Harvard) Teaching Fellow, Harvard University, Cambridge, Mass.

C. E. ROBERSON, Student, Carleton College, Northfield, Minn.

MAX ROSENBERG, M.S. (George Washington) Mathematician, Picatinny Arsenal, Dover, N. J.

E. M. SCHEUER, B.A. (Reed) Mathematician, Naval Ordnance Test Station, Pasadena, Calif.

E. M. SCHUSS, B.S. (St. John's) Corporal, U. S. Army, Camp Gordon, Ga.

J. A. SIMMONS, Student, University of California, Berkeley, Calif.

W. B. SIMMONS, JR., Student, Knox College, Galesburg, Ill.

D. A. SIPFLE, Student, Carleton College, Northfield, Minn.

SISTER M. DOMITILLA, Ed.M. (Marquette) Head, Mathematics and Physics Departments, Viterbo College, La Crosse, Wis.

SISTER MARY OF PERPETUAL HELP, M.A. (Catholic U.) Professor, Incarnate Word College, San Antonio, Tex.

J. H. SKELTON, M.A. (Missouri) Asst. Professor, Southwest Missouri State College, Springfield, Mo.

HAROLD SLATER, Student, Northeastern University, Boston, Mass.

F. B. SLOAT, M.A. (Arkansas) Asst. Professor, Kansas State College, Manhattan, Kan.

C. T. SMITH, B.A. (Buffalo) Grad. Student, University of Buffalo, N. Y.

P. L. SMITH, B.A. (Gustavus Adophus) Grad. Assistant, Kansas State College, Manhattan, Kan.

J. C. SOLLEY, Student, University of California, Berkeley, Calif.

D. R. SONDERGELD, Student, Bowling Green State University, Ohio

J. L. SPENCELEY, M.A. (Michigan) Instr., High School, Alpena, Michigan

W. L. STEINER, Student, Fordham University, New York, N. Y.

K. E. STOLL, Student, University of Kentucky, Lexington, Ky.

D. L. STURTZ, Student, Reed College, Portland, Ore.

NIYAZI TARIMER, Student, Massachusetts Institute of Technology, Cambridge, Mass.

G. T. THOMPSON, M.A. (Oregon) Grad. Assistant, Oregon State College, Corvallis, Ore.

M. W. TOLMAN, Student, University of South Dakota, Vermillion, S. D.

R. S. VARGA, M.A. (Harvard) Teaching Fellow, Harvard University, Cambridge, Mass.

J. F. WAMPLER, B.A. (Kansas) Instr., York College, Neb.

HELEN B. WARNER, B.S. (N.Y.S.C. for Teachers, Buffalo) Teacher, Board of Education, Buffalo, N. Y.

J. V. WHITTAKER, Student, University of California at Los Angeles, Calif.

R. E. WILD, M.S. (Iowa) Asst. Professor, University of Idaho, Moscow, Idaho

L. H. WILLIAMS, M.S. (Georgia) Mathematician, Redstone Arsenal, Huntsville, Ala.

J. K. WITTHAUS, Student, University of Oklahoma, Norman, Okla.

F. G. WITTMANN, B.A. (Buffalo) Grad. Student, University of Buffalo, N. Y.

A. G. WOOTTON, M.A. (Columbia) Asst. Professor, Champlain College, Plattsburg, N. Y.

FUMIO YAGI, Ph.D. (M.I.T.) Asst. Professor, University of Washington, Seattle, Wash.

LOIS J. YOUNGER, Student, William Jewell College, Liberty, Mo.

THE TRAVELING LECTURER PROGRAM OF THE SOUTHWESTERN SECTION

Since its inception in 1937 the Southwestern Section of the Association has supported a traveling lecturer program for the promotion of interest in mathematics among the institutions of the section. Each year two lecturers are appointed by the executive committee of the Section, one to visit as many of the institutions in New Mexico as possible, the other to do the same in Arizona, and

to present a lecture on some mathematical subject intelligible to undergraduates. Except for the war years this program has been in operation with the enthusiastic support of a majority of colleges in the area.

In 1952 Dr. Morris Hendrickson of the University of New Mexico lectured in New Mexico on the subject *The Technique of Generalization as Applied to the Concept of Continuity*. Dr. D. L. Webb of the University of Arizona spoke in Arizona on *The Characteristics of a Mathematician*.

The program is supported financially by the institutions at which the lecturer appears. An amount of $25 is contributed usually by the participating colleges. However, the lecturer will visit institutions which cannot contribute that much or, indeed, will visit those which can provide no money if it does not involve too great inconvenience.

The members of the section feel that interest in mathematics has been promoted by the program and that it has aided materially in maintaining cordial relationships between the colleges and universities of the Southwestern Section.

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The twenty-ninth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Northwestern Louisiana State College, Natchitoches, Louisiana, on February 15 and 16, 1952. Professor Arthur Ollivier, Chairman of the Section, presided at the Friday afternoon session. Professor C. G. Killen, Louisiana Vice-Chairman, presided at the Saturday morning session.

There were eighty-nine persons present including the following thirty-nine members of the Association:

T. A. Bickerstaff, Elsie T. Church, G. J. Corley, Margaret R. Davis, Virginia I. Felder, L. M. Garrison, M. E. Gillis, J. A. Hardin, Jessie M. Hoag, W. R. Hutcherson, L. H. Kanter, H. T. Karnes, C. G. Killen, S. R. Knox, Z. L. Loflin, A. C. Maddox, Betty McKnight, R. A. Miller, Benjamin Ernest Mitchell, S. B. Murray, M. M. Ohmer, Arthur Ollivier, R. L. O'Quinn, H. A. Palmer, P. K. Rees, T. L. Reynolds, F. A. Rickey, H. F. Schroeder, L. L. Scott, S. W. Shelton, P. K. Smith, W. H. Spragens, V. B. Temple, W. E. Timon, Jr., B. B. Townsend, G. J. Trammell, B. A. Tucker, B. O. Van Hook, W. M. Whyburn.

The following officers were elected for the coming year: Chairman, F. A. Rickey, Louisiana State University; Louisiana Vice-Chairman, Margaret M. LaSalle, Nicholls Junior College; Mississippi Vice-Chairman, T. L. Reynolds, Millsaps; Secretary-Treasurer, Z. L. Loflin, Southwestern Louisiana Institute.

The time and place of the next meeting were set as February 13–14, 1953 at Millsaps College, Jackson, Mississippi.

The invited speaker for the meeting was Professor W. M. Whyburn, University of North Carolina. His subject for an address at the Friday evening dinner held jointly with the Louisiana-Mississippi Branch of the National Council of Teachers of Mathematics was *Efficient Use of Mathematically Trained Manpower*. At the Saturday morning session Professor Whyburn spoke on *Critical Solutions of Differential Systems*.

The following short papers were presented:

1. *Minkowski inequality for the mixed area*, by Miss Margaret M. LaSalle, Nicholls Junior College.

The area of a third convex curve traced out by the resultant of two vectors, each of which traces out a convex curve is given by

$$F = \frac{1}{2}\int (x_1 dx_2 - x_2 dx_1).$$

Substituting for the x's in terms of $p(\phi) = x_1 \cos\phi + x_2 \sin\phi$, the integrand becomes $p^2 - p'^2$, $p(\phi)$ of period π, limits of ϕ from $-\pi$ to π. Upon setting $p(\phi) = c_0 p_0 = c_1 p_1$ and rearranging terms according to subscripts there results

$$F = c_0^2 F_{00} + 2c_0 c_1 F_{01} + c_1^2 F_{11}$$

where F_{01} is the "mixed area." It is proved in this paper that the F's satisfy the Minkowski inequality

$$F_{01}^2 - F_{00}F_{11} \leqq 0.$$

2. *On the impossibility of an odd perfect number N with six different prime divisors*, by Professor T. L. Reynolds, Millsaps College.

Some lemmas and applications were given illustrating the proof of the following theorem: "On the Impossibility of an Odd Perfect Number N not Divisible by 5 with Six Different Prime Divisors"—the main theorem of the author's doctoral dissertation at the University of North Carolina in 1950.

3. *A new chapter in the life of Gauss*, by Professor G. Waldo Dunnington, Northwestern Louisiana State College, introduced by the Secretary.

This paper takes up the friendship of Gauss with numerous scientists and mathematicians, as well as individuals outside that classification, in foreign nations, particularly Great Britain and America. Particular attention is paid to B. A. Gould, the American astronomer, who was one of Gauss' pupils, to William C. Bond and his son George P. Bond, and to F. R. Hassler. Gauss' especial interest in America is discussed. His role in the affairs of the University of Göttingen and various attempts to lure him away from Göttingen form the final section of the paper.

4. *A certain cubic transformation*, by Elsie T. Church, Northwestern Louisiana State College.

This paper was published in this MONTHLY, vol. 59, p. 314.

5. *Invariant curves of order eight*, by Professor W. R. Hutcherson, University of Florida.

When invariant eight degree surfaces cut a certain fourth degree surface, the imperfect point P_3 is either an eight, seven, four, or one-tuple point. (W. R. Hutcherson, *Point non parfait et courbes invariables*, Bulletin de la Société Royale des Sciences de Liège No. 11 (1950).)

6. *A use of matrices in multivariant analysis*, by Professor S. R. Knox, Millsaps College.

A means of obtaining moments for the multi-variate normal distribution is shown, using methods suggested by Professor P. S. Dwyer of the University of Michigan.

7. *On the ultraspherical Christoffel numbers*, by Professor L. H. Kanter, Mississippi State College.

Let $w(x, \tau)$ be positive and continuous for $a<x<b$. Further, $w_\tau(x, \tau)$ shall exist and be continuous for $a<x<b$, and the integrals $\int_a^b x^\nu w_\tau(x, \tau)dx$, $\nu=0, 1, \cdots, 2n-1$ shall be uniformly convergent in every closed interval $\tau' \leqq \tau \leqq \tau''$ of the open interval (τ_1, τ_2). Then differentiating the Gauss-Jacobi mechanical quadrature formula with respect to τ, we have $\int_a^b w_\tau(x, \tau)\rho(x)dx = \sum_{\nu=1}^n \lambda_\nu(\tau)\rho'(x_\nu)x_\nu'(\tau) + \sum_{\nu=1}^n \lambda_\nu'(\tau)\rho(x_\nu)$. Here $\rho(x)$ is an arbitrary polynomial of degree $2n-1$ and $\lambda_\nu(\tau)$ are the Christoffel numbers. If for $\rho(x)$ we choose the well-known Hermite polynomial of the first kind corresponding to the ultraspherical polynomial $P_n(x, \tau)$, it may be shown that $\lambda_\nu(\tau)$ is a decreasing function of τ for $-1<\tau\leqq 0$.

8. *N-section curves*, by Professor V. B. Temple, Louisiana College.

Circles with centers at O_1 and O_2 and radii a and r are tangent at A, internally or externally. They rotate around their centers with no slipping at A. H_1 and H_2 are points on the circles such that $AH_1=AH_2$. Let θ be the angle AO_1H_1 and α the angle AO_2H_2. Then, if $a=nr$, $n\theta=|\alpha|$, $(n>0)$. Now draw lines through H_1 and H_2 parallel to O_1O_2 to meet lines through H_1 and H_2 at right angles respectively in P_1 and P_2. The locus of each of these points is an *N-section curve.*

If O_1 is the origin and O_1O_2 the positive end of the x-axis, the parametric equations are:

$$P_1: \quad x = a\cos\theta, \quad y = \frac{a}{n}\sin n\theta, \qquad P_2: \quad x = \frac{a}{n}(n \pm \cos n\theta), \quad y = a\sin\theta,$$

where the upper and lower signs in x of P_2 are respectively for the conditions of external and internal tangency. Charts of these curves were shown for $n=2, 3, 4, 5$, and 6.

9. *Wrong method, right answer*, by Professor F. A. Rickey, Louisiana State University.

A student in elementary algebra "solved" the simultaneous pair

$$y = x + 2$$
$$(x+1)^2 + (y-5)^2 = 16$$

by replacing the second of these equations by the pair $x+1+y-5=\pm 4$. The four points obtained are actually the intersection points of the given pair.

In this paper an analysis of the conditions under which such incorrect procedure yields the correct answers was given. Other instances in which incorrect procedures result in the correct answers were discussed. A study of the Diophantine equation for the cancellation

$$\frac{1\not{6}}{\not{6}4} = \frac{1}{4}$$

leads to the interesting cancellation

$$\frac{270}{756} = \frac{20}{56}.$$

10. *Finite metabelian groups and lines of a projective 5-space*, by Professor L. L. Scott, University of Mississippi.

A finite metabelian group G is a group whose commutator subgroup K is contained in the central C. The metabelian groups considered are such that: $K=C$; all elements are of order p; each group is generated by six elements and is not the direct product of an abelian group and a

metabelian group with less than six generators. One-to-one correspondences were set up between certain abelian subgroups of G and the points of a projective 5-space S_5 and between the cyclic subgroups of order p of C and the points of a projective 14-space Σ_{14}. It was shown that an element of C is a commutator of G if and only if its corresponding point in Σ_{14} lies on the eight-dimensional manifold V determined as the locus of the Plücker coordinates of the lines of S_5.

F. A. RICKEY, *Secretary*

THE MARCH MEETING OF THE KANSAS SECTION

The thirty-seventh annual meeting of the Kansas Section of the Mathematical Association of America was held at Bethany College, Lindsborg, Kansas on March 29, 1952. Professor Anna Marm, Chairman of the Section, presided at both the morning and afternoon sessions.

One hundred seventy persons were present including the following forty-three members of the Association:

P. H. Arnold, Wealthy Babcock, Florence L. Black, Jeneva J. Brewer, J. E. Faulkner, D. H. Firl, W. C. Foreman, Abraham Franck, Albert Furman, W. H. Garrett, F. C. German, Laura Z. Greene, (Mrs.) Sabrina Hecht, D. M. Houser, W. C. Janes, L. E. Laird, K. E. Lake, Violet H. Larney, Anna Marm, Margaret E. Martinson, Thirza A. Mossman, Agnes E. Nibarger, V. D. Nyhoff, S. T. Parker, O. J. Peterson, C. B. Read, L. M. Reagan, Sister M. Helen, Sister Jeanette, Sister M. Nicholas, R. G. Sanger, W. R. Scott, G. W. Smith, R. G. Smith, E. C. Stopher, W. T. Stratton, C. B. Tucker, Gilbert Ulmer, Frances E. Walsh, A. M. Wedel, A. E. White, Ferna E. Wrestler, P. M. Young.

At the business session the following officers were elected: Chairman, P. M. Young, Kansas State College; Vice-Chairman, W. C. Foreman, Baker University; Secretary-Treasurer, Laura Z. Greene, Washburn University.

The program consisted of the following papers:

1. *Mathematics in the secondary school, 1952*, by Professor J. R. Mayor, University of Wisconsin.

In this paper an analysis was given of factors affecting the place and nature of mathematics instruction in the secondary schools and of the trends in secondary school mathematics. Special reference was made to the current shortage of scientific personnel, advantages of co-operation between college and secondary school teachers, inservice programs for teachers, curriculum plans which tend to de-emphasize mathematics, enrollment statistics, and course modifications at the secondary school level. The conclusion was drawn that there was evidence to support an optimistic point of view.

2. *Adequate opportunity for the gifted high school student*, by Mr. A. S. Richert, Wichita High School East, Kansas, introduced by Professor Anna Marm.

A democracy must provide educational opportunity commensurate with ability. About 10 per cent of the six million high school children have an I.Q. better than 120. These are "the gifted." Certain barriers hamper the gifted such as: inflexible curriculum and teaching methods; resentment of teachers toward the gifted; lack of time, energy, equipment for special help. Education for the gifted is improved by: improved methods of identification; improved counseling; a chance to work in chosen field before graduation; financial assistance even in high school; acceleration, homogeneous grouping, more electives, enrichment. The New York High School of Science is a good example of adequate opportunity. The plan of the Wisconsin University Correspondence School illustrates what can be done for smaller high schools.

3. *Opportunity for the gifted student in junior college*, by Miss Henrietta Courtright, Arkansas City Junior College, Kansas, introduced by Professor Anna Marm.

The best brains of society are required to meet the need for consecrated leadership in American civilization. Junior colleges can help by identifying gifted students, providing opportunities to develop their talents by acceleration, sectioning, enrichment and elective courses, guiding them to grasp their opportunities, and developing their sense of social responsibility. Scholarships should be attainable by financially handicapped gifted students. Teachers with imagination, enthusiasm and ability to transmit live, meaningful knowledge can help the gifted realize the responsibility of their talents and harmonize personality with intellectual development. Gifted students must be educated to the height of their powers for the realization of these ideals.

4. *Opportunities for the gifted student in the university and the college*, by Professor R. G. Sanger, Kansas State College.

In this paper the opportunities available in small colleges and large universities for students with special mathematical abilities were considered. The amount of individual attention given to a student in each type of school, the variety of offerings, physical facilities, and the stimulation gained by contact with similar interests were discussed.

5. *Certain functions defined by series*, by Professor Ferna E. Wrestler, University of Wichita.

This paper used infinite series to define two functions $C(x)$ and $S(x)$ and to develop several of their properties. It was then shown that these are the familiar properties of the functions cos x and sin x which are usually developed in terms of angles.

6. *Point sets and number theory*, by Professor E. C. Stopher, Fort Hays Kansas State College.

Certain analogies between point sets and numbers were discussed. Equations involving point sets, their sums and their products were presented, together with corresponding equations involving numbers, their lowest common multiples and their greatest common divisors.

7. *Remarks on Tchebycheff polynomials*, by Mr. P. H. Arnold, Kansas State College.

The speaker discussed the problem proposed by Tchebycheff in 1854: Given a polynomial in x of the form, $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$, to determine that one which in the interval, $-1<x<1$, possesses the smallest maximum absolute value. He outlined a general method of deriving these polynomials and presented a few special properties of them.

8. *Certain congruent figures*, by Professor J. D. Haggard, Kansas State Teachers College, introduced by the Secretary.

A number of examples and theorems concerning congruent figures in euclidean space were reviewed. The examples and theorems seem to violate one's intuition and were suggested as an antidote for the attitude often expressed by students that a certain theorem is "obvious" and therefore does not require a proof.

LAURA Z. GREENE, *Secretary*

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The annual meeting of the Southeastern Section of the Mathematical Association of America was held at Agnes Scott College and Georgia Institute of

Technology, Decatur, Georgia, on March 21–22, 1952. Professors B. G. Clark, Chairman of the Section, H. A. Robinson and J. W. Lasley, Jr., presided over the general sessions; and Professors B. G. Clark, H. K. Fulmer, Vice-Chairman of the Section, and W. L. Williams presided over subsections.

There were about two hundred fifty present including the following one hundred twenty-six members of the Association:

Edith W. Ainsworth, S. A. Anderson, P. L. Armstrong, A. H. Bailey, J. C. Barnes, D. F. Barrow, W. S. Beckwith, M. G. Boyce, Evelyn Boyd, G. M. Brown, J. W. Brown, R. K. Butz, H. E. Campbell, L. Virginia Carlton, Ella F. Casey, R. S. Christian, B. G. Clark, C. E. Clark, A. B. Coble, A. C. Cohen, Jr., R. M. Conkling, C. L. Cope, F. E. Cothran, R. W. Cowan, J. C. Currie, J. J. Dennis, Annie L. Dicks, V. E. Dietrich, N. E. Dodson, Patrick DuVal, L. A. Dye, J. C. Eaves, David Ellis, Trevor Evans, Floyd Field, Tomlinson Fort, R. W. Frankel, H. K. Fulmer, Barbara A. Futral, L. L. Garner, J. R. Garrett, Leslie J. Gaylord, V. D. Gokhale, D. B. Goodner, S. T. Gormsen, H. C. Griffith, S. W. Hahn, E. H. Hadlock, E. A. Hedberg, Melvin Henriksen, A. T. Hind, C. H. Holton, C. W. Hook, J. H. Hornback, J. V. Howell, H. B. Hoyle, Jr., C. W. Huff, G. B. Huff, S. L. Hull, L. P. Hutchinson, S. L. Jamison, Ayrlene M. Jones, F. W. Kokomoor, J. W. Lasley, Jr., J. W. LaGrone, C. G. Latimer, R. J. Levit, F. A. Lewis, Lee Lorch, R. A. Lytle, S. W. McInnis, Nathaniel Macon, J. D. Mancill, J. M. Marr, C. F. Martin, W. A. Martin, G. W. Medlin, Benjamin Evans Mitchell, J. C. Morelock, T. F. Mulcrone, W. V. Neisius, Sara L. Nelson, T. A. Newton, J. D. Novak, W. V. Parker, Erna H. Pearson, Lillian G. Perkins, C. G. Phipps, W. W. Rankin, Ellen F. Rasor, G. E. Reves, A. J. Robinson, H. A. Robinson, J. W. Sawyer, Carol S. Scott, C. L. Seebeck, Jr., E. B. Shanks, D. C. Sheldon, C. Eucebia Shuler, C. B. Smith, E. L. Stanley, L. W. Stark, F. W. Stallard, A. L. Starrett, J. M. Thomas, H. E. Taylor, S. L. Thompson, J. C. Thurman, H. S. Thurston, Lila P. Walker, D. T. Walker, J. G. Wall, Susie L. Ward, W. W. Weber, M. C. Wicht, L. H. Williams, W. L. Williams, Ernest Williams, L. E. Williams, R. L. Wilson, Herbert Wolf, G. N. Wollan, R. S. Wollan, F. L. Wren, J. W. Young, B. K. Youse.

At the business session the following officers were elected: Chairman, H. K. Fulmer, Georgia Institute of Technology; Vice-Chairman, W. V. Parker, Alabama Polytechnic Institute; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The Section voted to hold its next meeting in March, 1953, at Alabama Polytechnic Institute, and its 1954 meeting at University of South Carolina.

The program consisted of the following papers:

1. *Coordination of mathematics with industry*, by Professor W. W. Rankin, Duke University.

As chairman, Professor Rankin gave a preliminary report of a committee of twelve mathematicians from industry and six from teaching on the coordination of mathematics with industry.

2. *Historical outlines helpful in teaching mathematics*, by Professor G. E. Reves, The Citadel.

Samples of historical outlines for arithmetic, algebra and geometry were presented. A discussion was given of the usefulness of such outlines in helping the student understand both the development of these subjects and their subject matter. The importance of this approach in motivating the student in the teaching of college mathematics was emphasized.

3. *The merits and content of a freshman mathematics course*, by Professor F. L.

Wren, George Peabody College for Teachers.

Is our traditional program the best program possible in mathematics for college freshmen? There is a need for a more basic course for all people capable of taking a college education. There are certain fundamental postulates which constitute the foundation upon which such a course might be constructed.

4. *Mathematics in the research program of the Ordnance Corps*, by Dr. J. J. Gergen, Chief pro tem, Mathematics Division, Ordnance Research.

For many years a substantial amount of basic research has been carried out at Ordnance research and development installations. In 1950 the Ordnance Corps incorporated into its general program a program of sponsored research under the Office of Ordnance Research. This office is interested in basic research in essentially all mathematical fields. Of special interest are those topics in mathematics which have more immediate bearing on Ordnance related subjects such as ballistics, plasticity, and heat conduction. Proposals by qualified investigators on problems of particular interest to the investigators are encouraged.

5. *Unified mathematics since 1936*, by Professor B. G. Clark, Vanderbilt University.

Ninety-five schools were included in the survey with at least one from each state except one. In 1936, 20 of the schools offered a unified course; in 1941, 31; then perhaps due to the war the number decreased to 18; and for the last three years 21. It is estimated that not more than 7% of all freshmen now are in these courses. The fact that a unified course requires more experienced instructors, and the difficulty experienced when a student transfers has contributed to the hesitancy of most schools to adopt a unified approach to mathematics.

6. *One-sided maxima and minima of functions of two or more variables*, by Dr. J. D. Mancill, University of Alabama and Redstone Arsenal.

Dr. Mancill considered a real single-valued function $z=f(x, y)$ of two real variables, which is continuous on a closed region R, bounded by an ordinary curve C. The extreme values of $f(x, y)$ may be at points interior to R or at points on its boundary C. Necessary and sufficient conditions were determined in order that $f(x, y)$ have an extreme value at (x_0, y_0) on C relative to the region R of admissible points (one-sided extremes). These results were extended to functions of more than two variables.

7. *A representation of an analytic normal operator*, by Professor S. L. Jamison, Florida State University.

The following theorem was proved: If $N(z)$ is analytic and normal for $|z|<\rho$, then there is a self-adjoint operator A and a function $F(z, \lambda)$ such that $N(z)=F(z, A)$ where $F(z, \lambda)$ is analytic in z for each fixed λ and is A-integrable with respect to λ for $|z|<\rho$.

8. *Magic circles*, by Professor S. W. McInnis, University of Florida.

This paper classifies magic circles into simple, compound, and perfect. The development of the last two was demonstrated by the formula $S_r=c+(a+b)x/2$, S_r being the sum of the number along any radius, a the lower bound, b the upper bound, x the number of rings, and c an arbitrary constant.

9. *On the Euler method of summability for double series*, by Professor G. N. Wollen, Memphis State College.

In this paper the Euler method of summability was defined for double series and a few of its

properties discussed.

10. *The exponent of a product mod M*, by Mr. R. S. Wollen, University of Georgia.

The author discussed the known results on a special case of the exponent of a product modulo M and showed they might be extended to include other special cases.

11. *Applications of the stress and strain tensors to orthotropic materials*, by Professor C. B. Smith, University of Florida.

Since in the use of orthotropic material the applied forces may not act always in the direction of the axes perpendicular to the planes of elastic symmetry, the expression of the relations between stress and strain when referred to any orthogonal axes are sometimes needed. By a simple use of the stress and strain tensors, the author readily obtained the desired relations and gave some applications.

12. *The ratio equation of a conic*, by Professor W. L. Williams, University of South Carolina.

By the process of elimination among certain triangular area ratios, Professor Williams showed how the equation of a general conic could be written in terms of these ratios. As an immediate consequence, a theorem on a quadrilateral inscribed in a circle was extended to hold for any pentagon.

13. *On the element of* -1 *dimensions in projective geometry*, by Professor Patrick Du Val, University of Georgia.

A plea was made for the formal introduction in the presentation of projective geometry of a unique element of -1 dimensions through which all elements of the geometry are held to pass and is accordingly the intersection of two elements which in the usual sense do not intersect. This is required by the principle of duality; it is needed to complete the validity of the formula for the dimensions of the join and intersections of two given elements; and if the geometry is developed in terms of homogeneous coordinates the void corresponds to the set $(0, \cdots, 0)$ and if elements are regarded as aggregates of points it is the empty aggregate.

14. *Topological group isomorphisms*, by Mr. W. L. Strother, University of Alabama Center, Gadsden, introduced by the Secretary.

A new proof and a generalization were made of the first and second isomorphism theorems for topological groups as stated, for example, in Pontrjagin's *Topological Groups*.

15. *A note on the rotation of axes*, by Professor R. L. Wilson, University of Tennessee.

Professor Wilson reviewed familiar facts concerning the rotation of axes in analytic geometry as applied to conic sections, and suggested some labor saving devices in connection with the reduction of an equation to the standard form.

16. *Arithmetic—algebra—geometry—each an aid in the study of the other*, by Professor W. W. Rankin, Duke University.

The author exhibited several simple instruments for making the "any" concept more acceptable to students, also some instruments which would give the idea of " parameter" and "function" a more friendly welcome.

17. *Imbedding of normed Abelian groups*, by Mr. R. M. Conkling and Pro-

fessor David Ellis, University of Florida.

Any normed Abelian group which satisfied $\|ma\| = |m| \|a\|$, where m is any integer and a is any element of the group, may be isometrically isomorphically imbedded in a Banach space.

18. *Cross-associativity and essential similarity in double-sfields*, by Professor David Ellis, University of Florida.

In any double-sfield $R(+, *, \#)$, cross-associativity $[a\#(b*c) = (a\#b)*c; \forall a, b, c \epsilon R]$ and essential similarity $[\exists k \epsilon R; a*b = a\#k\#b; \forall a, b \epsilon R]$ are equivalent and each implies isomorphism of the two mutiplications $a\#b$ and $a*b$.

19. *Sylvester's technique for solving the equation $X^2+PX+Q=0$ in binary matrices*, by Professor H. S. Thurston and Miss Mary K. Alexander, University of Alabama, introduced by the Secretary.

Sylvester obtained a cubic equation from which the characteristic equations of all solutions of $X^2+PX+Q=0$ can be obtained. In this paper the special case in which P is a linear function of Q is discussed in detail.

20. *Reduction of a matrix to rational cononical form*, by Professor Lila P. Walker, Woman's College of University of North Carolina.

Comparisons of various methods are given for finding the matrix P such that $P^{-1}AP$ gives the rational cononical form for A. Certain observations are made concerning the minimum functions of the column vectors of the matrix P.

21. *Criteria for normal matrices*, by Professor Benjamin Evans Mitchell, Alabama Polytechnic Institute.

Fourteen necessary and sufficient conditions for a matrix to be normal were listed, including some new ones. Proofs for the latter were given.

22. *The derivatives of Lagrangian coefficients in terms of Vandermonde determinants*, by Professor C. L. Seebeck, Jr., University of Alabama.

Let $V = |(x-x_i)^j|$, where i and j range from 0 to n, be a Vandermonde determinant; V_i^j the cofactor of the element in the i-th row and j-th column; $L_i^j(x) =$ the j-th derivative of $L_i(x) = \prod_{i \neq k}(x-x_k)/(x_i-x_k)$. Then $L_i^j(x) = j!V_i^j/V$.

23. *A system with ternary operation*, by Professor J. C. Currie, Georgia Institute of Technology.

An algebra with one ternary operation was defined axiomatically, the axioms being chosen in analogy to the axioms for a group. Some additional axioms were proposed and a number of simple theorems developed.

24. *Properly primitive indefinite ternary quadratic genera of more than one class*, by Professor E. H. Hadlock, University of Florida.

Let certain restrictions be put on the invariants of a form and let the characters have assigned values. Then a form exists with certain conditions attached to its coefficients. By applying Siegel's method to the form, a method suggested by B. W. Jones, genera containing at least three classes are shown to exist.

25. *A new type of planar birational transformation*, by Emeritus Professor A. B. Coble, University of Illinois (visiting professor, University of North

Carolina).

The conventional method of transforming a curve with compound singularities into one with ordinary nodes is first to reduce the compound singularities to ordinary singularities (multiple points with distinct tangents) by a succession of quadratic transformations each of which may introduce additional ordinary singularities; and then by birational transformations to replace the ordinary singularities by nodes. In this paper a (1,3) transformation set up by a net of conics with a single base point (F-point), and thus only one P-curve, is applied to obtain the first stage in the reduction of a compound singularity at the expense of introducing nothing but ordinary nodes.

26. *An application of Laplace's transformation to a class of linear partial differential equations of the third order*, by Professor R. W. Cowan, University of Florida.

A change of independent variables is made so that the form of the differential equation is simplified. The form of the reduced equation depends upon the roots of a certain cubic equation. Several cases arise depending on the nature of these roots. If the coefficients of the reduced equation satisfy certain conditions, a general solution may be found by solving a Lagrange linear equation.

27. *A note on compound frequency distributions*, by Professor A. C. Cohen, Jr., University of Georgia.

This paper concerns the conditions under which two normal frequency distributions with different means may combine to produce a compound distribution that is unimodal rather than bimodal.

28. *On a class of polynomials*, by Professor F. A. Lewis, University of Alabama.

The purpose of this paper is to present some properties of the polynomials P_n, defined by

$$\frac{d^n}{dx^n}(1-x^2)^{-1/2} = (1-x^2)^{-(2n+1)/2}P_n.$$

29. *Lattices of integers*, by Professor J. M. Thomas, Duke University.

The points of the plane whose Cartesian coordinates are integers form a lattice. A necessary and sufficient condition that a lattice for an oblique system be also a lattice for a rectangular system is found.

30. *A finite formula for the Bernoulli numbers*, by Professor E. B. Shanks, Vanderbilt University.

By use of complete induction, it is shown that the determinant A_j given by the author in an article "Iterated sums of powers of the binomial coefficients," this MONTHLY, Vol. 58, No. 6, p. 405 has the value

$$A_j^k = \sum_{r=0}^{j-1} (-1)^r \binom{ik+1}{r} \binom{i+j-1-r}{i}^k.$$

If we now set $i=1$ in the latter formula and $i=p=1$ in formula (6) of the same article, then determine the coefficient of n, we obtain as a finite formula for the k-th Bernoulli number

$$B_k = \sum_{j=1}^{2k} (-1)^{j+1} A_j^{2k} (2k+1)^{-1} \binom{2k}{j-1}^{-1}.$$

31. *Expected increase in earning due to college training*, by Professor S. L. Thompson, Alabama Polytechnic Institute.

If one considers college training as only a financial investment that may be regained by increased earnings, he can express his potential annual earning E_x as a function of his age x, his number of years of college training, retirement age, probabilities of living each of the years until retirement, college fees paid, interest rate on investment, relative values of investments at different times and the average annual wages received by persons without college training. The fact that each year of college training represents an investment of a year of potential earning, gives the function E_x somewhat of a recursion relationship.

32. *Integral rational functions and irrational roots of integral rational equations in freshman algebra*, by Professor H. B. Hoyle, Jr., Queens College.

A method was presented for investigating algebraic functions and for finding irrational roots of an algebraic equation utilizing such parts of calculus and theory of equations as may be brought within the easy grasp of college freshmen.

33. *Remarks on radical equations*, by Professor G. B. Huff, University of Georgia.

This paper was published in this MONTHLY, vol. 59, p. 320.

34. *A geometric discussion of certain radical equations*, by Professor D. F. Barrow, University of Georgia.

This paper was published in this MONTHLY, vol. 59, p. 320.

35. *Partial differentiation in elementary calculus*, by Professor L. A. Dye, The Citadel.

A closely integrated treatment of the topics usually included in the theory of partial differentiation is developed without the use of infinitesimals or the Mean Value Theorem.

36. *A note on exact linear differential equations*, by Professor W. V. Neisius, Georgia Institute of Technology.

The principle of exact linear differential equations was discussed and a simple method of solving was shown which enables most students to test for exactness by inspection and, if exact, to write down the answer immediately.

37. *Matrices and polynomials*, by Professor W. V. Parker, Alabama Polytechnic Institute.

In this paper the "companion" matrix for a monic polynomial is considered. A proof of the Hamilton-Cayley Theorem is given for this matrix. Several other theorems concerning polynomials in this matrix are given.

38. *A heat flow problem*, by Mr. O. R. Ainsworth, University of Alabama, introduced by the Secretary.

Mr. Ainsworth presented a corrected solution to a heat flow problem which he felt was improperly reported in the January *Mathematical Reviews*.

H. A. ROBINSON, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-second annual meeting of the Southern California Section of the Mathematical Association of America was held at Occidental College, Los Angeles, on Saturday, March 8, 1952. Mr. William H. Glenn, Chairman of the Section, presided.

The attendance was one hundred fifteen, including the following eighty members of the Association:

O. W. Albert, Florence R. Anderson, T. M. Apostol, L. A. Aroian, H. M. Bacon, Leon Bankhoff, R. S. Barton, May M. Beenken, Clifford Bell, L. T. Black, Gertrude Blanch, H. F. Bohnenblust, Frances L. Campbell, L. M. Coffin, E. P. Coleman, P. H. Daus, Robert Davies, R. P. Dilworth, D. C. Duncan, H. P. Edmundson, W. V. Gamzon, F. W. Gibson, W. H. Glenn, M. L. Goldwater, C. J. A. Halberg, H. J. Hamilton, V. C. Harris, Robert Hartranft, A. F. Herbst, R. B. Herrera, M. R. Hestenes, P. G. Hoel, J. M. Howell, D. H. Hyers, P. B. Johnson, G. R. Kaelin, Rosella Kanarik, L. C. Lay, D. H. Lehmer, J. W. Lindsay, Sophia L. McDonald, Fred Marer, F. R. Morris, W. A. Nielsen, P. M. Niersbach, L. J. Paige, D. B. Perry, D. J. Peterson, W. T. Puckett, C. A. Pursel, H. R. Pyle, E. C. Rex, J. M. Robb, E. B. Roessler, J. M. Sandy, I. J. Schoenberg, G. E. F. Sherwood, Sister Rose Gertrude, Samuel Skolnik, James C. Smith, R. H. Sorgenfrey, I. S. Sokolnikoff, Robert Steinberg, J. D. Swift, T. E. Sydnor, Anna L. Taylor, D. L. Thomsen, Elmer Tolsted, C. W. Trigg, S. E. Urner, F. A. Valentine, D. D. Wall, Maria Weber, J. G. Wendel, R. L. White, A. L. Whiteman, Lois B. Whitman, B. R. Wicker, F. M. Yanari, E. M. Zaustinsky.

At the business meeting the following officers were elected for the next academic year: Chairman, C. W. Trigg, Los Angeles City College; Vice-Chairman, A. E. Taylor, University of California at Los Angeles; Secretary-Treasurer, P. H. Daus, University of California at Los Angeles; Program Committee: L. Clark Lay (Chairman), John Muir College; T. M. Apostol, California Institute of Technology; A. L. Whiteman, University of Southern California.

The next meeting was scheduled for March 14, 1953, at Los Angeles City College.

The following program was presented:

1. *The uniform step method for magic squares*, by Professor T. M. Apostol, California Institute of Technology.

This paper has appeared in the Proceedings of the American Mathematical Society, vol. 2 (1951), pp. 557–565, under the title *On Magic Squares Constructed by the Uniform Step Method.* The paper was written jointly with H. S. Zuckerman of the University of Washington.

2. A panel discussion: *What can collegiate mathematics contribute to the education and training of high school teachers?* Participants: Mr. Dale Carpenter, Los Angeles City Schools, introduced by Professor P. H. Daus; Dr. D. C. Duncan, East Los Angeles Junior College; Professor H. J. Hamilton, Pomona College.

Mr. Carpenter suggested the following procedures to college mathematics teachers: development of a specific plan for helping train mathematics teachers; a definite way of cooperating with the education and teacher training departments; cooperative planning directed toward the reduction of the number of hours now required for a teaching major; cooperation with secondary teachers in planning institutes, conferences, workshops; visitation of secondary school mathematics classes; cooperation with high school teachers in revising and improving mathematics content and

method at the secondary level; formulation and practice of good instructional procedures for college mathematics classes; familiarity with the applications of mathematics in consumer mathematics, business education, industrial arts, chemistry, physics; provision of courses pertaining to research in mathematics curricula, and mathematics instruction, methods of teaching arithmetic, general mathematics, algebra, geometry and trigonometry, arithmetic content, and basic relationships of arithmetic, algebra, geometry and trigonometry.

To enable and to encourage high school mathematics teachers to widen their mathematical culture, Dr. Duncan suggested that the local universities offer degrees of D.M. (Doctor of Mathematics) and D.M.S. (Doctor of Mathematical Sciences), upon the completion of 75 units of mathematics, or of mathematics and allied subjects, accumulated with grades of A or B within a period of 15 years. This suggestion was referred to the Education Committee of the Southern and Northern California Sections for study.

Professor Hamilton suggested that the Association would do well to define the teaching *minor* in mathematics as well as the teaching major; also, to recommend that potential high school teachers take courses in the history of mathematics, the teaching of mathematics and (perhaps in high school itself) a physical science, descriptive geometry, and public speaking. He urged college teachers to employ good teaching techniques.

3. *Machine development and research in pure mathematics*, by Professor D. H. Lehmer, Institute for Numerical Analysis, National Bureau of Standards.

The present day development of computing machinery was discussed from a purely mathematical point of view. The advantages and disadvantages of certain characteristic features of machines as applied to problems in pure mathematics were treated in some detail. The possible future influence of computing machinery upon research mathematics and mathematical points of view was discussed briefly.

4. *On bell-shaped curves*, by Professor I. J. Schoenberg, Institute for Numerical Analysis, National Bureau of Standards.

Let $f(x)$ ($\not\equiv 0$) be defined for all real x, be infinitely often differentiable and such that $f^{(n)}(x) \to 0$, as $x \to \pm\infty$, for $n = 0, 1, \cdots$. Let N_n denote the number of changes of sign of $f^{(n)}(x)$. Rolle's theorem readily implies that $N_n \geqq n$ for all values of n. The curve $y = f(x)$, and also the function $f(x)$, is called *bell-shaped* if $N_n = n$ for all values of n. The bell-shaped function $f(x)$ is said to be *bell-shaped in the interval* (a, b) provided $f(x) = 0$ outside (a, b). While many examples of functions are known which are bell-shaped in $(-\infty, \infty)$, $[\exp(-x^2), (1+x^2)^{-1}]$, or in $(0, +\infty)$, $[f(x) = x^{-1}\exp(-x^{-1})$ if $x > 0, f(x) = 0$ if $x < 0]$, I. I. Hirschman has proved the following theorem: *There exist no functions $f(x)$ which are bell-shaped in a finite interval (a, b)*. Hirschman's elegant and elementary proof (Proceedings American Mathematical Society, vol. 1 (1950), pp. 63–65) was presented.

5. *Experiences teaching mathematics in Japan*, by Professor Frances L. Campbell, George Pepperdine College.

The following observations were made from 1948 to 1950 while the speaker was teaching at Ibaraki Christian College, a small college in rural Japan.

Scientific studies are, with a few exceptions, about 20 years behind the present developments in the States. The students of mathematics seem to be well prepared in the elementary courses, with emphasis on computation. Formulas of courses also emphasize computational ability and rote learning, influenced perhaps by Japan's earliest contribution to mathematics, a well developed system of computation called *Wasan*. The Japanese in general have the ability but not the initiative to be creative. There is a national organization for mathematicians and teachers of mathematics. Most school libraries have few recent mathematical publications in English. The entire educational program is in a state of transition under the impact of the occupation. The old system

was very rigid with very strict entrance examinations and a set course of study, allowing no transfer of credit between courses or between universities. The new system is an attempt to integrate modern trends of the West with the good points of the old system. Accreditation to teach in Japanese colleges requires an advanced degree but only those subjects may be taught in which the prospective teacher has had articles published, even though the field may differ from that of the degree.

6. *Semi-groups and Laplace transforms*, by Professor R. S. Phillips, University of Southern California, introduced by Professor D. H. Hyers.

Let X be a Banach space and let $T(s)$ be a one-parameter family of linear bounded transformations on X to itself. If $T(s_1+s_2)=T(s_1)T(s_2)$, then these operators form a semi-group of transformations. It is further assumed that $\lim_{s\to 0}T(s)f=f$ for each $f\epsilon X$. If further f belongs to the domain $D(A)$ of the infinitesimal generator A, then $\lim_{\Delta\to 0}(T(s+\Delta)f-T(s)f)\Delta^{-1}$ exists and is equal to $AT(s)f$. The theory of semi-groups is applicable to partial differential equations in the following way. Consider for example $u_s=u_{xx}$, $-\infty<x<\infty$, $s\geqq 0$ and $u(0, x)=f(x)$, where say $f\epsilon L_1(-\infty, \infty)$. Then there exists a semi-group of operators such that $u(s, x)=T(s)f$. To establish the existence of this type of solution it is sufficient to show, according to a theorem due to Hille and Yosida, that $\|R(\lambda; A)\|\leqq(\lambda-\omega)^{-1}$ for $\lambda>\omega$. Here $R(\lambda; A)$ is the inverse of $(\lambda I-A)$. In general this criterion is not too difficult to verify.

7. *Geometric background of some theorems of Hardy, Littlewood, and Polya*, by Dr. Seymour Sherman, Lockheed Aircraft Corporation, introduced by Professor M. R. Hestenes.

The Muirhead theorem in Hardy, Littlewood and Polya's *Inequalities* concerning real numbers and its analogue for general vector space is related to the duality theory of convex polyhedral cones in finite dimensional spaces.

P. H. DAUS, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The thirty-ninth annual meeting of the Iowa Section of the Mathematical Association of America was held in conjunction with the annual meeting of the Iowa Academy of Science at Coe College, Cedar Rapids, Iowa, on April 18–19, 1952. Rev. L. E. Ernsdorff, Chairman of the Section, presided.

A total of forty-three persons registered their attendance including the following twenty-two members of the Association:

E. R. Bowersox, F. A. Brandner, E. L. Canfield, J. O. Chellevold, N. B. Conkwright, W. M. Davis, L. E. Ernsdorff, A. M. Feyerherm, B. E. Gillam, Cornelius Gouwens, J. J. L. Hinrichsen, Don Kirkham, C. H. Lindahl, R. B. McClenon, J. V. McKelvey, E. E. Moots, Fred Robertson, F. M. Stein, E. C. Stopher, H. P. Thielman, Roscoe Wood, and C. C. Wylie.

Officers elected at the meeting are: Chairman, L. A. Knowler, State University of Iowa; Vice-Chairman, J. O. Chellevold, Wartburg College; Secretary-Treasurer, Fred Robertson, Iowa State College. The next meeting is to be held at Cornell College, Mt. Vernon, Iowa, on April 17–18, 1953.

Professor H. P. Thielman was granted the award for the most meritorious paper. His paper is in competition with similar papers from the other sections of the Iowa Academy of Science for a fifty dollar prize.

The following papers were presented:

1. *On the zeros of* $J_n(x_p)\,Y_m(kx_p) - J_m(kx_p)\,Y_n(x_p)$, by Mr. Don Kirkham, Iowa State College.

Explicit expressions for kx_p as a function of x_p are derived for the indicated equation, in which J_n and Y_m are the first and second kind of Bessel functions of order n and m respectively, k is a parameter and p gives the order of the root x_p. Similar expressions are derived when the indicated equation is modified so that the inner two terms or the outer two terms or all four terms are primed, that is, differentiated with respect to the argument. The derived expressions are asymptotic and yield better values of kx when $|m|$ is small and x_p and p are large. The derivations utilize methods of Stokes and MacMahon (See Watson's *Bessel Functions*, p. 505). The results differ from MacMahon's expressions in that values $x_p=0$ are not excluded. In fact, when $m=n=0$ and $p=1$, kx_p is given to an accuracy of better than 1 part in 1500 by the formula when $x=0$. For cases when the inner and outer terms are primed, some earlier sets of roots are found than heretofore mentioned in the literature. For these roots, which may be termed zeroth order roots, the analytical expressions derived above are not valid for $x_p=0$. For calculation of the zeroth order roots graphical methods are presented.

2. *Laguerre series in the complex domain: The representation problem*, by Professor Nelson Yeardley, Iowa State College, introduced by the Secretary.

Let $\{L_n^{\alpha}(x)\}$ $(n=0, 1, 2, \cdots; \alpha > -1)$ represent a sequence of Laguerre polynomials of order α and degree n. To the function of a complex variable $f(z)$, $(z=x+iy)$, let correspond the general Fourier-Laguerre series of $f(z)$:

$$f(z) \sim \sum_{n=0}^{\infty} a_n^{\alpha} L_n^{\alpha}(z) \qquad (\alpha > -1)$$

where a_n^{α} represent the Fourier-Laguerre coefficients. If $d_{\alpha} = -\limsup (2n^{1/2})^{-1} \log |a_n^{\alpha}|$, then for $d_{\alpha}>0$ the general Fourier-Laguerre series of $f(z)$ converges absolutely inside and diverges outside the parabola of convergence $p(d_{\alpha})$: $y^2=4d_{\alpha}^2(x+d_{\alpha}^2)$.

The author proves the following theorem: *If the general Fourier-Laguerre series of* $f(z)$ *converges at a point* $z_0=x_0+iy_0 (y_0 \neq 0)$ *then to every* $b_0 (0<b_{\alpha}<d_{\alpha})$ *there corresponds a positive number* B (α, b) *such that for all points* z *(with* $|z|$ *large) inside the parabola of convergence* $p(d_{\alpha})$ *of the general Fourier-Laguerre series of* $f(z)$

$$\left|\sum_{n=0}^{\infty} a_n^{\alpha} L_n^{\alpha}(z)\right|^2 < B(\alpha, b)\,\frac{\sqrt{2}}{16}\,\frac{\Gamma(\alpha+1)}{q^{3/4}}\,\frac{x^{1/4}}{|z|^{\alpha+1/2}}\,e^{x-2\sqrt{qx}}\left(1+\frac{a_1}{x^{1/2}}+\frac{a_2}{x^{2/2}}+\cdots\right)$$

where

$$q = c_{\alpha}^2 - \tfrac{1}{2}(r-x); \qquad r = |z|; \qquad c_{\alpha} = \tfrac{1}{2}(b_{\alpha}+d_{\alpha}).$$

3. *Conjugate points of singular quadratic functionals for N dependent variables*, by Professor J. O. Chellevold, Wartburg College.

Morse and Leighton (*Singular quadratic functionals*, Trans. Amer. Math. Soc., vol. 40, 1936 pp. 252–286) gave a systematic approach to the problem of minimizing a singular quadratic functional for one dependent variable. They defined the first conjugate point of the singular point $x=0$ and determined the analogue of the Jacobi condition. In this paper the definition of the conjugate point is extended to n dependent variables and resulting theorems are proved. The analogue of the Jacobi condition as a necessary condition for a minimum limit for a class of curves defined as C-admissible is also extended to n dependent variables.

4. *Probability of success in business mathematics*, by Professor E. L. Canfield,

Drake University.

Predicting achievement in a business mathematics course at Drake University involves use of a test designed to be given at the beginning of a semester. Test scores for 118 students were used to develop a discriminant function under the assumptions of biserial correlation, which function, in the case of one variable, may be expressed in the form: bx. The coefficient b may be found by solving the equation $D=b\sum x^2$, where D is the difference between the means of the test scores in the unsuccessful and successful groups of business mathematics students, and the right-hand member of the equation is the same as in regression analysis using deviation values from the general mean.

By using an appropriate transformation of the function, a value in sigma units can be computed for any particular student, given his predictive test scores, that yields a probability upon consulting a table of the normal curve. A tabulation of such probabilities may be used in advising a student of his chances for success in business mathematics. Such probabilities might be developed using several variables, as in multiple regression.

5. *The application of overlapping Pfaffians to utility theory*, by Professor C. H. Lindahl, Iowa State College.

A sub-set of the integrability conditions is imposed upon overlapping sub-Pfaffians of length greater than two, and the coefficients restricted sufficiently to make feasible an analysis of common class two. It is proved that these conditions imply exactness when the coefficients are polynomials of the first and second degree or general functions under a specific restriction.

Properties of the above Pfaffians are applied to utility theory in economics. The distinction between integrability and non-integrability is considered as a distinction between rational and irrational behavior on the part of the consumer. Make the general hypotheses: (a) a consumer will be rational in an economic transaction if the number of variables involved is sufficiently, but not trivially, restricted, and (b) a consumer cannot only order the variables of the above transaction and therefore group them in harmony with such an order but there exists an ordering such that he can deal rationally within the groups provided the size of the groups is less than four. Then the following conclusions are drawn: (a) the consumer's utility Pfaffian in general is not only integrable, in the large, but is exact, and (b) the implication is that the behavior of the consumer, in the large, is rational in general.

6. *A note on autocorrelation for an oscillating series*, by Professor F. A. Brandner, Iowa State College.

For oscillating series it is difficult to find what periods exist, or whether any definite periodicity prevails. A number of methods have been used to gain information for that type of a time series curve. Among those methods are two which have proved of greater use than any of the others, autoregression and serial correlation. However, the two methods are used separately and have not been interrelated.

In this paper a method of finding autocorrelation by use of the autoregression equation is given. The results are summarized as to what was accomplished, with its advantages and disadvantages.

7. *Equivalent functions of strategies in the theory of games*, by Mr. Harlan Mills, Iowa State College, introduced by the Secretary.

By use of the restrictions on the strategy vectors, x, y, in the bilinear payoff form, xAy, we obtain a relation which holds for every pair of strategies, x, y,

$$xAy = uAv + (x - u)B(y - v),$$

where u, v are functions of the coefficients of A, and B is a somewhat arbitrary linear combination of matrices derived from the coefficients of A. If A is square and non-singular, the condition

for existence of the form is that the sum of the cofactors of A be non-zero.

When $u^i \geqq 0$, all i, and $v^j \geqq 0$, all j, u and v are optimal and uAv is the value of A. B has sufficient freedom to include any arbitrary pair of strategies in its own optimal sets, and have arbitrary value. B also includes A. When a Taylor expansion exists with all first derivatives zero, it is included in the form above.

8. *Distribution of Kronecker products*, by Mr. A. M. Feyerherm, Iowa State College.

Let $K = B \times C$ where K, B and C are square matrices with real elements and $(\times)$ denotes the Kronecker product of B and C. The elements of a particular $K_{n\times n}$ may be considered as the coordinates of a point in n^2 dimensional space. The distribution of the points representing $K_{n\times n}$'s is considered in this paper. Insight into the density of such points is obtained by finding the K or K's which minimize the tr $(A-K)(A-K)'$ where A is a given matrix which is not an exact Kronecker product. Results are derived for a number of specific forms of A and K.

9. *Pathological functions*, by Professor H. P. Thielman, Iowa State College.

Let f be a function defined on X onto Y. Let P be a point property of the function f. The function f is said to be peculiar or pathological with respect to the property P if there exists a partition of X into two subsets X_1 and X_2 each of which is everywhere dense in X, and the property P holds at every point of X_1 and fails to hold at every point of X_2. Examples were given of functions which were pathological with respect to the properties of continuity, neighborliness and differentiability. For the definitions of neighborliness of a function see an article by W. W. Bledge; (Proc. Amer. Math. Soc., vol. 3, 1952, pp. 114–115). Cliquishness of a function was defined as follows: Let the domain of definition of a function f be a neighborhood space X with neighborhoods N_x. Let the range of the function f be a metric space with metric P. Then the function f is said to be cliquish at a point x_0 if for every neighborhood N_{x_0} of x_0 and for every positive number ϵ there exists a neighborhood N_x contained in N_{x_0} (but not necessarily containing x_0) such that for every two elements x_1 and x_2 of N_x it is true that $P(f(x_1), f(x_2)) < \epsilon$. It was proved that there exists no function which is pathological with respect to cliquishness. The concept of neighborliness of a function was extended to functions whose domains of definition and ranges are general neighborhood spaces. Among the results on cliquish functions were the following: The set of points where a pointwise cliquish function is cliquish is nowhere dense. The limit of a sequence of cliquish functions may have no points of cliquishness, and hence may be totally discontinuous. Every cliquish function is at most pointwise discontinuous.

10. *Regularization of systems of differential equations*, by Professor J. J. Hinrichsen, Iowa State College.

In this paper the transformation method applied so successfully by Levi Civita (*Acta Mathematica*, Vol. XXX (1906)) and G. D. Birkhoff (*Rendiconti del Circolo Matematico di Palermo*, Vol. XXXIX (1914)) in regularizing the equations of motion of the restricted problem of three bodies, is used to regularize the equations of motion of certain other dynamical problems.

11. *The path and orbit of the Oklahoma detonating meteor of November 7, 1951*, by Professor C. C. Wylie, State University of Iowa.

Throughout the week of November 18–25, 1951, the author, assisted most of the time by Professor B. S. Whitney of the University of Oklahoma, interviewed observers of this meteor, having them stand exactly where they were when the meteor was seen and measuring the angles of the path as recalled. Some additional measures were made by O. E. Monnig of Fort Worth, Texas.

From these measures and from stop watch timings of "reenactions of the scene," the path through the atmosphere and the orbit in which the meteor had been traveling about the sun have been computed.

12. *An experiment in teaching algebra and trigonometry*, by Professor Fred Robertson, Iowa State College.

The author discussed an experiment in teaching in which an attempt was made to teach the usual material in college algebra and trigonometry in one quarter instead of the customary two. He described the methods used and gave some of the results in statistical form. The results indicate the older method is the preferred one.

13. *Notes on the mean value theorem*, by Reverend L. E. Ernsdorff, Loras College.

Very often students who study the mean value theorem are puzzled about how the usual function $F(x)$ is derived. Geometric ways of deriving the function which is the difference between the ordinate and chord on the curve were explained.

14. *An aid to teaching the metric system*, by Professor Boyd Henry, Parsons College, introduced by the Secretary.

The author described the method of constructing a motion picture accompanied by sound which he uses to teach the metric system of weights and measures. The cost is not prohibitive since the total outlay for material was $12.50. The film was run and the section saw it from the viewpoint of the students in class.

FRED ROBERTSON, *Secretary*

THE APRIL MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky, Lexington, on April 19, 1952. Dr. J. A. Ward, Chairman of the Section, presided at the morning and afternoon sessions.

There were seventy-six persons in attendance, including the following thirty-six members of the Association:

H. H. Berry, M. C. Brown, R. C. Brown, H. W. Burnette, Esther A. Compton, H. H. Downing, R. I. Fields, Clarence Ford, A. W. Goodman, Beulah Graham, Charles Hatfield, Aughtum S. Howard, Tadeusz Leser, A. G. McGlasson, W. L. Moore, E. J. Musch, R. S. Park, Sallie E. Pence, Mary Pettus, D. W. Pugsley, Sara L. Ripy, G. G. Roberts, W. J. Robinson, Sister M. Rosalin, D. C. Rose, F. E. Ross, W. C. Royster, D. E. South, Guy Stevenson, R. P. Tapscott, J. T. Vallandingham, J. A. Ward, R. H. Wilson, H. A. Wright, R. C. Yates, W. M. Zaring.

Officers elected at the meeting were: Chairman, Professor W. L. Moore, University of Louisville; Secretary, Professor J. A. Ward, University of Kentucky.

The annual meeting in 1953 is to be held at the University of Louisville.

The following papers were given:

1. *The development of operational calculus*, by Professor Tadeusz Leser, University of Kentucky.

The author described briefly the Heavyside operational calculus and mentioned the earlier operational methods. The Heavyside expansion formula for $1/\phi(p)$ operating on the unit function $H(t)$ was justified through the Laplace transformation. Attention was drawn to the fact that the Heavyside operator p with rational exponents positive or negative is not justified as yet. The out-

line of the present state of the operational calculus, based on the Laplace transformation, was also presented.

2. *A note on the teaching of plane analytic geometry*, by Professor Aughtum S. Howard, Kentucky Wesleyan College.

As an aid in curve sketching in plane analytics, it is suggested that $y=f(x)$ and $y=1/f(x)$ be consistently paired for study when ever feasible. By drawing the lines $y=1$ and $y=-1$, a visual demonstration of the relationship between the zeros of $f(x)$ and the vertical asymptotes of $y=1/f(x)$ is possible. In polar coordinates, consideration of $r=1$ along with a pairing of $r=f(\theta)$ and $r=1/f(\theta)$ shows graphically the relationship between the zeros of $f(\theta)$ and the oblique asymptotes of $r=1/f(\theta)$. This approach also makes possible an elementary introduction to the study of inversion.

3. *Ceva's point for a problem in geometry*, by Professor W. J. Robinson, Centre College.

The following problem was considered. For a given triangle ABC, find point O such that $OA:OB:OC=r_1:r_2:r_3$. The solution of this problem depends on the location of Ceva's point P. An analytic derivation of the locus of P yields four conics. Each of these passes through A, B, and C. No two intersect otherwise. They are tangent by twos at A, B, and C and the tangent lines are the angle bisectors of triangle ABC.

4. *Drawings illustrating the problem of Apollonius*, by Professor H. A. Wright, Transylvania College.

After a brief description of methods of constructing circles tangent to three given circles, a series of large scale drawings was exhibited. The drawings consisted of twelve relative positions of the given circles with all possible (zero to eight) tangent circles constructed. The actual constructions were done by a student, Mr. D. C. Rose, as a project in college geometry.

5. *The use of calculus for solving heat problems*, by Professor R. H. Wilson, University of Louisville.

Integration concepts essential to chemical thermodynamics were exemplified as most cogent reasons for the study of calculus by chemistry majors. In the relation $\Delta H=\int_{T_1}^{T_2} C_p dT$, H represents the heat content, C_p the heat capacity constant pressure, which is a function of T, the absolute temperature of the given mass. Such an integral may add understanding even to the homely process of cooking, where, since the top limit on T is usually a definite boiling point, the degree of cooking is a simple function of the more easily measured element, time. In extreme-temperature chemical reactions, the top limit on T may be the most inaccessible quantity in this relation, which thus becomes an integral equation. Such was the case for the hydrogen-fluorine flame, recently investigated by the speaker as consultant of the Research Institute of Temple University. For this extreme temperature, 4300° K, the problem is complicated by the necessity of considering, simultaneously with the integral equation, the relations giving equilibrium constants for the three reversed reactions, as functions of the unknown partial pressures of the three substances thus formed. A trial-and-error solution indicated the world's highest flame temperature, which was later confirmed experimentally only by direct comparison with the sun.

6. *Giant brains*, by Professor D. W. Pugsley, Berea College.

The author became interested in so called "giant brains" while studying at the Watson Scientific Computing Laboratory, New York City, when on sabbatical leave. In this paper he listed some of the calculators that are in operation or under construction, with brief statistics about them. The several types of problems solvable by such machines were described. Then followed a discussion of several systems of numbers commonly employed for computations in giant

brains, namely, the binary, the bi-quinary, and the octal number systems. The author concluded with a statement of the prospects for the future, both as to machines and personnel, and with a plea to other teachers to encourage promising students to investigate this field.

7. *The straight line*, by Professor A. W. Goodman, University of Kentucky.

Using the Zermelo well-ordering theorem, discontinuous solutions of the functional equation $f(x+y)=f(x)+f(y)$ can be constructed in the manner of G. Hamel, *Math. Ann.* vol. 60, 1905, pp. 459–462.

8. *The place of mathematics in the life adjustment program*, by Sister M. Matilda, Loretta Junior College, introduced by the Secretary.

From the very nature of the life-adjustment program no definitive place can be assigned to any particular subject, but as the study of mathematics in some form or other prepares for intelligent and happy daily living for all, it should rank high in the curriculum. Gauging space, size, measure, distance come constantly into daily life as also do buying, selling, computing, accounting. In business or in leisure, in urban or in rural living, at any age, all need mathematics; hence its importance. Steam, internal combustion, hydroelectric and atomic power projects call for experts with upper-bracket scientific education and technical training based on highly specialized mathematical foundations, but this concerns the few. What of the others? Besides its practical usefulness mathematics exerts a disciplinary influence on the mind, training to habits of thought that carry over into other areas, and, if properly taught even to those on the lower levels, it can afford pleasure and entertainment, and can lead to appreciation of higher things and provide wholesome and worthy use of leisure time.

9. *Measure theory*, by Mr. F. E. Ross of Lybrand, Ross Brothers and Montgomery, Louisville.

After stating the problem of measure, the more significant theories which endeavor and purport to find its solution are discussed in historical sequence. Thus, there are defined the theories due to Hankel-Stolz-Harnack, Cantor, Peano-Jordan, Borel, Lebesgue, Caratheodory and Haar, and theorems are proven regarding these measures and measure-systems. The properties stressed in this presentation give the extent of the class of sets measurable under these various theories and attempt to clarify the relationships among the measures discussed.

10. *The probability of x successes in n trials*, by Mr. R. E. Wheeler, University of Kentucky, introduced by the Secretary.

This paper is concerned with finding the probability of x successes in n trials, $P(n, x)$, where the probability of success on a single trial varies depending on the number of the trial and on the number of previous successes. A probability function is algebraically formulated and $P(n, x)$ is obtained in terms of this function. The Bernoulli probability is considered to illustrate the general procedure.

11. *Models, methods and the stimulation of interest in the teaching of college mathematics*, by Dr. R. C. Yates, United States Military Academy.

A nucleus of ideas intended to stimulate and maintain interest in various phases of high school and college mathematics was presented. Items discussed in connection with algebra, geometry, analytics, calculus, mechanics, and optics included: (1) the crossed parallelogram linkage; (2) the ladder and rolling circles; (3) roses and the hypocycloids; (4) maximum-minimum solutions by elementary methods; (5) caustics and their reflecting curves; (6) l'Hospital's rule; (7) asymptotes and tangents at the origin; (8) sums of powers of integers; (9) normal and tangential acceleration; (10) Euler's exponential formula.

A variety of models and diagrams were used to emphasize these ideas and to give point to the general theme. It was recommended that a small laboratory space be established and supplied with inexpensive materials for classroom activities to foster and encourage the spirit of "to have and to hold."

12. *A study of caustic curves*, by Mr. A. G. McGlasson, Eastern Kentucky State College.

A caustic curve is the envelope of light rays emitted from a radiant point source after reflection or refraction by a given curve which may be considered as the boundary separating two optical media. The idea of caustics originated with Huygens and Tschirnhausen about 1678. It was remarked that the actual determination of the caustic of a general plane curve often presents great difficulty but that the calculation can be performed for simpler curves. Gergonne has shown that each caustic is the evolute of an algebraic curve and it was noted that it is often more convenient to express the caustic in this manner. Some of the more interesting caustics were discussed and illustrated.

13. *Failures caused by fear*, by Sister M. Fides, St. Catharine Junior College, introduced by the Secretary.

Many failures of mathematics students are caused, not by lack of ability, but by fear of the subject or some phase of it. Considered as a disease, a term, mathemaphobia, is coined for this attitude. Causes of mathemaphobia are: (1) unpleasant childhood experiences connected with the subject, (2) absences from school, causing a gap in successive stages of development on which satisfactory performance of later stages depend, (3) faulty teaching methods in which stress is placed on practice rather than on understanding of techniques, (4) failure to make the effort to master certain phases of the subject, and (5) practice of teachers whereby skills are shown without any explanation of the application and thought necessary to acquire such skills. Some suggestions for diagnosis and treatment of mathemaphobia are drawn from a long teaching experience. These are, briefly: (1) getting the student to look at his own case objectively and work toward a cure, (2) helping a student to bridge the gap after absences, (3) emphasis on teaching *why* rather than *how*, (4) recognition of those certain areas which cause fear with special patience and understanding of students in these difficulties, and (5) ability on the part of teacher to show that skills are not acquired ready-made but are purchased by thought and application.

14. *On the product representation of sin* πx, by Professor V. F. Cowling, University of Kentucky.

In this paper the product representation of sin πx is developed without the aid of the theory of functions of a complex variable.

J. A. WARD, *Secretary*

THE APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Virginia Military Institute, Lexington, Virginia, on April 26, 1952. Professor E. R. Sleight, Chairman of the Section, presided at the morning and afternoon sessions.

There were sixty-five persons in attendance, including the following thirty-four members of the Association:

W. E. Byrne, H. H. Campaigne, G. R. Clements, Georgie T. Davis, L. R. Ford, Herta T. Freitag, C. H. Frick, F. S. Goepper, Michael Goldberg, E. S. Grable, M. Gweneth Humphreys,

J. E. Ikenberry, Mary A. Lee, Florence P. Lewis, R. H. Kasriel, F. B. Key, Ella C. Marth, F. H. McGar, Jr., Carol V. McCamman, C. G. Maple, T. W. Moore, E. K. Paxton, K. S. Purdie, L. V. Robinson, W. G. Saunders, Veryl G. Schult, E. R. Sleight, R. E. Smith, R. F. Steward, Mildred E. Taylor, Marian M. Torrey, R. E. Walters, P. M. Whitman, G. T. Whyburn.

During the business meeting the following officers were elected: Chairman, Professor Marian M. Torrey, Goucher College; Vice-Chairman, Professor M. Gweneth Humphreys, Randolph-Macon Woman's College; Vice-Chairman, Professor Mildred E. Taylor, Mary Baldwin College; Secretary, Professor C. H. Frick, Mary Washington College.

It was announced that the next meeting of the Section will be held at Howard University, Washington, D. C., on December 6, 1952. The Section voted to amend the By-Laws of the Section by replacing the words "additional members" by "Vice-Chairmen" on line two in section one of Article III. It was further voted that the Committee on proposals for future activities of the Section be thanked for their efforts and that authority be given to proceed with steps that might be necessary.

The following papers were presented at the meeting:

1. *The partial differential equations of visco-elastic materials*, by Professor R. E. Smith, College of William and Mary.

The partial differential equations of visco-elastic media and the solution of certain boundary value problems were considered. The materials considered were isotropic. Both compressible and incompressible materials were considered giving Alfrey's incompressible results as a special case. Changes of state were isothermal and only small strains were considered. The stress-strain relations used were

$$Ps_{ik} = 2Qe_{ik}, \qquad \sigma = K\epsilon, \text{ where } K \text{ is a constant.}$$

The s and e are the deviatoric stress and strain, σ and ϵ are the mean stress and strain, and P and Q are differential operators in t.

2. *Some novel sequences*, by Professor E. S. Grable, University of Richmond.

The method of iteration, commonly used to approximate roots of an equation, was shown to be a source of unusual sequences when applied to suitably chosen equations with known roots. In particular the equation $X=\sqrt{2^x}$ leads to a sequence, involving repeated extraction of roots, whose limit is 2, 4, or ∞ depending upon the initial value of X used. The equation $aX^2-bX=0$ in the form $X=\sqrt{(a/b)X}$ leads to another sequence involving the repeated extraction of roots. The equation $X=\int_0^x t^{-1/2}dt$ leads to a sequence involving repeated integration. Functions defined in terms of the limits of these sequences were considered.

3. *Solution of a most general form of the Clairaut equation by differential operator methods*, by Dr. L. V. Robinson, Aberdeen Proving Ground.

Defining

$$\lambda \sim P_1 \frac{\partial}{\partial x_1} + P_2 \frac{\partial}{\partial x_2} + \cdots + P_n \frac{\partial}{\partial x_n},$$

where P_k, $(k=1, 2, \cdots, n)$ are functions of x_k, it was shown how equations of the type

$$u = (\lambda^{-1} \cdot 0)\lambda u + F(\lambda u)$$

can be solved by operator methods.

4. *A proposed course for college freshmen*, by Professor W. E. Byrne, Virginia Military Institute.

The ideas of rational numbers, enumerable and non-enumerable sets, arithmetic progressions, geometric progressions, two-dimensional coordinate systems, vectors, definition of the trigonometric functions in terms of the general angle, complex numbers, and partial fractions in forms suitable for presentation to freshmen were discussed.

5. *Proposals for future activities of the section*, by Professor S. B. Jackson, University of Maryland, Professor Ella C. Marth, Wilson Teachers College, Professor Florence M. Mears, George Washington University, and Miss Veryl G. Schult, District of Columbia Public Schools; presented by Miss Schult.

The desirability of sponsoring contests for high school and college students, preparing a bulletin for high school students showing the mathematics requirements for various college courses offered in the area, expanding our relations with industry, having group meetings on the teaching of college mathematics, and participation of graduate students in section programs was discussed.

6. *Some remarkable theorems about areas*, by Dr. L. R. Ford, Emeritus Professor of Mathematics, Illinois Institute of Technology.

This was the invited lecture. A number of general theorems were presented. Application of some of these to the figures studied in a first course in calculus was made.

C. H. FRICK, *Secretary*

THE APRIL MEETING OF THE MICHIGAN SECTION

The spring meeting of the Michigan Section of the Mathematical Association of America was held on April 12, 1952 at the University of Michigan in Ann Arbor in conjunction with the annual meetings of the Michigan Academy of Science, Arts, and Letters. Professor Norman Anning, Chairman of the Section, presided at both morning and afternoon meetings as well as at the business meeting held immediately following the luncheon at the Michigan Union.

A total of seventy-four persons attended the meetings including the following fifty-four members of the Association:

M. W. Al-Dhahir, H. W. Alexander, N. H. Anning, J. W. Baldwin, W. D. Baten, F. A. Beeler, J. H. Bell, J. W. Bradshaw, G. U. Brauer, D. M. Brown, C. H. Butler, R. V. Churchill, Nathaniel Coburn, S. D. Conte, A. H. Copeland, J. W. Coy, D. E. Deal, M. L. DeMoss, P. S. Dwyer, C. M. Erikson, J. S. Frame, E. W. Goings, Betty Grossman, G. W. Grotts, V. G. Grove, A. E. Hallerberg, H. H. Hannon, Louise F. Hanson, G. E. Hay, Fritz Herzog, E. E. Ingalls, P. S. Jones, Wilfred Kaplan, A. E. Lampen, H. D. Larsen, R. D. Lowe, L. E. Mehlenbacher, D. M. Mesner, H. W. Nace, A. L. Nelson, R. S. Pate, Mary H. Payne, J. E. Powell, G. Y. Rainich, E. D. Rainville, P. H. Raker, C. C. Richtmeyer, E. H. Rothe, Arthur Saastad, J. L. Spenceley, E. M. Steinbach, H. E. Stelson, B. M. Stewart, D. T. Teodoro.

At the business meeting the nominating committee, consisting of Professors Fritz Herzog and A. L. Nelson, reported the names of Professor H. D. Larsen for Chairman and Professor P. S. Jones for Secretary-Treasurer for the year 1952–53. They were elected unanimously.

The 1953 meeting of the Section was set for April 11 in Kalamazoo at Western Michigan College of Education.

The following papers were presented at the morning and afternoon meetings:

1. *Complex numbers as an example of the development of mathematical ideas*, by Professor P. S. Jones, University of Michigan.

The role of a course in the history of mathematics in displaying the growth of and changes in fundamental concepts from their inception until the present was illustrated by tracing the development of complex numbers from Graeco-Arabic times through W. R. Hamilton's formulation of complex numbers as pairs or "couples." The interest and value to be derived by referring to the formulations of the early authors were pointed out by means of slides showing selected pages from original works.

2. *The simple geometric proof of the minimax theorem*, by Professor A. H. Copeland, Sr., and Dr. Howard Raiffa, University of Michigan.

The minimax theorem is a consequence of the fact that in Euclidean n-space two convex sets can be separated by a hyperplane provided they have at most boundary points in common and at least one of the sets contains interior points. The authors gave a simple geometrical proof of such a separation. The proof is valid for Hilbert space. The results are applied to the study of convex (polyhedral) cones and to the theory of linear programming without recourse to the equivalence of the sum and intersection definitions of convex polyhedral cones. For completeness sake an alternate proof of this equivalence was given.

3. *The parabolic correlation coefficient*, by Professor W. D. Baten, Michigan State College.

This paper showed that a non-linear correlation coefficient pertaining to a polynomial predicting equation, for normally distributed independent variables, can be expressed as a function of the square of the averages of the derivatives of the predicting equations for the given values.

4. *Discontinuities in compressible fluid flow*, by Professor Nathaniel Coburn, University of Michigan.

By following a method developed by R. K. Luneberg for studying electromagnetic waves (*Asymptotic Development of Steady State Electromagnetic Fields*, N.Y.U., Mathematics Research Group, Research Report No. EM 14, July 1949), a unified theory of discontinuities (shock and contact manifolds, characteristic manifolds) for the case of compressible fluids is obtained. Essentially, the method depends upon writing the equations of motion, continuity, and energy in divergence free form in a space-time manifold. By applying Stokes' theorem, one may express the previous equations as integrals over hypersurfaces of space-time. These integrals are assumed to be the fundamental equations of the theory. In regions where the density, velocity, *etc.* are continuous with continuous derivatives, these equations reduce to the original equations of motion, continuity, and energy. If the density, velocity, *etc.* are discontinuous over certain hypersurfaces, then the equations furnish relations for the various discontinuities along contact and shock manifolds. By use of a differentiation process, discontinuities of the derivatives of density, *etc.* along characteristic manifolds can be studied.

5. *Teaching a blind student in a regular college course in algebra*, by Mr. E. M. Steinbach, University of Detroit.

The problems involved in teaching a completely blind student in a regular class in intermediate algebra were discussed, some of the solutions arrived at by the student and instructor working together were outlined, and some sources of materials helpful in such a situation were mentioned.

6. *The Waring problem in the dissection of polygons*, by Professor B. M. Stewart, Michigan State College, and Mr. Michael Goldberg, Bureau of Ordnance, Navy Department.

Given n congruent polygons of shape P the problem is to dissect each polygon in the same way into K parts so that the total set of Kn parts may be composed, without reflection, exactly to fill a large polygon, directly similar to the original smaller polygons. The authors succeeded in finding a value for K *independent* of n; in fact if one of the given polygons can be divided into e triangles, it was shown that $K(n, P) \leqq 7e$. For a parallelogram M, $K(n, M) \leqq 6$. For a square S, $K(n, S) \leqq 4$ and for many values of n they are able to produce interesting three-part and two-part dissections of the square.

7. *Finitely normal numbers*, by Mr. H. A. Hanson, Michigan State College, introduced by the Chairman.

The solution of problem 4385 in the MONTHLY, [1951, 573], in which a number is constructed to the base 2, in which every sequence of n digits occurs exactly once, suggests the extension of such a construction to an arbitrary base $B \geqq 2$. The construction consists in writing n digits $B-1$, followed at each position thereafter by the smallest digit which does not repeat a sequence of n digits previously occurring. A finitely normal number of order n is then defined as a decimal, expressed in some base B, in which every sequence of n digits occurs with a limiting frequency of $1/B^n$. If the first B^n digits of the number constructed as above are taken as the period of a repeating decimal, the resulting decimal is finitely normal of order n, but of no order greater than n.

8. *Modifications of Fourier integrals*, by Professor R. V. Churchill, University of Michigan.

Let $z(x, y)$ denote the static transverse displacements in a stretched membrane covering the quadrant $x>0$, $y>0$. The edge $y=0$ has a prescribed displacement $F(x)$. The edge $x=0$ is attached to the center line of a stretched metal strap that is rigid in the x-direction but bends like a stretched string in the yz-plane, producing a boundary condition of the type $z_{xx}(0, y) = kz_x(0, y)$. Separation of variables in this boundary value problem leads to a solution provided that $F(x)$ can be represented by a modification of the Fourier cosine integral formula in which the kernel function $\cos \lambda x$ is replaced by the function $\cos(\lambda x+\beta)$, where $\tan \beta = \lambda/k$. A short derivation of that formula was presented. The corresponding modified Fourier transforms and their applications were noted, as well as other trigonometric modifications of Fourier integrals.

9. *The solution of the bilateral matrix equation*, by Professor J. H. Bell, Michigan State College.

This paper deals with the bilateral matrix equation $\sum_{m=0}^{s} \sum_{i=0}^{m} X^{m-i} A_{m-i,i} X^i = 0$, where all matrices involved are square of order n and have elements in a field $\mathfrak{F}$. A brief summary of the work done in special cases of this problem was given. Necessary conditions were developed which suggested a method of solution of the general equation. Let $Y=(Y_{ij})$ where $Y_{ij}=X^{i+j-2}$, and $B=(B_{ij})$ where $B_{ij}=A_{i-1,\,j-1}$ $(i=0, 1, 2, \cdots, s; j=0, 1, 2, \cdots, s; A_{kj}=0, j>k)$. If B is nonsingular and X is a solution of the equation then Y must be such that $Y=ZB^{-1}$ where $Z^2=0$. If B is singular, Y must be such that $Y=ZQP$ where

$$PBQ = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix},$$

I and Z_{11} are $r \times r$ submatrices; P, Q are non-singular, $Z_{11}^2=0$ and $Z_{21}Z_{11}=Z_{11}Z_{12}=Z_{21}Z_{12}=0$.

10. *A reducible case*, by Professor N. H. Anning, University of Michigan.

It was suggested as a classroom device that, after a cubic equation has been prepared for

solution, it be identified with

$$x^3 - 3m^2x - 2n^3 = 0.$$

The solution of this "by Cardan" is tidy. Moreover if there are three real roots the students can put $n^3 = m^3 \cos 3A$ and by an easy application of De Moivre's Theorem go ahead to the solutions: $x_1 = 2m \cos A$, $x_2 = 2m \cos (A + 120°)$, $x_3 = 2m \cos (A + 240°)$.

P. S. JONES, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The thirty-sixth annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Saturday, April 19, 1952. Professor R. F. Rinehart, Chairman of the Section, presided at the morning and afternoon sessions.

Ninety-seven persons registered in attendance, including the following seventy-seven members of the Association:

J. E. Adney, Jr., A. G. Anderson, W. E. Anderson, Grace M. Bareis, I. A. Barnett, William Beck, Theodore Bennett, G. M. Bloom, Foster Brooks, Dorothy I. Carpenter, A. B. Carson, V. B. Caris, W. G. Clark, Florentina M. Clinton, W. F. Cornell, H. K. Crowder, J. E. Darraugh, C. H. Denbow, B. B. Dressler, O. L. Dustheimer, E. R. Epperson, D. T. Finkbeiner, II, M. P. Fobes, H. G. Harp, Frances Harshbarger, R. G. Helsel, Carl Holtom, R. Y. Iwanchuk, E. D. Jenkins, Margaret E. Jones, Chosaburo Kato, K. D. Kelly, L. C. Knight, D. M. Krabill, D. H. Kraft, Nathan Lazar, F. C. Leone, H. D. Lipsich, C. P. Louthan, L. L. Lowenstein, W. J. McCallion, S. W McCuskey, H. R. Mathias, Margaret E. Mauch, L. H. Miller, Max Morris, O. M. Nikodym, S. R. Orr, P. M. Pepper, H. S. Pollard, W. O. Portmann, Tibor Rado, R. F. Reeves, P. R. Rider, L. G. Riggs, Louis Ross, H. J. Ryser, Samuel Selby, Sister M. Constantia, Ruth B. Smyth, E. T. Stapleford, Andrew Sterrett, C. W. Topp, E. P. Vance, W. R. VanVoorhis, E. H. Wang, D. R. Whitney, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, E. H. Wohler, Alberta Wolfe, G. F. Woodson, Jr., C. H. Yeaton, Marie M. Yeaton, B. J. Yozwiak, A. D. Ziebur.

The following officers were elected for the coming year: Chairman, E. J. Mickle, The Ohio State University; Secretary-Treasurer, Foster Brooks, Kent State University; third member of the Executive Committee, H. J. Ryser, The Ohio State University; Program Committee: W. R. Transue, Kenyon College, Chairman; L. L. Lowenstein, Kent State University; Marion Wetzel, Denison University.

A special committee on methods for promoting interest in mathematics among secondary school students, Professor F. B. Wiley, Ashland College, chairman, reported on methods in use at present, and recommended, first, that a project for this purpose be launched on a nation-wide basis as a joint effort of the Mathematical Association of America and the National Council of Teachers of Mathematics; and, second, that such a joint project be planned at once for Ohio.

The following papers were presented:

1. *Functions of matrices*, by Professor R. F. Rinehart, Case Institute of Technology.

A system of first order linear homogeneous differential equations with constant coefficients,

when written as a single matric differential equation, gives rise to the search for a matric solution analogous to that of the similar appearing single scalar differential equation, namely the matric exponential function. A discussion of the power series representation of the matric exponential function was followed by a discussion of the general matric functions defined by power series, including conditions for convergence. Since every convergent power series in a square matrix Z can be written as a polynomial in Z, it was shown that this polynomial was necessarily that given by the Sylvester-Bucheim definition of a function of a square matrix, and hence that the latter definition is more inclusive than the power series definition.

The definitions of matric function given by Schwerdtfeger, Fantappie, Jordan, Giorgi, and Cipolla were discussed. The equivalences of the Schwerdtfeger and Sylvester-Bucheim definitions, and of the Fantappie and Jordan definitions were noted. It was remarked that the combinatory criteria laid down by Fantappie are satisfied by all of the definitions except, under certain conditions, that of Cipolla. These criteria insure that a polynomial identity among scalar functions will be preserved by their matric analogues, according to any of the above definitions (with the exception noted). Thus, for example, $\sin^2 Z+\cos^2 Z=1$, $e^{iZ}=\cos Z+i \sin Z$, hold for square matrices Z.

2. *Rigid membranes*, by Professor T. Rado, The Ohio State University.

It has been known for over thirty years that a flexible and inextensible membrane of *convex* shape is infinitesimally rigid if its boundary B is held point-wise fixed. In cooperation with T. Minagawa, the speaker obtained theorems to the effect that the same rigidity phenomenon occurs for wide categories of non-convex shapes, and that in many instances it is sufficient to keep fixed only an appropriately chosen arc on the boundary.

3. *A step function solution of the bounded variation moment problem*, by Professor G. M. Bloom, Miami University.

G. Polya has shown that for arbitrary real sequences $\{\mu_n\}_0^\infty$ and $0=b_0<b_1<\cdots\rightarrow\infty$ there exist infinitely many step functions $\Phi(t)$ with points of change restricted to the set $\{b_n\}_0^\infty$ for which $\int_0^\infty t^n d\Phi(t)=\mu_n$, $\int_0^\infty |t^n d\Phi(t)| = B_n<\infty$, $n=0, 1, 2, \cdots$. Here for $\epsilon>0$, $a>1$, and $b_n=a^n$, $n\geqq 1$, is given a solution $\Phi(t)$ for which $B_0<|\mu_0|+\epsilon$.

4. *A probability distribution of random matches*, by Professor W. R. Van Voorhis, Fenn College.

If n numbered wires are attached at random one each to n correspondingly numbered terminals the probability that x of the numbers on the wires match the numbers on the terminals is $1/x!\sum_{r=0}^{n-x}(-1)^r/r!$, and the expectation of the number of matches is 1. This problem, expressed in other words, was first discussed by Montmort, and was later generalized by DeMoivre, Laplace, and Euler. Using a table for calculating the necessary moments, Professor Van Voorhis pointed out that the probability distribution has variance and skewness each equal to 1, as well as the mean, and that these statistics are independent of the integer n. Some properties of the distribution were discussed, and probability levels were given for exceeding certain numbers of matches.

5. *A new 23-place logarithm table to base 10 and e*, by Mr. E. R. Epperson, Miami University.

When accuracy to approximately twenty significant figures is desired in finding logarithms or antilogarithms, the factorizing method provides a procedure that decreases the arithmetical labor involved in such calculations. The new *Smithsonian Logarithmic Tables to Base e and Base 10*, Smithsonian Miscellaneous Collections, 118 (published by the Smithsonian Institution as publication 4054, 1952), prepared by G. W. Spenceley, R. M. Spenceley, and E. R. Epperson, is devoted exclusively to the factorizing method. This paper described the new volume containing the twenty-three decimal place values of the natural and common logarithms of integers from one to 10,000

and of appropriate decimal numbers and illustrated the application of the factorizing method to the volume.

6. *Social security and college retirement programs*, by Dr. O. L. Dustheimer, Cleveland, Ohio.

The author presented the results of a survey of pension plans now in effect in the colleges and universities of the United States and possessions. Of 815 institutions replying to the study questionnaire 81 per cent indicated that faculty members are now under social security.

7. *Some mathematical problems encountered in Air Force Research and Development work*, by Brigadier General L. I. Davis, Air Research and Development Command U.S.A.F. (By invitation).

Service research and development programs become involved in many problems requiring intensive mathematical treatment. The majority of the problems are common to the problems of science and engineering in any line of endeavor. A few examples of problems in mechanics requiring the use of differential equations, vector analysis, and probability relationships are given. The truly military problems involving tactics and strategy are held to be rather unsuitable for exact mathematical treatment. The many factors that enter into any attempt to make a model of strategic warfare are pointed out, using as an example a description of some of the components of an electronic war game built at the U.S.A.F. Institute of Technology, Wright Field, Dayton, Ohio.

8. *The law of inconsistency—a new approach to indirect proofs in mathematics*, by Professor Nathan Lazar, The Ohio State University.

The author discussed substitution for a proposition *A implies B* the related proposition *A and the denial of B are inconsistent*, and the application of this substitution to theorems in high school geometry where proofs are usually made by the indirect method.

9. *An interesting matrix*, by Mr. Fenton Stancliff, Akron, Ohio, introduced by the Secretary.

The author expressed e^x in a series of determinants, the n-th term of which was a determinant of the n-th order. When a positive integer K was added to each individual element of each determinant, the limit of the series reduced to $e^x/(K+1)$. This result was ascribed to a new invariant, tentatively referred to as the "lambda" number of a square array. When a single unit is added to each element of a determinant, this invariant fixes the amount of change in the determinant. Since the lambda number may be regulated at pleasure, the limit of any series of determinants may be regulated. The terms of the series for e^x were derived from "An interesting matrix," the subject of the talk, which was of such form, that all the characteristic roots and eigen-vectors were rational integral functions of the elements, and despite the possible high order of the matrix, were available by inspection.

FOSTER BROOKS, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The twelfth annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of Arizona, Tucson, Arizona, on April 11 and 12, 1952.

Dr. Charles Wexler, Chairman of the Section, presided. Forty persons attended the sessions including the following twenty-six members of the Association:

J. W. Beach, Louis Brand, J. H. Butchart, B. R. Cato, Louise H. Chin, R. B. Crouch, G. A. Culpepper, W. W. Denton, D. G. Duncan, R. S. Fouch, R. F. Graesser, R. E. Graves, Max Kramer, H. B. Leonard, B. C. Meyer, W. W. Mitchell, J. L. Olpin, E. J. Purcell, Henry Schutzberger, Andrew Sobczyk, Deonisie Trifan, Earl Walden, D. L. Webb, R. L. Westhafer, Charles Wexler, Dale Woods.

At the business meeting the following officers were elected for the year 1952–1953: Chairman, J. H. Butchart, Arizona State College, Flagstaff; Vice-Chairman, M. S. Hendrickson, University of New Mexico. The Section accepted the invitation from the New Mexico College of A. and M. A. to hold the annual meeting at State College, New Mexico, January 2–3, 1953.

On Friday evening Dr. Louis Brand of the University of Cincinnati delivered an address entitled *The Inquiring Mind.*

The following papers were read in the two-day session:

1. *A theorem on double series*, by Professor B. C. Meyer, University of Arizona.

Let $\sum a_{ij}$ be a double series such that $|a_{ij}| \leqq |a_{mn}|$ for $i \geqq m, j \geqq n$ and such that each row and column is an alternating series. A necessary and sufficient condition that it be Sheffer convergent is that it be summable by diagonals. A comparison is made between Sheffer and regular convergence for series of this type.

2. *A minimum principle of plasticity*, by Professor Deonisie Trifan, University of Arizona.

For a given state of strain ϵ_{ij} throughout a body, and given velocities $\dot{v}_i$ on the surface such that the surface integral of the normal component vanishes, the rate of change of strain $\dot{\epsilon}_{ij}$, uniquely determined throughout the body, and $\dot{\sigma}_{ij}$ the corresponding rate of change of stress—$(\dot{\sigma}_{ij} - 1/3\delta_{ij}\dot{\sigma}_{kk} = 2G_0\dot{\epsilon}_{ij} - p(E)\dot{E}\epsilon_{ij}$ where $E = 2\epsilon_{ij}\epsilon_{ij}/3$, G_0 the shear modulus in the plastic range, and $p = p(E)$ an arbitrary function depending on the material)—satisfy the following minimum principle $\int_v \dot{\epsilon}_{ij}\dot{\sigma}_{ij}dv \leqq \int_v \dot{\epsilon}'_{ij}\dot{\sigma}'_{ij}dv$, where $\dot{\epsilon}'_{ij}$ are suitably selected strain rates and $\dot{\sigma}'_{ij}$ the corresponding stress rates.

3. *Conjugate functions in Boolean algebra*, by Professor Louise H. Chin, University of Arizona.

Let A be the set of elements in a Boolean algebra, and consider functions whose domain is A. A function g is said to be a conjugate function of a function f if the equations $f(x) \cdot y = 0$ and $g(y) \cdot x = 0$ are equivalent for any elements x and y in A. This concept of conjugate functions was introduced by A. Tarski and many consequences result from this definition. For example, this notion is especially useful when applied in the discussion of relation algebras and cylindric algebras.

4. *Length of a vertical column to a parabolic bridge*, by Professor W. W. Mitchell, Jr., Phoenix College.

In the design of a steel bridge with a parabolic arch it is necessary to calculate the length of the supporting columns from the arch to the roadbed. For an arch whose center line is a true parabola and whose cross-section is uniform, the center line of the curved surface upon which the vertical supporting columns rest is a curve at a uniform perpendicular distance from the parabolic center line of the arch. The equation of this curve was found, also an approximation, and the error made by using the approximate equation.

5. *The pi theorem of dimensional analysis*, by Professor Louis Brand, University of Cincinnati.

A dimensionally homogeneous or *isobaric* function $f(x_1, \cdots, x_n)$, in which the n variables x_i have the $m \times n$ dimensional matrix (a_{ij}) with respect to m fundamental units U_j, satisfies a certain identity in $t_1, t_2, \cdots, t_m$ when the units are changed from U_j to $U_j/t_j (t_j > 0)$. Ordinary homogeneous functions may be regarded as isobaric functions which have the dimensional matrix $(1, 1, \cdots, 1)$ with respect to a single unit U_1; in this case the corresponding identity in t is $f(tx_1, \cdots, tx_n) = t^a f(x_1, \cdots, x_n)$ where a is the degree of homogeneity. Putting $t = 1/x_1$ shows that the *equation* $f(x_1, \cdots, x_n) = 0$ is equivalent to $f(1, x_2/x_1, \cdots, x_n/x_1) = 0$, in which the $n-1$ arguments are dimensionless. A natural extension of this reasoning yields a proof of the pi theorem: *If the function f in an equation $f(x_1, \cdots, x_n) = 0$ with n arguments is isobaric with respect to m fundamental units $U_1, \cdots, U_m$, and if the $m \times n$ dimensional matrix of the arguments is of rank r, then if $n > r$, the given equation is equivalent to another $F(\pi_1, \cdots, \pi_{n-r}) = 0$ in $n-r$ independent and dimensionless products π_i formed from the original arguments.* The proof is constructive; and a simple matrix equation gives all the dimensionless products $\pi_1, \cdots, \pi_{n-r}$.

6. *The congruence of bisecants of the normal rational ruled surface of order 4 in* [5], by Professor E. J. Purcell, University of Arizona.

7. *The analysis and algebra of linear transformations*, by Dr. Andrew Sobczyk, Boston University and Los Alamos Scientific Laboratory.

The author reviewed the salient facts of the theory of (1) linear transformations between infinitely dimensional linear spaces, and (2) algebras of such transformations, contained in many published contributions on Hilbert space, Banach spaces and algebras. He then indicated his recent solutions of some outstanding problems of the theory.

8. *Torque and thrust on bounding cylinders in a two-dimensional flow problem*, by Professor J. W. Beach, University of New Mexico.

By starting with the known solution of $\nabla^4 \Psi = 0$ in the region between infinitely long eccentric cylinders in slow steady motion, the torque and thrust on a unit height of these cylinders was found.

9. *A note on the divergence of a certain Dirichlet series*, by Professor Dale Woods, Texas Western College.

The author gave a simple proof of the divergence of the series $\sum 1/n^z$ for $z = a + ib$ and $a \leqq 1$.

10. *Complex treatment of trochoidal curves*, by Professor J. H. Butchart, Arizona State College, Flagstaff.

The author presented trochoids in general form with arbitrary radii of the fixed and rolling circles, using the *turn* of inversive geometry as the parameter. The self conjugate equation of the tangent was specialized for the deltoid and cardioid. The isotopic with a 60° angle was shown to be the trifolium and the orthoptic of the cardioid was shown to consist of a circle and a nodal limacon. The length of an arch and area under an arch of the general epicycloid were shown to be $8(a+b)b/a$ and $\pi b^2(3a+2b)/a$ respectively.

11. *Parabolic singular solutions of a differential equation of second degree*, by Professor R. L. Westhafer, New Mexico College of A. and M. A.

The p-discriminant locus for the differential equation $(a_0x + a_1y + a_2)p^2 + (b_0x + b_1y + b_2)p + (c_0x + c_1y + c_2) = 0$ is a second degree expression in x and y. The only cases in which the p-discrimi-

nant locus contains a non-degenerate conic for a singular solution of the differential equation are for certain parabolas.

12. *Statistical analysis of a department's grades*, by Professor R. F. Graesser, University of Arizona.

A record was kept of the distributions of semester grades in all classes in a department for a considerable period of time. These serve to establish a norm which may be adopted by the department as its ideal distribution. The combined grade distribution for all instructors in a given semester may be tested by the chi-square function for compatibility with this norm. If these distributions are incompatible, the instructors' individual distributions may be tested for homogeneity by the same chi-square function in order to determine whether this incompatibility is due to a change in the severity of grading by the whole department or by nonconformity of a few individuals. If the latter is suspected, then the distributions of individuals may be tested for compatibility with the norm. Changes in grading standards on the part of the whole department may be discovered by testing for trends with the theory of runs or with the serial correlation coefficient.

13. *Demonstration of space curves by means of light projection*, by Professors R. F. Graesser and E. J. Purcell, University of Arizona.

Space curves were beautifully demonstrated by use of planes and cones of light and string models of quadric, cubic, and quartic surfaces.

14. *Report of committee on high school cooperation*, by Professors Earl Walden, Max Kramer, New Mexico College of A. and M. A., and Charles Wexler, Arizona State College, Tempe.

Meetings have been held with the mathematics and science sections of the New Mexico Education Association and the Arizona Education Association. All agreed that better subject matter training was needed in the preparation of teachers. Plans have been set up for mathematical contests to be held in one of the New Mexico districts in 1953. If successful, expansion of these contests is anticipated.

15. *A subject matter course for the training of mathematics teachers*, by Professor Max Kramer, New Mexico College of A. and M. A.

A course was described, now being offered at New Mexico A. and M., in which subject matter on fundamentals from Courant and Robbins book *What Is Mathematics* is studied concurrently with the applications to the teaching of secondary school mathematics.

16. *Motor analysis*, by Professor Louis Brand, University of Cincinnati.

Dr. Brand gave an illuminating exposition of the theory of Motor Analysis and its application to the dynamics of a rigid body.

R. L. WESTHAFER, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

On April 25–26, 1952, the annual meeting of the Texas Section of the Mathematical Association of America was held in Commerce, Texas. East Texas State Teachers College served as sponsor.

About ninety persons attended the sessions of the Association including the following fifty-three members of the Association:

R. E. Baker, J. C. Bradford, Ina M. Bramblett, H. E. Bray, L. A. Colquitt, M. L. Coffman,

J. V. Cooke, A. H. Cowling, Don Cude, J. A. Daum, Jim Douglas, Jr., G. H. Dubay, Terrell Ellis, R. L. Glass, L. D. Gregory, W. T. Guy, Jr., E. H. Hanson, E. A. Hazlewood, E. R. Heineman, J. M. Hurt, E. C. Kennedy, H. A. Luther, E. K. McLachlan, W. K. McNabb, Hazel L. Mason, V. A. Miculka, Harlan C. Miller, E. D. Mouzon, Jr., M. E. Mullings, C. A. Murray, H. C. Parrish, R. B. Pinson, C. J. Pipes, Henry Rainbow, L. W. Ramsey, C. L. Riggs, R. Q. Seale, C. R. Sherer, F. W. Sparks, C. E. Springer, D. W. Starr, W. G. Stokes, E. J. Stulken, Jennie L. Tate, W. W. Taylor, M. E. Tittle, F. E. Ulrich, R. S. Underwood, Margaret M. Welch, Mabel Williams, H. A. Wood, A. W. Wortham, C. B. Wright.

At the business meeting the following officers were elected for the next academic year: Chairman, Professor C. B. Wright, East Texas State Teachers College; Vice-Chairman, Professor L. A. Colquitt, Texas Christian University; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University.

A banquet was given on Friday night; President J. G. Gee of East Texas State Teachers College was the principal speaker.

The program included the following papers:

1. *Some characterizations of certain set properties*, by Professor H. C. Parrish, North Texas State College.

For each of several (planar) set properties, a new characterization was given in terms of a (conditioned) region or sector property. Included were characterizations of a non-dense set; a non-dense, closed set; and a voidal set. A planar (linear) set S was termed voidal if for every point P of S there exists an open sector (interval) with P as vertex (end-point) and containing no point of S.

2. *The relaxation method as a projection method*, by Professor Samuel Agmon, The Rice Institute, introduced by the Secretary.

It is shown that the relaxation method used in the numerical solution of linear equations can be interpreted as a projection method. From this a simple geometric proof of the convergence of the relaxation method for a consistent system of linear equations is derived.

3. *On exhaustibility and certain related properties of a real function*, by Mr. J. C. Bradford, North Texas State College.

The speaker reported on a study made of certain function properties which were defined in terms of an exhaustible set, *i.e.* a set of the first category. In particular, functions which satisfy the property of Baire were studied. Most of the results were analogues of certain theorems on metric properties of functions.

4. *Stirling's approximation formula for* $n!$, by Professor C. L. Riggs, East Texas State Teachers College.

This is a development of Stirling's formula based upon the calculus of finite differences.

5. *An equivalence method for sequences of random variables*, by Mr. Jim Douglas, Jr., The Rice Institute.

A generalization of the classical truncation device due to Khintchine for the comparison of two sequences of independent random variables is introduced. The distance between two distribution functions $F(x)$ and $G(x)$ is defined to be the least upper bound of $|F(x)-G(x)|$ for all x. If $\{F_k(x)\}$ and $\{G_k(x)\}$ are, respectively, the sequences of distribution functions corresponding to two sequences of independent random variables, and if the sum of the distances between $F_k(x)$ and

$G_k(x)$ converges, the sequences are said to be equivalent. Applications to probability limit laws are discussed.

6. *Potential functions for particles moving with relativistic velocities in given orbits*, by Mr. M. L. Coffman, Texas Agricultural and Mechanical College.

The relativistic Lagrangian is used to determine the differential equation satisfied by the potential function required for particles moving in given orbits on given surfaces. The potential function satisfies a linear partial differential equation of second order. As examples, the potential functions are determined for two cases: where the curves are spirals on a cylinder, and where the curves are equilateral hyperbolas on a hyperbolic paraboloid.

7. *Union curves in differential geometry*, by Professor C. E. Springer, University of Oklahoma (by invitation).

A union curve on a surface relative to a given rectilinear congruence has the property that its osculating plane at every point contains the line of the congruence through the point. The union curvature vector of a curve on the surface is obtained. It is a zero vector in case the curve is a union curve, and it is the geodesic curvature vector in case the congruence is that of normals to the surface. A union affine connection appears, and its components reduce to the Christoffel symbols of the second kind for the congruence normal to the surface. The concepts of parallelism of surface vectors, intrinsic differentiation, and covariant derivative are generalized to union parallelism of surface vectors, union intrinsic differentiation, and union covariant derivative. An analogue of the Gauss-Bonnet theorem involving union curvature is introduced.

8. *Loci on the four-axes plane*, by Professor R. S. Underwood, Texas Technological College.

This paper discusses a coordinate system for four variables in which the positive halves of the z and u axes coincide respectively with the negative halves of the x and y axes. This scheme leads to very simple equations of transformation from the two-axes system. Also, since it provides the convenient lattice points of plane analytic geometry, it is well adapted to graphical discussions of simultaneous Diophantine equations in four variables. A method of finding typical, or area, loci is discussed, and a test for degenerate, or curve, loci is obtained.

9. *Differential equations of infinite order*, by Mr. T. M. Gallie, Jr., The Rice Institute, introduced by the Secretary.

This paper summarizes some interesting properties of solutions of homogeneous differential equations of infinite order with constant coefficients. These solutions are compared with those of the equation of finite order and a fundamental difference between the two is demonstrated. The existence of exponential solutions is shown and the possibility of expanding every solution throughout its domain of existence in a series of exponential solutions is discussed.

10. *A little-used method of solving a system of linear equations*, by Professor E. R. Heineman, Texas Technological College.

Let A be the matrix of a system of m linear equations in n unknowns. By performing elementary operations on the rows, but not the columns, of A (*i.e.* adding to any row a multiple of another, interchanging rows, multiplying a row by a constant), it is possible to obtain an equivalent matrix B, in which all elements below the principal diagonal of the coefficient matrix are zeros. The matrix B yields information as to the rank r of A, the consistency or inconsistency of the system, and the number of solutions. If the system is inconsistent, this completes the solution of the problem. For the consistent case, by manipulating the rows of B, it is possible to get an equivalent matrix C, in which all elements above the principal diagonal and appearing in the first r columns are zeros.

From matrix C, one can read the complete solution of the system. Slight variations of the method may be necessary in exceptional cases.

11. *Fractional integration and differentiation*, by Professor W. T. Guy, Jr., The University of Texas (by title).

Certain "fractional" operators were defined and applications of their use were given. The examples were chosen as being suitable for inclusion in the usual Laplace transform course for science majors.

12. *Computing the Kendall tau coefficient*, by Mr. H. F. Bright, The University of Texas (by title).

The assumptions implicit in the use of correlation techniques cannot always be justified, particularly for small samples. The Kendall tau coefficient offers a method of dealing with this difficulty since no assumptions regarding the shape of the distributions are necessary. However, the computation of this coefficient as ordinarily accomplished is laborious. In this note, a mechanical device is described for expediting the computation.

13. *The place of the mean value theorem in the differential calculus course*, by Mr. J. M. Hurt, The University of Texas.

The mean value theorem is fundamental in the study of increasing and decreasing functions, maxima and minima, and Newton's method, as well as in the development of L'Hospital's rule. Thus its early introduction into the differential calculus course strengthens the treatment of these important topics, gives the student a powerful tool with which to make his arguments more rigorous and complete, and obviates the temptation to give discussions that are vaguely intuitive, incomplete, or incorrect.

14. *A program of correlated mathematics in secondary schools*, by Sister Antonietta, Incarnate Word College, introduced by the Secretary.

By its nature mathematics is cumulative. What is studied in one branch should help towards understanding the next. While arithmetic, algebra, geometry, trigonometry are distinct to the extent that all subject matter boundaries cannot be dissolved, yet the connections between them not only unify the whole field of mathematics but also give a better grasp of each phase. To make these connections where they occur naturally is the aim of correlated mathematics. This program is worked out on the basis of the recommendations of the 1940 Reorganizations Committee which was appointed by the National Council of Mathematics Teachers.

15. *What do mathematicians do?* by Professor W. T. Guy, Jr., The University of Texas.

Since no adequate definition of mathematics has been given (nor is likely to be given) the author made an attempt to delineate what mathematicians do. The role of the pure and the applied mathematician was examined. It was suggested that such an orientation during the usual college course would help the non-mathematics majors (as well as the mathematics majors) in understanding and appreciating the place of mathematics in their own fields.

16. Panel discussion: *Certification of teachers in Texas*, by Professor J. E. Franklin and Professor Everett Shepherd, East Texas State Teachers College; Professor H. E. Bray, The Rice Institute; Professor C. R. Sherer, Texas Christian University; Professor E. A. Hazelwood, Texas Technological College, Moderator.

C. R. SHERER, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS

KENTUCKY, University of Louisville, Spring, 1953.

LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Howard University, Washington, D. C., December 6, 1952.

METROPOLITAN NEW YORK

MICHIGAN, Western Michigan College of Education, Kalamazoo, April 11, 1953.

MINNESOTA, College of St. Scholastica, Duluth, October 11, 1952.

MISSOURI, William Jewell College, Liberty, Spring, 1953.

NEBRASKA

NORTHERN CALIFORNIA

OHIO

OKLAHOMA, Oklahoma City, October 31, 1952.

PACIFIC NORTHWEST

PHILADELPHIA, University of Delaware, Newark, November 29, 1952.

ROCKY MOUNTAIN

SOUTHEASTERN, Alabama Polytechnic Institute, Auburn, March, 1953.

SOUTHERN CALIFORNIA, Los Angeles City College, March 14, 1953.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanic Arts, State College, January 2–3, 1953.

TEXAS

UPPER NEW YORK STATE, United States Military Academy, West Point, Spring, 1953.

WISCONSIN, Mount Mary College, Milwaukee, May, 1953.

THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and acientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9 and 10, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. These numbers have been issued to date:

No. 1 *Calculus of Variations* by G. A. Bliss, xii+189 pages.

No. 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages.

No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.

No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.

No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.

No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.

No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.

No. 8. *Rings and Ideals* by N. H. McCoy, xii +216 pages.

No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.

No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

New

Price-Knowler: Basic Skills in Mathematics

A careful presentation, emphasizing meaning of arithmetic, geometric figures, simple algebra, graphs, elementary finance, and statistics. For those with insufficient training in mathematics.

Mouzon-Rees: Mathematics of Finance

A sound presentation of problems of business and finance with material on perpetuities, interest annuities, life insurance, etc.

Longley-Smith-Wilson: Analytic Geometry and Calculus

A reasonably rigorous treatment which includes material on solid analytic geometry and differential equations. Early introduction of integration.

Fulmer-Reynolds: College Algebra

A clear, concise treatment of essentials, which is ideal for a short course. Includes a chapter on the theory of equations. Many illustrative examples.

Ginn and Company Home Office: Boston Sales Offices: New York 11

Chicago 16 Atlanta 3 Dallas 1 Columbus 16 San Francisco 3 Toronto 5

Now available:

PROFESSIONAL OPPORTUNITIES IN MATHEMATICS

The report of a Committee of the

Mathematical Association of America,

reprinted from the January 1951 AMERICAN MATHEMATICAL MONTHLY

24 pages, paper covers

25¢ for single copies; 10¢ each for orders of ten or more

Send orders to: HARRY M. GEHMAN, Secretary-Treasurer

Mathematical Association of America

University of Buffalo

Buffalo 14, New York

Timely McGRAW-HILL Books

GENERAL COLLEGE MATHEMATICS

By W. L. AYRES, C. G. FRY, and H. F. S. JONAH, Purdue University. 288 pages, $3.75

A new approach to freshman mathematics, designed to help the student in his application of mathematical topics to the biological and social sciences. Keyed primarily for those students who will take only one year of college mathematics, this text examines such subjects as ratios and percentages, linear and quadratic equations, trigonometry, interest and its application to installment buying, laws of growth, statistics, etc.

STATISTICAL THEORY IN RESEARCH

By R. L. ANDERSON, University of North Carolina, and T. A. BANCROFT, Iowa State College. 399 pages, $7.00

A combined text and reference work, this book presents basic statistical theory, including the elementary principles of probability, population and sampling theory with moment-generating functions, orthogonal linear forms, and the theory of estimation and tests of significance. The last half of the book covers the theory of least squares and its use in the analysis of actual experimental data, including multiple regression, experimental design and variance component models.

AN INTRODUCTION TO THE THEORY OF DIFFERENTIAL EQUATIONS

By WALTER LEIGHTON, Washington University. *International Series in Pure and Applied Mathematics.* 174 pages, $3.50

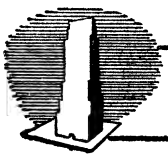
A *unified* treatment, unusually clear and rigorous, this text covers the material customarily given in a first course—with considerable attention devoted to underlying theory. The author has concentrated on making the *fundamental existence theorem* a central idea and one which becomes part of the student's experience.

DIFFERENTIAL EQUATIONS

By ROBERT C. YATES, United States Military Academy. 215 pages, $3.75

This text is designed to prepare the student for work in modern engineering practice and theory and to present the basic mathematical tools necessary for analysis and solution of problems leading to differential equations. The book ranges from first order equations through general linear ordinary equations with constant coefficients, special equations mostly of a nonlinear character, numerical solutions and solutions by series, and then into the solution of the wave equation by separation of variables.

Send for copies on approval



McGRAW-HILL BOOK COMPANY, INC.

330 WEST 42ND STREET, NEW YORK 18, N. Y.



Three Important New Texts

FUNDAMENTAL PROCEDURES OF FINANCIAL MATHEMATICS

●

MERRILL RASSWEILER and IRENE RASSWEILER

This book requires no algebra or higher mathematics thus making it possible to offer this important subject to larger groups of students. Its treatment of the usual topics of business and investment mathematics is thorough. Ready for Fall classes.

CALCULUS

●

JOHN F. RANDOLPH

A student-directed text that shows clearly the difference between an intuitive discussion and a proof. An unusual technique of anticipating a subject is used throughout the book so that a concept is subtly suggested before it is crystallized in a definition or application.

483 pages $5.00

THE THEORY OF NUMBERS

●

B. M. STEWART

This book was written to fill a long-felt need for a text that seemed neither too short nor too long, too easy nor too advanced, for a short course, involving approximately thirty lessons and offered to a mixed group of undergraduate and beginning graduate students. Ready for Fall classes.

THE MACMILLAN COMPANY

60 FIFTH AVE., NEW YORK 11, N.Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 59 NUMBER 8

CONTENTS

The Problem of Simplifying Truth Functions . . . W. V. QUINE 521
The Differential Equation of a Conic and Its Relation to the Aberrancy . . . . . A. W. WALKER 531
The William Lowell Putnam Mathematical Competition . . . . . L. E. BUSH 538
Mathematical Notes . . . VICTOR THÉBAULT, C. M. FULTON 542
Classroom Notes. . . C. S. OGILVY, ROGER OSBORN, B. B. MISRA 547
Elementary Problems and Solutions 551
Advanced Problems and Solutions 554
Recent Publications 558
Clubs and Allied Activities 561
News and Notices 563
The Mathematical Association of America 568
New Members 568
May Meeting of the Allegheny Mountain Section 571
May Meeting of the Illinois Section 573
May Meeting of the Indiana Section 576
May Meeting of the Metropolitan New York Section 578
May Meeting of the Minnesota Section 580
May Meeting of the Missouri Section 582
May Meeting of the Nebraska Section 584
May Meeting of the Rocky Mountain Section 585
May Meeting of the Upper New York State Section 587
May Meeting of the Wisconsin Section 590
June Meeting of the Pacific Northwest Section 592

OCTOBER 1952

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	J. W. GIVENS	GEORGE SPRINGER
PATRICK Du VAL	L. M. KELLY	E. P. STARKE
HOWARD EVES	H. D. LARSEN	G. B. THOMAS
F. A. FICKEN	IVAN NIVEN	E. P. VANCE

EDITH R. SCHNECKENBURGER

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of College Mathematics)

OFFICERS OF THE ASSOCIATION

President, SAUNDERS MACLANE, University of Chicago
Honorary President, W. D. CAIRNS, Oberlin College
First Vice-President, F. L. GRIFFIN, Wesleyan University
Second Vice-President, JEWELL H. BUSHEY, Hunter College
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo
Editor, C. B. ALLENDOERFER, University of Washington

Additional Members of the Board of Governors: I. A. BARNETT, C. F. BARR, T. A. BICKERSTAFF, J. C. BRIXEY, B. H. BROWN, R. H. CAMERON, G. R. CLEMENTS, T. F. COPE, H. S. M. COXETER, W. M. DAVIS, J. M. EARL, L. R. FORD, J. S. FRAME, A. E. GAULT, R. F. GRAESSER, E. H. HANSON, M. R. HESTENES, A. S. HOWARD, C. G. JAEGER, R. D. JAMES, B. W. JONES, R. E. LANGER, D. H. LEHMER, F. A. LEWIS, C. C. MACDUFFEE, R. J. MICHEL, W. E. MILNE, C. W. MUNSHOWER, H. P. PETTIT, J. C. POLLEY, G. B. PRICE, G. E. RAYNOR, E. B. ROESSLER, F. H. STEEN

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

THE PROBLEM OF SIMPLIFYING TRUTH FUNCTIONS

W. V. QUINE, Harvard University

The formulas of the propositional calculus, or the logic of truth functions, are here to be understood as built up of the statement letters 'p', 'q', 'r', · · · by just the notations of negation, conjunction, and alternation (or disjunction), *viz.* '$\bar{p}$', 'pq', and '$p \vee q$', to any degree of iteration. A formula is *valid* if it comes out true under all assignments of truth values to the letters, and *consistent* if it comes out true under some. One formula *implies* another if there is no assignment of truth values which makes the first formula true and the second false. Two formulas are *equivalent* if they imply each other. Implication and equivalence, so defined, are relations of formulas; they are not to be confused with the conditional and biconditional, commonly expressed by '$\supset$' and '$\equiv$'. These latter notations will be omitted, being translatable into terms of negation, conjunction, and alternation in familiar fashion.

It will be convenient to use the words 'conjunction' and 'alternation' in slightly extended senses. Ordinarily one speaks of a conjunction of two or more formulas; but I shall speak also of a conjunction of one formula, meaning thereby simply the formula itself. Thus every formula is a conjunction at least of itself. Correspondingly for alternation.

Letters and negations of letters will be spoken of collectively as *literals*. A conjunction of literals will be called a *fundamental* formula if no letter appears in it twice. Literals themselves count as fundamental formulas, in view of my broad use of the word 'conjunction'. Finally any alternation of fundamental formulas will be called a *normal* formula, and the fundamental formulas of which it is an alternation will be called its *clauses*. Fundamental formulas themselves count as normal, in view of my broad use of the word 'alternation'; a fundamental formula is a one-clause normal formula. In general, thus, normal formulas are simply what have been known in the literature as *disjunctive normal forms*, or *alternational normal forms*, except that normal formulas are subject to one additional requirement: no letter can occur in a clause twice. A normal formula cannot contain 'pqp' nor '$\bar{p}q\bar{p}$' nor '$pq\bar{p}$'. Mechanical routines are well known for transforming any consistent formula into an equivalent which is normal.*

But there remains a problem which, despite the trivial character of truth-function logic, has proved curiously stubborn; *viz.*, the problem of devising a general mechanical procedure for reducing any formula to its *simplest* equivalent. Since Shannon's correlation of the formulas of truth-function logic with electric circuits,† this problem of simplification has taken on significance for engineering; for, a technique for simplifying truth-functional formulas would be

* See, *e.g.*, §10 of my *Methods of Logic*, Henry Holt & Co., 1950.

† C. E. Shannon, A symbolic analysis of relay and switching circuits, Trans. Amer. Inst. of Electrical Engineers, vol. 57, 1938, pp. 713–723.

a technique for simplifying circuits. It is noteworthy that the staff of the Computation Laboratory of Harvard University have found it worth while to set forth elaborate procedures for simplifying truth functions, and even to tabulate all the simplest equivalents of formulas involving four or fewer letters.‡

In a certain theoretical sense, indeed, there is no problem. Given any formula, we can, in principle, survey the totality of simpler formulas involving no additional letters; for this totality is finite. By truth tables or otherwise, we can test each of these simpler formulas for equivalence to the given formula, and thus pick out the simplest equivalent. This procedure is mechanical; what is wanted, however, is a mechanical procedure which is short enough to be practical.

Because of the perspicuity and general convenience of normal formulas, an interesting specialization of the simplification problem is the problem of finding a simplest *normal* equivalent. In fact we may limit our problem to normal formulas from start to finish, since the preliminary step of converting a given formula into *some* normal equivalent, not necessarily a simplest, presents no problem. By limiting our consideration thus to normal formulas we are indeed disregarding inconsistent formulas, but this is no real limitation, since a shortest equivalent of any inconsistent formula can be supplied out of hand: '$p\bar{p}$'. So the problem which I shall examine is that of converting any normal formula into a simplest normal equivalent. This is not the most general form of the simplification problem from the point of view of engineering, since it can happen that some short non-normal formula represents a still cheaper electric circuit than any normal equivalent. But it will be more than enough to occupy us on the present occasion.

Limiting ourselves to normal formulas, we still have some choice as to our measure of simplicity. We might simply count all occurrences of literals and alternation signs, or we might put a premium on fewness of clauses and so resort to a count of occurrences of literals only when comparing formulas which are alike in number of clauses. What I shall have to say in this paper will not require any decision, however, between these or other reasonable standards of simplicity.

Let us use the Greek letter 'ζ' to refer to any literal, and 'ϕ', 'ψ', 'χ' to refer to any fundamental formulas, and 'Φ' and 'Ψ' more generally to refer to any normal formulas. In order to refer to compounds of formulas which are severally referred to thus by Greek letters, let us use corresponding compounds of the Greek letters themselves; thus where ζ is taken as '$\bar{p}$' and ϕ as '$p\bar{q}$', $\zeta \vee \phi$ is to be understood as '$\bar{p} \vee p\bar{q}$'.

Now it can happen that some clause ϕ is superfluous in a normal formula $\phi \vee \Psi$; *i.e.*, that $\phi \vee \Psi$ is equivalent to Ψ alone. It can also happen that an occurrence of a literal ζ is superfluous in a normal formula $\phi\zeta \vee \Psi$; *i.e.*, that $\phi\zeta \vee \Psi$ is equivalent to $\phi \vee \Psi$ alone. Weeding out such superfluous clauses and literals is

‡ *Synthesis of Electronic Computing and Control Circuits*, by the aforementioned staff, headed by Howard H. Aiken. Harvard University Press, 1951.

the obvious way of reducing normal formulas to simpler normal equivalents. To implement this sort of reduction, all we need are convenient techniques for spotting superfluous clauses and literals. Now such techniques are readily devised, as follows.

To say that $\phi \vee \Psi$ is equivalent to Ψ is the same as saying that ϕ implies Ψ. Also, as is slightly less evident but readily verifiable, to say that $\phi\zeta \vee \Psi$ is equivalent to $\phi \vee \Psi$ is the same as saying that ϕ implies $\zeta \vee \Psi$. To test a clause ϕ for superfluousness in a normal formula, therefore, we have only to see whether ϕ implies the rest of the normal formula; and to test an occurrence of a literal ζ in a clause $\phi\zeta$ of a normal formula for superfluousness, we have again only to see whether ϕ implies the rest of the normal formula. Now the ϕ in either problem is a fundamental formula; and any question of implication on the part of a fundamental formula ϕ is always quickly settled. To find whether ϕ implies any given formula we have merely to mark as true, throughout the given formula, all the letters which occur affirmatively in ϕ, and as false all the letters which occur negated in ϕ, and then see whether the given formula thereupon comes out true (for all values of any remaining letters).

Example 1: We find the clause '$p\bar{r}$' of '$pq \vee p\bar{r} \vee \bar{q}\bar{r}$' superfluous by testing to see if it implies the rest, '$pq \vee \bar{q}\bar{r}$'. The test of implication consists in putting 'T' for 'p' and 'F' for 'r' (conformably with '$p\bar{r}$') in '$pq \vee \bar{q}\bar{r}$'; the result is '$Tq \vee \bar{q}T$', which reduces to '$q \vee \bar{q}$'.

Example 2: We find the first occurrence of '$\bar{q}$' in '$pq \vee p\bar{q}r \vee \bar{p}\bar{q}\bar{r}$' superfluous by testing to see if 'pr' implies the rest, '$pq \vee \bar{q} \vee \bar{p}\bar{q}\bar{r}$'. The test of implication consists in putting 'T' for 'p' and 'r' in '$pq \vee \bar{q} \vee \bar{p}\bar{q}\bar{r}$'; the result '$Tq \vee \bar{q} \vee F\bar{q}F$' reduces to '$q \vee \bar{q}$'.

Let us call a normal formula *irredundant* if it has no superfluous clauses and none of its clauses has superfluous literals. We now have a mechanical routine for reducing any normal formula to an irredundant equivalent. Summed up, it runs as follows. First try each clause in turn to see whether it implies the rest of the formula; whenever any clause is found which does imply the rest of the formula, delete it once and for all before continuing the survey. After all reductions of this type are at an end, then try each "immediate subclause" (a clause minus one of its literals) to see whether it implies the rest of the formula; if it does, delete the superfluous occurrence of the literal. When this process can be carried no farther, we have an irredundant formula.

It seems reasonable to hope that this procedure of simplification, issuing as it does in a normal equivalent which is irredundant, may solve our original problem; namely, the problem of reducing any normal formula to a simplest normal equivalent. The procedure leads to a normal formula in which no clause is superfluous and no occurrences of literals within clauses are superfluous; and it seems reasonable to suppose that such a normal formula is as simple as any equivalent normal formula can be.

But this is not so. Consider the normal formula '$p\bar{q} \vee \bar{p}q \vee q\bar{r} \vee \bar{q}r$'. This is irredundant; no clause can be dropped, nor can any occurrences of literals be

dropped, without breach of equivalence. Yet this formula has simpler normal equivalents, indeed two: both '$p\bar{q} \vee \bar{p}r \vee q\bar{r}$' and '$\bar{p}q \vee p\bar{r} \vee \bar{q}r$', as can be checked by truth tables. These are simpler than the original by any conceivable standard of simplicity, but they cannot be got from the original by any process of dropping or curtailing clauses.

The routine of eliminating redundancies by dropping or curtailing clauses remains useful, for it is quick and easy and it brings gains in simplicity wherever it can be used. But it does not assure us always of a simplest result. The remainder of this paper will be devoted to presenting a general procedure for finding a really simplest normal equivalent. The procedure will be laborious, but not to the point of unmanageability.

A normal formula is called *developed* if all of its letters appear in each of its clauses; e.g., '$pq\bar{r} \vee p\bar{q}r$'. Any normal formula can be turned into a developed equivalent by an obvious procedure: any clause ϕ which lacks a letter ζ can be supplanted by its equivalent $\phi\zeta \vee \phi\bar{\zeta}$, and the process can be continued until each clause contains each letter, duplicate clauses being dropped as they arise. Example: the normal formula '$pqr \vee r\bar{s}$' becomes '$pqrs \vee pqr\bar{s} \vee pr\bar{s} \vee \bar{p}r\bar{s}$', which in turn becomes '$pqrs \vee pqr\bar{s} \vee p\bar{q}r\bar{s} \vee \bar{p}qr\bar{s} \vee \bar{p}\bar{q}r\bar{s}$', a developed normal formula. The procedure for finding simplest normal equivalents will take developed normal formulas as its point of departure. Meanwhile a couple of auxiliary notions must be defined, and their properties established.

DEFINITIONS: ϕ will be said to *subsume* ψ if and only if all the literals whereof ψ is a conjunction are among the literals whereof ϕ is a conjunction. ϕ will be called a *prime implicant* of Ψ if and only if ϕ implies Ψ and subsumes no shorter formula which implies Ψ. ϕ will be called a *completion* of χ with respect to Ψ if and only if ϕ subsumes χ and contains all letters of Ψ and no others.

THEOREM 1. *Any simplest normal equivalent of* Φ *is an alternation of prime implicants of* Φ.

Proof. Every clause ψ of a normal equivalent Ψ of Φ implies Ψ and therefore Φ. So, if ψ is not a prime implicant of Φ, then ψ subsumes a shorter formula ψ' which implies Φ and therefore Ψ. But then Ψ has one or more redundant occurrences of literals, in ψ, which could be deleted (as noted in earlier pages); so Ψ is not a simplest normal equivalent.

The above theorem brings out the relevance, to our simplification project, of listing the prime implicants of a formula Φ. The way to obtain such a list will become evident, in the case of developed Φ, after the next three theorems.

THEOREM 2. *No prime implicant of* Φ *contains letters foreign to* Φ.

Proof. If $\psi\zeta$ implies Φ and the letter in ζ is foreign to Φ, then any assignment of truth values which makes ψ true will make Φ true, regardless of ζ; *i.e.*, ψ will imply Φ, and hence $\psi\zeta$ will not be a prime implicant of Φ.

THEOREM 3. *If Φ is a developed normal formula and contains all letters of ψ, then ψ implies Φ if and only if all completions of ψ with respect to Φ are clauses of Φ.*

Proof. If ψ has a completion ψ' which is not a clause of Φ, then each clause of Φ contains a letter affirmatively which is negated in ψ' or vice versa. Then the assignment of truth values to letters which makes ψ' true makes all clauses of Φ false, though making ψ true; so ψ does not imply Φ. Conversely, each assignment of truth values to the letters of Φ which makes ψ true makes some completion of ψ true; so, if all the completions of ψ are clauses of Φ, then each assignment which makes ψ true makes Φ true, and hence ψ implies Φ.

From Theorems 2 and 3 and the definition of prime implicant, there follows this corollary:

THEOREM 4. *ψ is a prime implicant of a developed normal formula Φ if and only if all letters of ψ are among those of Φ and all completions of ψ with respect to Φ are clauses of Φ and there is no shorter formula ψ', subsumed by ψ, such that all completions of ψ' with respect to Φ are clauses of Φ.*

Theorem 4 enables us, given a developed normal formula $\phi_1 \vee \cdots \vee \phi_n$, to arrive at its prime implicants by the following mechanical routine. We make a growing list which does not begin as a list of prime implicants, but begins rather with $\phi_1, \cdots, \phi_n$ and is extended according to the following principle: whenever two entries can be found in the list which are related as $\chi\zeta$ and $\chi\bar{\zeta}$ (thus identical except for a negation sign), add their common part χ as a new entry in the list. Check marks are to be applied to any entries $\chi\zeta$ and $\chi\bar{\zeta}$ which thus generate new entries, but a check mark is not to be treated as disqualifying an entry from reuse; thus '$pqrs$' can be used once with '$pqr\bar{s}$' to generate 'pqr' and once with '$p\bar{q}rs$' to generate 'prs'. When the list has been extended as far as possible by the above process, we can read off the prime implicants of $\phi_1 \vee \cdots \vee \phi_n$ from it thus: they are the entries which bear no check marks.

Example: Suppose $\phi_1, \cdots, \phi_n$ are '$pqrs$', '$\bar{p}\bar{q}rs$', '$pq\bar{r}s$', '$p\bar{q}\bar{r}s$', '$pq\bar{r}\bar{s}$', and '$p\bar{q}\bar{r}\bar{s}$'. The first and third of these six yield 'pqs' as a seventh entry in our list; the third and fourth yield '$p\bar{r}s$' as an eighth; the third and fifth yield '$pq\bar{r}$' as a ninth; the fourth and sixth yield '$p\bar{q}\bar{r}$' as a tenth; and the fifth and sixth yield '$p\bar{r}\bar{s}$' as an eleventh. Of the original six entries, all but the second receive check marks in the process. Proceeding now to generate still further entries from the five added ones, we get '$p\bar{r}$' twice and nothing more; and accordingly we apply check marks to '$p\bar{r}s$' and '$p\bar{r}\bar{s}$', and also to '$pq\bar{r}$' and '$p\bar{q}\bar{r}$'. (Note the necessity of applying check marks to all four, despite the duplicate nature of their yield.) Surveying the finished list, we find just these entries without checks: '$\bar{p}\bar{q}rs$', 'pqs', '$p\bar{r}$'. These are the prime implicants of '$pqrs \vee \bar{p}\bar{q}rs \vee pq\bar{r}s \vee p\bar{q}\bar{r}s \vee pq\bar{r}\bar{s} \vee p\bar{q}\bar{r}\bar{s}$'.

How to use the list of prime implicants, in order to obtain a simplest normal

equivalent of a developed normal formula, is suggested by the next theorem together with Theorem 1.

THEOREM 5. *If Ψ is a simplest normal equivalent of a developed normal formula Φ, then each clause of Φ subsumes a clause of Ψ.*

Proof. Consider any clause ϕ of Φ. By Theorems 1 and 2, Ψ contains no letters foreign to Φ, nor, therefore, to ϕ. Hence any clause of Ψ which ϕ does not subsume must contain a letter affirmatively which is negated in ϕ or vice versa. Hence the assignment of truth values to letters which makes ϕ true will make all clauses of Ψ false except those which ϕ subsumes. Hence ϕ must subsume a clause of Ψ if ϕ is to imply Ψ. But ϕ does imply Ψ, since Φ is equivalent to Ψ.

In view of Theorems 1 and 5, we can obtain a panorama of all simplest normal equivalents of a developed normal formula $\phi_1 \vee \cdots \vee \phi_n$ as follows. First we list the prime implicants, as seen earlier. Then we survey the various subsets of the list, such that ϕ_i for each i subsumes a member of the subset. Each simplest such subset, written as an alternation, is a simplest normal equivalent of $\phi_1 \vee \cdots \vee \phi_n$.

The survey is facilitated by constructing what I shall call the *table of prime implicants* of $\phi_1 \vee \cdots \vee \phi_n$. The abscissas of the table, inscribed across the top, are $\phi_1, \cdots, \phi_n$. The ordinates of the table, inscribed down the left side, are the prime implicants of $\phi_1 \vee \cdots \vee \phi_n$. In the interior of the table we enter crosses in those positions whose abscissas subsume their ordinates.

For the example '$pqrs \vee \bar{p}\bar{q}rs \vee pq\bar{r}s \vee p\bar{q}\bar{r}s \vee pq\bar{r}\bar{s} \vee p\bar{q}\bar{r}\bar{s}$', whose prime implicants were derived earlier, the table is this:

	$pqrs$	$\bar{p}\bar{q}rs$	$pq\bar{r}s$	$p\bar{q}\bar{r}s$	$pq\bar{r}\bar{s}$	$p\bar{q}\bar{r}\bar{s}$
$\bar{p}\bar{q}rs$		×				
pqs	×		×			
$p\bar{r}$			×	×	×	×

Once we have the table of prime implicants, we canvass all ways of so selecting ordinates as to *represent* all abscissas; *i.e.*, to show crosses under all abscissas. We settle upon a selection such that the alternation of the selected ordinates will be as simple as possible. In the above example there is no choice; no selection of rows, short of all three, exhibits crosses in all columns. So in this example the simplest normal equivalent is '$p\bar{q}rs \vee pqs \vee p\bar{r}$', which uses all the prime implicants.

For another example let us return to '$p\bar{q} \vee \bar{p}q \vee q\bar{r} \vee \bar{q}r$', which was cited earlier to show that irredundant formulas could have simpler equivalents. To find the simplest normal equivalents of this example by our new general method, we must first expand the formula into a developed normal formula, then derive the list of prime implicants, and finally form the table. The table turns out thus:

	$p\bar{q}r$	$p\bar{q}\bar{r}$	$\bar{p}qr$	$\bar{p}q\bar{r}$	$pq\bar{r}$	$\bar{p}\bar{q}r$
$p\bar{q}$	×	×				
$\bar{q}r$	×					×
$p\bar{r}$		×			×	
$\bar{p}q$			×	×		
$\bar{p}r$			×			×
$q\bar{r}$				×	×	

Survey of the table shows two ways of so picking three rows as to represent all columns, so we come out with two simplest normal equivalents, '$\bar{p}q \vee \bar{q}r \vee p\bar{r}$' and '$p\bar{q} \vee \bar{p}r \vee q\bar{r}$'.

Incidentally the list of prime implicants of a formula Φ has other uses besides its use in obtaining the simplest normal equivalents of Φ. It provides a panorama of all the fundamental formulas which imply Φ; for, the fundamental formulas which imply Φ are simply the prime implicants and all other fundamental formulas which subsume any of them.

So far as concerns the topic of the present paper, however, the use of the table is in finding shortest normal equivalents. As described thus far, the use of the table for this purpose proceeds by exhaustion: trying all the combinations of ordinates which represent all abscissas, and comparing all the resulting alternations for simplicity. Now this process of canvassing the table can be speeded up in many examples (though not in the above two) by the following routine of preparatory reduction.*

(i) If any columns of the table of a formula Φ contain only one cross apiece, then record for future reference the alternation of the ordinates of those crosses. Let us call this alternation the *core* of Φ. (The clauses of the core are bound, by Theorem 5, to be clauses of any simplest normal equivalent of Φ.)

(ii) Reduce the table by deleting the ordinates concerned in (i), and deleting also all abscissas represented by those ordinates. (These abscissas need no further consideration because they will be represented by clauses of our final simplification of Φ anyway as long as we take care to include the core as part of that final simplification.)

(iii) Wherever in the surviving table there are abscissas ϕ_i and ϕ_j such that ϕ_i has crosses only in rows in which ϕ_j has crosses, delete ϕ_j. (For, our final formula is bound to represent ϕ_j anyway, through representing ϕ_i.)

* Note the resemblance of the ensuing operations to the operations on "minimizing charts" which are set forth in pp. 56 ff. of *Synthesis* (see preceding footnote). The clauses of what I call the *core* (below) correspond to what are called "essential combinations" in *Synthesis*. More accurately, the clauses of the core are the duals of the essential combinations; for the minimizing charts produce conjunctional normal forms, in effect, rather than alternation ones. Between the minimizing charts and the tables of the present paper there are profound differences, however, beyond that of duality. A minimizing chart begins as a fixed form which depends only on the multiplicity of letters concerned, and not on the particular formula at hand. It tends in consequence to be more elaborate than the table of prime implicants.

(iv) Delete any ordinates whose crosses have all been lost through the cancelling of abscissas in (ii) and (iii).

The *reduced table of prime implicants* thus achieved can now be subjected to the process, described earlier, of canvassing the ways of selecting ordinates and singling out the most economical. Each end result thus obtained must be supplemented by adjoining the core to it, in alternation.

Example: $pqr \vee p\bar{r} \vee pq\bar{s} \vee \bar{p}r \vee \bar{p}\bar{q}\bar{r}\bar{s}$.

The table of prime implicants turns out as follows:

	$pqrs$	$pqr\bar{s}$	$pq\bar{r}s$	$pq\bar{r}\bar{s}$	$p\bar{q}\bar{r}s$	$p\bar{q}\bar{r}\bar{s}$	$\bar{p}qrs$	$\bar{p}qr\bar{s}$	$\bar{p}q\bar{r}s$	$\bar{p}q\bar{r}\bar{s}$	$\bar{p}\bar{q}\bar{r}\bar{s}$
pq	×	×	×	×							
qr	×	×					×	×			
$p\bar{r}$			×	×	×	×					
$\bar{p}r$							×	×	×	×	
$\bar{p}\bar{q}\bar{s}$										×	×
$\bar{q}\bar{r}\bar{s}$						×					×

Now we apply (i); *i.e.*, observing that the fifth and ninth columns contain only one cross apiece, we record the alternation of the ordinates of those two crosses; *viz.*, '$p\bar{r} \vee \bar{p}r$'. This is the core. Then, applying (ii), we cancel the ordinates '$p\bar{r}$' and '$\bar{p}r$' of the table, and also the eight columns (*viz.*, third through tenth) in which those cancelled rows contained crosses. To what is left of the table, we apply (iii); this enables us to cancel the first or second column at will, say the second. We find no way of applying (iv), so we are now down to our reduced table of prime implicants, which is just this:

	$pqrs$	$\bar{p}\bar{q}\bar{r}\bar{s}$
pq	×	
qr	×	
$\bar{p}\bar{q}\bar{s}$		×
$\bar{q}\bar{r}\bar{s}$		×

Inspection of this table shows just four shortest alternations of ordinates representing both abscissas. They are:

$$pq \vee \bar{p}\bar{q}\bar{s}, \qquad pq \vee \bar{q}\bar{r}\bar{s}, \qquad qr \vee \bar{p}\bar{q}\bar{s}, \qquad qr \vee \bar{q}\bar{r}\bar{s}.$$

Adjoining any of these by alternation to the core '$p\bar{r} \vee \bar{p}r$' gives a simplest normal equivalent of the original formula. We thus end up with four simplest normal equivalents:

$$p\bar{r} \vee \bar{p}r \vee pq \vee \bar{p}\bar{q}\bar{s}, \qquad p\bar{r} \vee \bar{p}r \vee pq \vee \bar{q}\bar{r}\bar{s},$$
$$p\bar{r} \vee \bar{p}r \vee qr \vee \bar{p}\bar{q}\bar{s}, \qquad p\bar{r} \vee \bar{p}r \vee qr \vee \bar{q}\bar{r}\bar{s}.$$

Sometimes the reduced table of prime implicants turns out to be an utter blank, so that the core stands alone as the simplest normal equivalent. An example is '$p\bar{r} \vee \bar{p}\bar{q}r\bar{s} \vee \bar{p}\bar{q}\bar{r}$'. Here the table of prime implicants is:

	$pq\bar{r}s$	$pq\bar{r}\bar{s}$	$p\bar{q}\bar{r}s$	$p\bar{q}\bar{r}\bar{s}$	$\bar{p}\bar{q}rs$	$\bar{p}\bar{q}\bar{r}s$	$\bar{p}\bar{q}\bar{r}\bar{s}$
$p\bar{r}$	×	×	×	×			
$\bar{q}\bar{r}$			×	×		×	×
$\bar{p}\bar{q}\bar{s}$					×		×

Applying (i) to this, we obtain '$p\bar{r} \vee \bar{q}\bar{r} \vee \bar{p}\bar{q}\bar{s}$' as core. All rows and columns disappear under (ii), so that we are left with '$p\bar{r} \vee \bar{q}\bar{r} \vee \bar{p}\bar{q}\bar{s}$' itself as the simplest normal equivalent of '$p\bar{r} \vee \bar{p}\bar{q}rs \vee \bar{p}\bar{q}\bar{r}$'.

The method admits of one further refinement which, though irrelevant to all the foregoing examples, saves much labor where applicable. I am indebted for it in part to Nelson Goodman. Given a formula for which a simplest normal equivalent is wanted, the new tactic begins by transforming the formula not into a developed normal formula, but rather into an irredundant normal formula by the routine of the early pages of this paper. This irredundant alternation is then separated into as many subsidiary alternations as possible such that no two of them have letters in common. (*E.g.*,' $p\bar{q} \vee \bar{r}s \vee \bar{p}q \vee t$' would be separated into '$p\bar{q} \vee \bar{p}q$', '$\bar{r}s$', and 't'.) Then we expand each of these subsidiary alternations independently into developed normal form and proceed to find a simplest normal equivalent for it, by use of a reduced table of prime implicants as hitherto explained. Finally we make a single alternation of the several results, and this is a simplest normal equivalent of the original formula.

The value of this separation expedient, where applicable, is evident: it saves the exorbitant development of all clauses with respect to all missing letters. But we must prove that the modified method always leads to the simplest normal equivalents. This will be proved as Theorem 8 below; the two intervening theorems are needed as lemmas.

THEOREM 6. *The only irredundant normal formulas which are valid are* '$p \vee \bar{p}$', '$q \vee \bar{q}$', *etc.*

Proof. Let $\phi\zeta \vee \Psi$ be a valid normal formula. Consider then any assignment of truth values to letters which makes ϕ true. It makes $\phi\zeta \vee \Psi$ true, since this is valid; moreover ϕ, being true under this assignment, can be deleted from $\phi\zeta \vee \Psi$ and the result $\zeta \vee \Psi$ will still be true. Thus every assignment which makes ϕ true makes $\zeta \vee \Psi$ true; *i.e.*, ϕ implies $\zeta \vee \Psi$. But this implication was seen, in early pages of the paper, to be the criterion of superfluousness of the occurrence of ζ in $\phi\zeta \vee \Psi$. We see therefore that no valid irredundant normal formula can have the form $\phi\zeta \vee \Psi$. Still every valid normal formula is obviously an alternation of at least two clauses; any single clause is falsifiable. Therefore every valid irredundant normal formula must be an alternation of clauses none of which is of the form $\phi\zeta$; each of which, in other words, is a single literal. Every valid irredundant normal formula has, in short, the form $\zeta_1 \vee \cdots \vee \zeta_n$. But obviously two of $\zeta_1, \cdots, \zeta_n$ must, for validity of $\zeta_1 \vee \cdots \vee \zeta_n$, be negations one of the other. But then each of $\zeta_1, \cdots, \zeta_n$ other than those two is superfluous; or rather there are no others, since $\zeta_1 \vee \cdots \vee \zeta_n$ is supposed to be irredundant. So

$\zeta_1 \vee \cdots \vee \zeta_n$ is just '$p \vee \bar{p}$', or perhaps '$q \vee \bar{q}$', *etc.*

THEOREM 7. *If no two of* $\Phi_1, \cdots, \Phi_n$ *have letters in common, and* ϕ *is a prime implicant of* $\Phi_1 \vee \cdots \vee \Phi_n$, *then* ϕ *contains letters exclusively of* Φ_i *for some* i.

Proof. By Theorem 2, there is an i such that some of the letters of ϕ appear in Φ_i. Now suppose (which will be proved impossible) that there are also letters in ϕ foreign to Φ_i; *i.e.*, that ϕ is $\psi\chi$ where all the letters of ψ but none of the letters of χ are letters of Φ_i. Since ϕ is a prime implicant, neither ψ nor χ implies $\Phi_1 \vee \cdots \vee \Phi_n$. Hence ψ does not imply Φ_i, and χ does not imply $\Phi_1 \vee \cdots \vee \Phi_{i-1} \vee \Phi_{i+1} \vee \cdots \vee \Phi_n$. Hence there is an assignment of truth values to the letters of Φ_i which makes ψ true and Φ_i false, and there is an assignment of truth values to the rest of the alphabet which makes χ true and $\Phi_1 \vee \cdots \vee \Phi_{i-1} \vee \Phi_{i+1} \vee \cdots \vee \Phi_n$ false. Pooling the two assignments (which we can do since the sets of letters concerned are mutually exclusive), we have an assignment which makes ϕ true and $\Phi_1 \vee \cdots \vee \Phi_n$ false. But this is impossible, since ϕ by hypothesis implied $\Phi_1 \vee \cdots \vee \Phi_n$.

THEOREM 8. *If* $\Phi_1 \vee \cdots \vee \Phi_n (n > 1)$ *is irredundant and no two of* $\Phi_1, \cdots, \Phi_n$ *have letters in common, then any simplest normal equivalent of* $\Phi_1 \vee \cdots \vee \Phi_n$ *will be of the form* $\Psi_1 \vee \cdots \vee \Psi_n$ *where* $\Psi_1, \cdots, \Psi_n$ *are equivalent respectively to* $\Phi_1, \cdots, \Phi_n$.

Proof. Let Ψ be a simplest normal equivalent of $\Phi_1 \vee \cdots \vee \Phi_n$. Then Φ_i, for each i, implies Ψ. Now suppose that Φ_i has no letters in common with Ψ. Implication can occur without common letters only in the extreme cases where the implying formula is inconsistent or the implied one is valid. But Φ_i, being normal, is consistent. So Ψ would have to be valid. Then its equivalent $\Phi_1 \vee \cdots \vee \Phi_n$ would be valid, and hence, by Theorem 6, would be simply '$p \vee \bar{p}$' or '$q \vee \bar{q}$' or the like. But by hypothesis this is impossible; for by hypothesis $n > 1$, and therefore $\Phi_1 \vee \cdots \vee \Phi_n$ contains two or more distinct letters. We conclude, therefore, that Φ_i for each i has letters in common with Ψ. Conversely, by Theorems 1 and 7, each clause of Ψ contains letters exclusively of Φ_i for some i. Therefore Ψ has the form $\Psi_1 \vee \cdots \vee \Psi_n$ where Ψ_i, for each i, contains letters exclusively of Φ_i. It remains to show that Φ_i implies Ψ_i and vice versa. We saw that $\Phi_1 \vee \cdots \vee \Phi_n$ is not valid; neither, therefore, is its equivalent $\Psi_1 \vee \cdots \vee \Psi_n$ valid. So there is an assignment $\mathfrak{A}$, of truth values to the letters of $\Psi_1 \vee \cdots \vee \Psi_{i-1} \vee \Psi_{i+1} \vee \cdots \vee \Psi_n$, which makes the latter formula false. Consider now any assignment $\mathfrak{B}$, of truth values to the letters of Φ_i, which makes Φ_i true. We can combine $\mathfrak{B}$ with $\mathfrak{A}$, since the sets of letters concerned are mutually exclusive; and the combined assignment makes Φ_i true and $\Psi_1 \vee \cdots \vee \Psi_{i-1} \vee \Psi_{i+1} \vee \cdots \vee \Psi_n$ false. Since the combined assignment makes Φ_i true, it must make $\Psi_1 \vee \cdots \vee \Psi_n$ true (for this latter is equivalent to $\Phi_1 \vee \cdots \vee \Phi_n$). More particularly then it must make Ψ_i true (for we just noted that it made the rest of $\Psi_1 \vee \cdots \vee \Psi_n$ false). But the letters of Ψ_i receive truth values

only from $\mathfrak{B}$, not $\mathfrak{A}$; and $\mathfrak{B}$ was any assignment which makes Φ_i true. Therefore Φ_i implies Ψ_i. An exactly parallel argument, interchanging the roles of $\Phi_1, \cdots, \Phi_n$ with those of $\Psi_1, \cdots, \Psi_n$, shows conversely that Ψ_i implies Φ_i, thus completing the proof of Theorem 8.

Summarized, our results are as follows. We found, to begin with, a fairly rapid method of reducing any normal formula (and therefore any consistent formula) to the extent of locating and cancelling any redundancies. But we found also that an irredundant normal equivalent was not necessarily a simplest normal equivalent. Accordingly, taking a fresh start, we worked out a routine which could be depended upon to reveal a simplest normal equivalent, and indeed all the simplest normal equivalents. This routine, though not unmanageable, turned out to be far more laborious than the method of merely locating and cancelling redundancies. Moreover, the two methods are almost independent. The laborious method of finding simplest normal equivalents depends on a preliminary expansion into a developed normal formula, and this expansion is not affected by any previous cancelling of redundancies. The only way in which the cancelling of redundancies contributes to the ultimate technique is in connection with the auxiliary expedient of separation developed in these last few pages. Clearly it would be desirable to find a quicker way of getting simplest normal equivalents, say by gearing the whole routine to irredundant formulas rather than to developed formulas. I have not seen how to manage this.

It may be useful to note one particular class of normal formulas which can be exempted from the foregoing procedures altogether; *viz.*, those normal formulas in which no one letter occurs both affirmatively and negatively. Such a formula is already reduced to simplest normal form as soon as we have merely deleted those of its clauses that subsume others of its clauses. I have proved this fact elsewhere,* for the case where all letters are affirmative; and the present extension then followed by substitution of negations of letters for letters.

* *Dos teoremas sobre funciones de verdad*, Memoria del Congreso Científico Mexicano, (año de 1951), vol. 1 (*ciencias físicas y matemáticas*). At press.

THE DIFFERENTIAL EQUATION OF A CONIC AND ITS RELATION TO THE ABERRANCY

A. W. WALKER, University of Toronto

1. Introduction.

(i) *General remarks.* In this paper, $\theta \equiv \tan^{-1}(y')$ is the angle which the tangent at a general point of a plane curve $y = y(x)$ makes with the x-axis; accents and dots denote differentiation with respect to x and θ respectively. The notation $\xi \equiv \phi^2 \equiv \rho^{-2/3}$ is used, where ρ is the radius of curvature; any equation relating

ρ, ϕ, or ξ to θ will be called an *intrinsic* equation. It is assumed that the portion of the curve $y=y(x)$ under consideration is such that all processes used are valid; in particular, we assume that $y''\neq 0$ and $\rho \neq \infty$, and that the term "conic" implies a non-linear conic. Certain sign conventions are frequently assumed; these should be clear from the context and diagrams.

(ii) *Summary.* The usual method of deriving the Cartesian differential equation of a conic is to eliminate the constants from the general equation of the conic by successive *implicit* differentiation, thus obtaining the familiar Monge form [**1, 8**]. A much more compact and significant form results from an *explicit* approach [**10**], as shown in Section 2. (For this I am indebted to Mr. E. E. Noonan.)

For the general conic and the circle there is no obvious connection between the *Cartesian* forms of the equation of the curve and the corresponding differential equation. For the circle, the connection between the *intrinsic* forms $\rho=$constant, $\dot{\rho}=0$ is immediate; this suggests that it will be profitable to investigate the intrinsic forms for the conic, and this is done in Section 3. A remarkably compact Cartesian-intrinsic functional identity is established which is useful in identifying the results of Sections 2 and 3 and thus proving the sufficiency of the intrinsic forms of the equations.

The full significance of these differential equations does not appear until we interpret them geometrically. The condition that a curve should be a circle is that $\dot{\rho}$, the radius of curvature of its evolute, must be zero for all points of the curve, and this is the geometrical interpretation of the differential equation of the circle. Only the concept of curvature is required here; to interpret similarly the differential equation of a conic we need the additional concept of *aberrancy* [**3, 4, 5**] defined and applied in Section 4. The special case of the rectangular hyperbola is treated in Section 5 by an extension of this concept.

Most of these results are not new, but they do not appear to have been previously stated as compactly, or collected to present as complete a picture.

2. The Cartesian differential equation of a conic. If $b\neq 0$, the general equation of a conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \tag{1}$$

can be expressed in the explicit form

$$y = mx + n + (px^2 + 2qx + r)^{1/2}. \tag{2}$$

It follows that

$$y'' = (pr - q^2)\cdot(px^2 + 2qx + r)^{-3/2} \tag{3}$$

and therefore (if $y''\neq 0$) we have, with Halphen [**10**],

$$(y''^{-2/3})''' = 0. \tag{4}$$

If (2) represents a parabola, then (transposing and squaring) the second degree terms must form a perfect square, *i.e.* $p=0$. Hence, from (3)

$$(y''^{-2/3})'' = 0. \tag{5}$$

If $b=0$, $h\neq 0$, then (1) can be expressed in the form $y=(\alpha x+\beta)+\gamma/(hx+f)$, and (4) is satisfied. If $b=h=0$, then $y'''=0$ and (4) and (5) are satisfied.

The differentiations performed above are clearly reversible, so that (4) and (5) are also *sufficient* conditions. We can say, then, that (4) and (5) are the differential equations of any non-linear conic and any parabola respectively.

Expansion of (4) gives us the more familiar (but less significant) Monge form [1, 8] of the differential equation, usually obtained by eliminating the constants from (1) by successive implicit differentiation.

3. The intrinsic equations. It is convenient to introduce the notation

$$\xi \equiv \phi^2 \equiv \rho^{-2/3} \equiv y''^{2/3}/(1+y'^2) \tag{6}$$

where ρ is the radius of curvature of the curve $y=y(x)$.

By a special choice of axes, the equation of any conic can be expressed in the form $y^2=L+Mx+Nx^2$, the x-axis being an axis of symmetry. It follows that

$$yy'' + y'^2 = N \tag{7}$$

and therefore $yy'''+3y'y''=0$, *i.e.* $y^3y''=$constant; and this, with (7), gives y'' in terms of y'. Substituting in (6), we then have, as the intrinsic equation of any conic,

$$\xi \equiv \phi^2 = A + B\cos 2\theta \tag{8}$$

where $y'\equiv\tan\theta$ and A, B are constants. The corresponding intrinsic differential equation (not involving θ explicitly) is then

$$\dddot{\xi} + 4\xi = 0 \tag{9}$$

which can be expressed in the alternative form

$$\frac{d}{d\theta}\left[\phi^3(\ddot{\phi}+\phi)\right] = 0. \tag{10}$$

Note that (8), (9) are much more closely related than are the Cartesian forms (1), (4).

If (8) represents a *parabola*, then $\phi\to 0$ as $\theta\to 0$; hence $B=-A$, $\phi^2=2A\sin^2\theta$, and

$$\ddot{\phi} + \phi = 0. \tag{11}$$

If (8) represents a *rectangular hyperbola*, then $\xi\to 0$ as $\theta\to\pi/4$; hence $A=0$ and

$$\ddot{\xi} + 4\xi = 0. \tag{12}$$

Less compact forms of (9)–(12) are obtained by Perrin [8], using a different

approach. (See also Cesàro [7].)

Differentiating the relation $y' \equiv \tan\theta$, we find that

$$\dot{x} \equiv (1 + y'^2)/y''. \tag{13}$$

Using this where necessary, we can show, by direct calculation from (6), that for any curve

$$*\phi^3(\ddot{\phi} + \phi) \equiv \tfrac{1}{3}(y''^{-5/3}y''')' \equiv -\tfrac{1}{2}(y''^{-2/3})'', \tag{14}$$

an interesting Cartesian-intrinsic functional identity, the existence of which is strongly indicated by a comparison of (4), (5) with (10), (11). This identity, together with the results of Section 2, shows at once the *sufficiency* of the intrinsic forms (8)–(12).

We note, in passing, that it follows from (14) that $(y''^{-2/3})''$ is invariant under any translation and rotation of rectangular Cartesian axes. Note also that from (6), (13) we have $\dot{x} \equiv y''^{-1/3}/\phi^2$, so by differentiation of (14) we see that $y''^{-1/3}(y''^{-2/3})'''$ is invariant. (These results can also, of course, be obtained by direct transformation, without reference to the intrinsic forms.) Note particularly the Cartesian identities which result from a 90° rotation, *i.e.* from replacing y by x, x by $-y$.

4. The concept of aberrancy and its applications. Using only the concept of curvature, we can make the following geometrical statements:

(a) The center of curvature of a circle is the (fixed) center of the circle.

(b) For a circle, the radius of curvature of the evolute is zero.

(c) For a straight line, the radius of curvature is infinite.

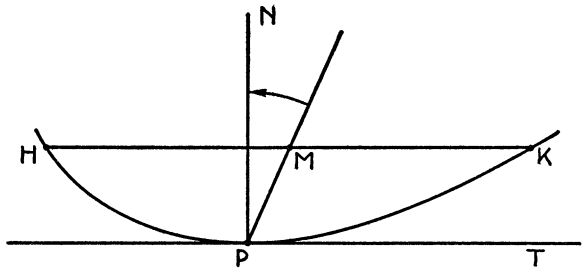


FIG. 1

To make similar statements for a conic, we require the additional concept of aberrancy, defined for any curve as follows. In Fig. 1, HK with midpoint M is a chord of any curve $y = y(x)$ parallel to the tangent at P. As the chord approaches the tangent, the angle between PM and the normal PN may approach a definite limit χ; following Salmon [3], we call χ the *aberrancy* of the curve at P. The limiting position of PM is the *axis of aberrancy*; the limiting position C (Fig. 2) of the point of intersection of neighboring axes of aberrancy is the

* This quantity is the *affine curvature*; from (10), (11) it is constant for a conic and zero for a parabola. (Blaschke [9], pp. 32, 18, 28).

center of aberrancy; the length $PC=r$ is the *radius of aberrancy*; and the locus of C is the *curve of aberrancy*.

Noting that for a conic the axis of aberrancy will be the diameter which bisects *all* chords parallel to the tangent, we see that

(15)
(a) The center of aberrancy of a conic is the (fixed) center of the conic.
(b) For a central conic, R (Fig. 2), the radius of curvature of the curve of aberrancy, is zero.
(c) For a parabola, the radius of aberrancy r is infinite.

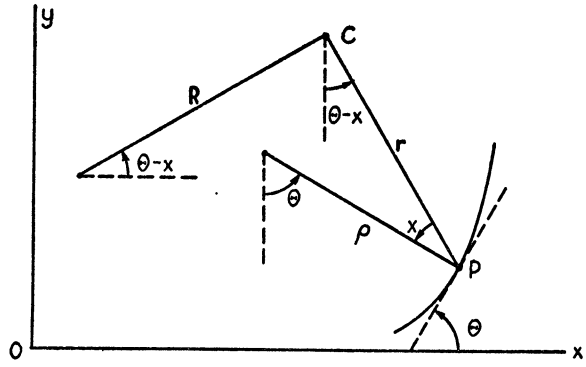


FIG. 2

The concept of aberrancy, then, plays the same role‡ in the discussion of the conic that curvature does for the circle. It was originally introduced by Transon [2] (using the name *deviation* for χ) to give a measure of the departure of a curve from its circle of curvature; more recent developments have been in the field of affine geometry, the axis of aberrancy being the *affine normal* (Blaschke [9], p. 15, 16 fn.).

In order to obtain the relations which must exist between R, r and (10), (11), we require an expression for $\tan\chi$. In Fig. 1, take PT, PN as axes of x and y; then for any point on the curve

(16)
$$x = \int_0^\theta \rho(\theta)\cdot\cos\theta d\theta = \int_0^\theta (\rho + \dot\rho\theta + \cdots)d\theta$$
$$= \rho\theta + \frac{1}{2}\dot\rho\theta^2 + \cdots$$
$$y = \int_0^\theta \rho(\theta)\cdot\sin\theta d\theta = \int_0^\theta (\rho\theta + \dot\rho\theta^2 + \cdots)d\theta$$
$$= \frac{1}{2}\rho\theta^2\left(1 + \frac{2}{3}\frac{\dot\rho}{\rho}\theta + \cdots\right)$$

‡ In a slightly different sense, this is also true for the concept of affine curvature (see previous footnote).

where ρ, $\dot{\rho}$, —, — are values at $\theta=0$, *i.e.* at the point P. Assuming an expansion for x as a power series in $\sqrt{y}$, *i.e.* $x=Ay^{1/2}+By+Cy^{3/2}+\cdots$, substituting from (16), and equating coefficients, we have $A=\pm\sqrt{2\rho}$, $B=\dot{\rho}/3\rho$; and therefore (Fig. 1)

$$\tan\chi=\lim_{y\to 0}(x_H+x_K)/2y=B=\dot{\rho}/3\rho=-\dot{\xi}/2\xi=-\dot{\phi}/\phi. \tag{17}$$

By comparison of Fig. 2 with a neighboring configuration, we have, to the first order

$$r\delta(\theta-\chi)=\rho\delta\theta\cdot\cos\chi, \quad \text{and} \quad \delta r=R\delta(\theta-\chi)-\rho\delta\theta\cdot\sin\chi$$

and therefore

$$r\phi^3(1-\chi)=\cos\chi \tag{18}$$

and

$$r\phi^3\dot{r}=r\phi^3R(1-\dot{\chi})-r\sin\chi=R\cos\chi-r\sin\chi. \tag{19}$$

Now from (17) we have $\dot{\phi}=-\phi\tan\chi$, and therefore

$$\ddot{\phi}=-\dot{\phi}\tan\chi-\phi\sec^2\chi\cdot\dot{\chi}=-\phi+\phi\sec^2\chi(1-\dot{\chi}). \tag{20}$$

It follows from (18), (20) that for any curve

$$\ddot{\phi}+\phi=\sec\chi/\phi^2r, \tag{21}$$

and we see how (15c) gives the geometrical significance of (11). A less compact form of (21) has been given by Transon [2].

Again, from (17), (18) we have $d(\phi\sec\chi)/d\theta=-\phi\sec\chi\tan\chi(1-\dot{\chi})=-\tan\chi/\phi^2r$, and this, together with (19), (21), gives for any curve

$$\frac{d}{d\theta}\left[\phi^3(\ddot{\phi}+\phi)\right]=-\frac{R}{\phi^2r^3}, \tag{22}$$

and in (15b) we have the geometrical significance of (10). According to Edwards [1], some form of (22) has been obtained previously [6]; the writer has not been able to check this reference.

We note that: (a) The *sufficiency* of the conditions (15) follows from (21), (22) and the results of Section 3. (b) As Salmon [3] remarks: (i) Any conic which has third order (*i.e.* 4-point) contact with a curve at P will have at P the same ϕ and $\dot{\phi}$ as the curve, and therefore, from (17), the same χ; hence its center will lie on the axis of aberrancy. (ii) In particular, the conic which has fourth order contact will have the same ϕ, $\dot{\phi}$, and $\ddot{\phi}$, and therefore, from (21), the same r; hence its center will be the center of aberrancy. (c) From (21), (10) we have the known result $\phi/p=$ constant, where $p\equiv r\cos\chi$ is the length of the perpendicular from the center of a conic to the tangent.

There are, incidentally, many additional ways of obtaining equations (9)–(11). For example, if m_1 and m_2 are the slopes (relative to an axis of the conic) of

the diameter and tangent through P, we have the familiar property of a conic

$$m_1 m_2 = \text{constant}, \tag{23}$$

and therefore the relation $\cot(\theta-\chi)\tan\theta = \text{constant}$ should, and does, lead to (9), as follows. By logarithmic differentiation, we have $(1-\dot{\chi})\sin 2\theta - \sin 2(\theta-\chi)=0$; dividing by $\sin 2\theta \sin 2\chi$, rearranging, and making use of (17), we have

$$\cot 2\theta = \frac{1}{2}\frac{d}{d\theta}(\log\tan\chi) - \tan\chi = \ddot{\xi}/2\dot{\xi},$$

which gives (9).

Alternatively, if O is any fixed point and t and p are the projections of PO along and perpendicular to the tangent to a curve at P, we can show that $\dot{p} = -t$, $\dot{t} = p - \rho$; from this, (10) and (11) can be deduced in several ways (using (17) or the (p, θ) equation of the conic, and taking O as the center or focus).

5. The rectangular hyperbola. For the rectangular hyperbola, (23) becomes $m_1 m_2 = 1$, from which it is easy to deduce that the bisector of the angle between the diameter and the normal is parallel to an axis of the hyperbola. Hence, if for any curve PH (Fig. 3) is the bisector of the angle of aberrancy χ and H

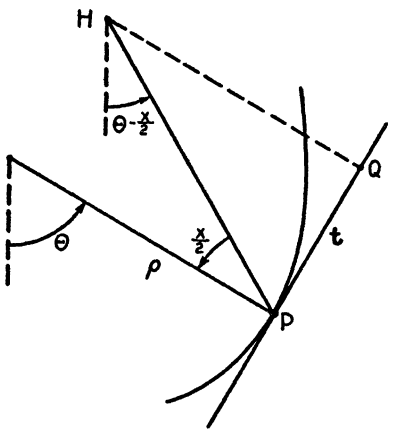


FIG. 3

is the limiting position of the point of intersection of PH with a neighbor, then H will be at infinity if the curve is a rectangular hyperbola. To the first order, we have $PH\cdot\delta(\theta-\frac{1}{2}\chi) = \rho\delta\theta\cos\frac{1}{2}\chi$, so if $PQ \equiv t$ is the projection of PH on the tangent, then $\phi^3 t(2-\dot{\chi}) = \sin\chi$. But just as we obtained (20) from (17), so we find that $\ddot{\xi} = -4\xi + 2\xi\sec^2\chi(2-\dot{\chi})$, and therefore for any curve,

$$\ddot{\xi} + 4\xi = \frac{2\sec\chi\tan\chi}{\phi t}, \tag{24}$$

from which we see the geometrical significance of (12), *i.e.* $t = \infty$ for a rectangular hyperbola.

References

1. Edwards, Differential Calculus, 3rd. ed., 1915, pp. 435, 436.
2. Transon, Journal de Math., Series 1, 6, 1841, pp. 191–197.
3. Salmon, Higher Plane Curves, 3rd ed., 1879, pp. 368, 369.
4. Encyk. der Math. Wiss., III, 3, p. 40.
5. Wilczynski, Bull. Amer. Math. Soc., 22, 1915–1916, pp. 320–324.
6. Mukhopadhyay, Journal of the Asiatic Society of Bengal, 58, I.
7. Cesàro, Lezioni di Geometria Intrinseca (German translation, Vorlesungen über Natürliche Geometrie, 2nd ed., 1926, p. 73.)
8. Perrin, Bull. Soc. Math. France, 31, 1903, pp. 54–64, 275–285.
9. Blaschke, Differentialgeometrie, Vol. II, 1923, Chapter 1.
10. Halphen, Bull. Soc. Math. France, 7, 1879, pp. 83, 84.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, College of St. Thomas*

The following results of the twelfth annual William Lowell Putnam Mathematical Competition held March 22, 1952, have been determined in accordance with the constitution of the Competition. This Competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of The Mathematical Association of America.

The first prize, four hundred dollars, is awarded to the Department of Mathematics of Queen's University, Kingston, Ontario. The members of the team were G. R. Cowper, A. H. Reddoch, H. F. Trotter; to each of these a prize of forty dollars is awarded.

The second prize, three hundred dollars, is awarded to the Department of Mathematics of Brooklyn College, Brooklyn, New York. The members of the team were Paul Cohen, Alan Goldman, Jack Towber; to each of these a prize of thirty dollars is awarded.

The third prize, two hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Henry Landau, Richard Palais, Gerhard Rayna; to each of these a prize of twenty dollars is awarded.

The fourth prize, one hundred dollars, is awarded to the Department of Mathematics of Massachusetts Institute of Technology, Cambridge, Massachusetts. The members of the team were Walter L. Baily, John E. Kimber, Jr.,

* Director of the competition.

Redmond O'Brien; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, were Walter L. Baily, Massachusetts Institute of Technology; James B. Herreshoff, IV, University of California at Berkeley; Gerhard Rayna, Harvard University; Eugene R. Rodemich, Washington University, St. Louis; Richard G. Swan, Princeton University. Each of these will receive a prize of fifty dollars.

The five succeeding persons ranking highest in the examination, named in alphabetical order, were Paul Cohen, Brooklyn College; Arthur P. Dempster, University of Toronto; Herbert C. Kranzer, New York University; Jon Mathews, Pomona College; H. F. Trotter, Queen's University. To each of these a prize of twenty dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: Carnegie Institute of Technology, Pittsburgh, Pennsylvania, the members of the team being Robert H. Boyer, John E. Meyer, John S. Nodvik; The City College of New York, New York, New York, the members of the team being Arnold Benson, Jacob Bricker, Irwin Friedman; Cornell University, Ithaca, New York, the members of the team being Alex Colvin, David Hertzig, Michael Horowitz; University of Toronto, Toronto, Ontario, the members of the team being Fred G. Brauer, Arthur P. Dempster, James R. Jamieson.

Eleven individuals were given honorable mention. The names are listed in alphabetical order. Norman Bauman, Harvard University; Alan F. Berndt, Cooper Union; Louis deBranges, Massachusetts Institute of Technology; Jerrold Franklin, Cooper Union; Marshall Freimer, Harvard University; Robert C. Gunning, University of Colorado; Michael Horowitz, Cornell University; Ralph M. Krause, Harvard University; Leon J. Lander, University of California at Los Angeles; Andre Robert, Laval University; Jack Towber, Brooklyn College.

The following is a list of all colleges and universities which entered teams in the Competition. The list, in alphabetical order, is: Arizona State College, Boston University, Brooklyn College, Brown University, California Institute of Technology, Carleton College, Carnegie Institute of Technology, Catholic University of America, City College of New York, College of the Holy Cross, College of St. Thomas, Columbia University, Concord College, Cooper Union, Cornell University, Harvard University, Haverford College, Knox College, Laval University, Loyola College (Montreal), Massachusetts Institute of Technology, Memphis State College, McGill University, McMaster University, Morris Brown College, New York University, Northwestern University, Polytechnic Institute of Brooklyn, Polytechnic Institute of Puerto Rico, Pomona College, Princeton University, Purdue University, Queen's University, Stanford University, Tennessee Polytechnic Institute, University of British Columbia, University of California (Berkeley), University of California (Los Angeles), University of Detroit, University of Kentucky, University of Miami, University of Minnesota, University of Montreal, University of Nebraska, University

of Oregon, University of Rochester, University of Toronto, University of Washington (Seattle), Wabash College, Washington University (St. Louis), Wayne University, Wheaton College.

The following additional colleges and universities entered individual contestants only: Bowling Green State University, Clemson Agricultural College, Chestnut Hill College, Eastern Nazarene College, Hartwick College, Lehigh University, Rosary College, Rutgers University, Texas College, U. S. Naval Academy, University of Colorado, University of Cincinnati, University of Florida, University of Illinois.

A total of 295 undergraduates representing 66 institutions took part in the Competition.

The departments of mathematics of any of the competing institutions may obtain the rankings of their individual contestants (except that the relative rankings of the first five will not be given) by writing to the Director. These rankings may now be communicated to the individual contestants by their departments. Any other departments of mathematics may obtain individual rankings for the purpose of selecting graduate students.

Participants in the Competition were given the following lists of problems:

PART I

MORNING SESSION: 9:00 A.M. TO 12:00 NOON

Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.

Omit one question. You must indicate which question is omitted.

1. Let

$$f(x) = \sum_{i=0}^{i=n} a_i x^{n-i}$$

be a polynomial of degree n with integral coefficients. If a_0, a_n, and $f(1)$ are odd, prove that $f(x) = 0$ has no rational roots.

2. Show that the equation

$$(9 - x^2)\left(\frac{dy}{dx}\right)^2 = (9 - y^2)$$

characterizes a family of conics touching the four sides of a fixed square.

3. Develop necessary and sufficient conditions which ensure that r_1, r_2, r_3 and r_1^2, r_2^2, r_3^2 are simultaneously roots of the equation $x^3+ax^2+bx+c=0$.

4. The flag of the United Nations consists of a polar map of the world, with the North Pole as center, and extending approximately to 45° South Latitude.

The parallels of latitude are concentric circles with radii proportional to their co-latitudes. Australia is near the periphery of the map, and is intersected by the parallel of latitude 30° S. In the very close vicinity of this parallel how much are East and West distances exaggerated as compared to North and South distances?

5. Let $a_j (j = 1, 2, \cdots, n)$ be entirely arbitrary numbers except that no one is equal to unity. Prove

$$a_1 + \sum_{i=2}^{n} a_i \prod_{j=1}^{i-1} (1 - a_j) = 1 - \prod_{i=1}^{n} (1 - a_i).$$

6. A man has a rectangular block of wood m by n by r inches (m, n, and r are integers). He paints the entire surface of the block, cuts the block into inch cubes, and notices that exactly half the cubes are completely unpainted. Prove that the number of essentially different blocks with this property is finite. (Do *not* attempt to enumerate them.)

7. Directed lines are drawn from the center of a circle, making angles of 0, ± 1, ± 2, ± 3, $\cdots$ (measured in radians from a prime direction). If these lines meet the circle in points P_0, P_1, P_{-1}, P_2, P_{-2}, $\cdots$, show that there is no interval on the circumference of the circle which does not contain some $P_{\pm i}$. (You may assume that π is irrational.)

PART II

AFTERNOON SESSION 2:00 TO 5:00 P.M.

Answer the questions in any order and by any method. Show all of your work in logical sequence and indicate your answers clearly. No tables or other books may be used. Use the right hand pages of your examination booklet for your solutions, use the left hand pages for scratch work. Cross out any work which you do not wish to have considered. Partial credit may be given on a question, even when the solution is not completed.

Omit one question. You must indicate which question is omitted.

1. A mathematical moron is given two sides and the included angle of a triangle, and attempts to use the Law of Cosines: $a^2 = b^2 + c^2 - 2bc \cos A$, to find the third side a. He uses logarithms as follows. He finds log b and doubles it; adds to that the double of log c; subtracts the sum of the logarithms of 2, b, c, and cos A; divides the result by 2; and takes the anti-logarithm. Although his method may be open to suspicion, his computation is accurate. What are the necessary and sufficient conditions on the triangle that this method should yield the correct result?

2. Find the surface generated by the solutions of

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy},$$

which intersects the circle $y^2 + z^2 = 1$, $x = 0$.

3. Develop necessary and sufficient conditions that the equation

$$\begin{vmatrix} 0 & a_1 - x & a_2 - x \\ -a_1 - x & 0 & a_3 - x \\ -a_2 - x & -a_3 - x & 0 \end{vmatrix} = 0 \qquad (a_i \neq 0)$$

shall have a multiple root.

4. A homogeneous solid body is made by joining a base of a circular cylinder of height h and radius r, and the base of a hemisphere of radius r. This body is placed with the hemispherical end on a horizontal table, with the axis of the cylinder in a vertical position, and then slightly oscillated. It is intuitively evident that if r is large as compared to h, the equilibrium will be stable; but if r is small as compared to h, the equilibrium will be unstable. What is the critical value If the ratio r/h which enables the body to rest in neutral equilibrium in any position?

5. If the terms of a sequence, a_n, are monotonic, and if $\sum_1^\infty a_n$ converges, show that $\sum_1^\infty n(a_n - a_{n+1})$ converges.

6. Prove: the necessary and sufficient condition that a triangle inscribed in an ellipse shall have maximum area is that its centroid coincide with the center of the ellipse.

7. Given any real number N_0. If $N_{j+1} = \cos N_j$, prove that $\lim_{j\to\infty} N_j$ exists and is independent of N_0.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

A NOTE ON ORTHOPOLAR TRIANGLES*

VICTOR THÉBAULT, Tennie, Sarthe, France

It is known that *if the sides of a triangle ABC, inscribed in a circle O, have a common orthopole with respect to a second triangle A'B'C', inscribed in the same circle, then the sides of the second triangle have a common orthopole with respect to the first triangle, and the two orthopoles coincide at the midpoint of the line segment joining the orthocenters of the two triangles.* This theorem, due to S. Kantor [1], has been established very simply synthetically by M. Absolome, who has,

* Translated from the French by Howard Eves.

in addition, found an expression for the radius of the circle involved in the proposition. It has also been established by T. Lalescu [2] and by F. D. Murnaghan [3], and then generalized by R. Godeau [4], who, moreover, brought to light the following proposition: *A necessary and sufficient condition for two triangles ABC and $A'B'C'$, inscribed in the same circle, to be orthopolar is that, in both magnitude and sign,*

$$\text{arc } AA' + \text{arc } BB' + \text{arc } CC' = 0.$$

The ensuing remarks were suggested by this fundamental relation.

THEOREM. *If Γ and Γ' are two intersecting conics with parallel axes, and O is a circle passing through one of their common points D and cutting Γ in A, B, C and Γ' in A', B', C', then triangles ABC and $A'B'C'$ are orthopolar.*

For, from a known property, the chords AD and BC common to Γ and O are equally inclined to the axes of Γ, and the chords $A'D$ and $B'C'$ common to Γ' and O are equally inclined to the axes of Γ'. Since the axes of Γ and Γ' are parallel we have, for some integer k, the angular relation

$$(DA, DA') + (BC, B'C') = k\pi,$$

whence

$$(OA, OA') + (OB, OB') + (OC, OC') = 2\pi k,$$

or

$$\text{arc } AA' + \text{arc } BB' + \text{arc } CC' = 2k\pi,$$

and our theorem is established.

COROLLARY. *If the circle O does not pass through a common point of the conics Γ and Γ', but cuts Γ in A, B, C, D and Γ' in A', B', C', D', then*

$$\text{arc } AA' + \text{arc } BB' + \text{arc } CC' + \text{arc } DD' = 0.$$

This equality is obtained from the angular relation

$$(AB, A'B') + (CD, C'D') = k\pi,$$

or by observing that its form does not change if the chords AB, CD, $A'B'$, $C'D'$ are displaced parallel to themselves. In the event of a tangency the modification of the relation is evident.

References

1. Mathesis, 1907, p. 23.
2. Gazeta Matematica, 1915, p. 213.
3. Mathesis, 1927, p. 24.
4. Mathesis, 1927, p. 70.

WEIERSTRASS TENSORS

C. M. FULTON, University of California, Davis

In this paper we present: (1) a definition of what we call Weierstrass tensors, *i.e.*, tensors relative to transformations connecting two Weierstrass coordinate systems in a non-Euclidean space of spherical type; and (2) a differentiation process which has the property that when applied to a Weierstrass tensor the result is such a tensor.

We begin with a brief description of Weierstrass coordinates, an account of which may be found in [5, pp. 204–207] and [2, pp. 133–177]. The range of all subscripts will be 1, 2, 3, 4. We make use throughout of the customary summation convention, Kronecker deltas, and permutation symbols. If we consider a system of rectangular Cartesian coordinates x_i in a Euclidean 4-space, then $x_ix_i=1$ is the equation of the surface of a unit hypersphere. The latter is a 3-space for which the x's constitute a special type of coordinates, called Weierstrass coordinates, subject to the relation

$$x_ix_i = 1. \tag{1}$$

The transformations connecting Weierstrass coordinate systems are obviously those of the original Cartesian coordinates. Thus we have [5, p. 205]

$$x_i = a_{ir}x_r', \quad a_{ir}a_{is} = \delta_{rs}, \quad \det a_{ir} = 1. \tag{2}$$

In order to find a suitable definition of Weierstrass tensors, we note that the key transformation laws of ordinary tensor analysis in a space of arbitrary dimensions are those of a tangent vector and of a gradient. For a tangent vector, which is a contravariant vector, we have

$$\frac{dx_i}{dt} = \frac{\partial x_i}{\partial x_r'}\frac{dx_r'}{dt},$$

and for a gradient, which is covariant,

$$\frac{\partial U}{\partial x_i} = \frac{\partial x_r'}{\partial x_i}\frac{\partial U}{\partial x_r'}.$$

We want to retain the tensor character of these quantities with due regard to the restriction (1). To begin with, we have from (2)

$$\frac{\partial x_i}{\partial x_r'} = \frac{\partial x_r'}{\partial x_i} = a_{ir}.$$

Hence no distinction between covariance and contravariance can be made. If now $x_i=x_i(t)$ is a curve in the spherical 3-space, its tangent vector transforms according to

$$\frac{dx_i}{dt} = a_{ir}\frac{dx_r'}{dt}.$$

Also, differentiating (1) we get

$$x_i\frac{dx_i}{dt} = 0.$$

On the other hand consider a scalar function $U(x_1, x_2, x_3, x_4)$. The relation (1) may be used to change the functional form of U. Thus the partial derivatives $D_hU = \partial U/\partial x_h$, which obey the desired transformation law, do not take into account the coordinate restriction $x_ix_i = 1$. We therefore have to find a new type of derivative independent of the particular form of U. Suppose that $U = 0$ identically if x_4 is eliminated using (1). Then

$$D_jU = \frac{x_j}{x_4}D_4U = 0$$

and, multiplying by x_j and summing for j,

$$x_jD_jU - \frac{1}{x_4}D_4U = 0.$$

Elimination of the terms to which the special subscription 4 is attached yields the symmetric form

$$D_hU - x_hx_jD_jU = 0.$$

We now define the gradient of a scalar U as

$$\nabla_hU = D_hU - x_hx_jD_jU, \tag{3}$$

and this definition can be shown to be consistent as follows. If U and V are two forms of the same function, according to the above reasoning $\nabla_h(U - V) = 0$. Hence, $\nabla_hU = \nabla_hV$. Our gradient can also be obtained as the part of the ordinary gradient in the imbedding 4-space which belongs to the subspace in the sense of being tangent to the hypersphere. Again, as can be seen quite easily, ∇_hU follows the same transformation equation as the x's and $x_h\nabla_hU = 0$. Since the tangent vector had like properties we are led to the general definition of a Weierstrass vector. Following Eisenhart [5, p. 205] we define a vector ξ_i at a certain point x_i by means of the relations

$$\xi_i = a_{ir}\xi_r', \qquad x_i\xi_i = 0. \tag{4}$$

The magnitude ξ of this vector or tensor of order one is defined by $\xi^2 = \xi_i\xi_i$. Let us confine the definition of Weierstrass tensors to the typical case of a second-order tensor τ_{ik} at a point x_i. The relations

$$\tau_{ik} = a_{ir}a_{ks}\tau_{rs}', \qquad x_i\tau_{ik} = 0, \qquad x_k\tau_{ik} = 0 \tag{5}$$

will characterize such a tensor, being immediate generalizations of (4).

We insert a few examples to illustrate the close analogies between Weierstrass tensors and ordinary tensors. Using (5) it can be shown easily that $\delta_{ik}-x_ix_k$ is a tensor of order two. Likewise with the permutation symbol ϵ_{iklm} [3, p. 93] we can construct the third-order tensor

$$\epsilon_{iklm}x_m. \tag{6}$$

The value of the scalar product of two vectors ξ_i, η_i at the point x_i is defined as $\xi_i\eta_i$. Furthermore, because of the known inequality $(\xi_i\eta_i)^2 \leqq \xi_r\xi_r\eta_s\eta_s$ the equation $\xi_i\eta_i=\xi\eta\cos\theta$ is a suitable definition of the angle θ of the two vectors. With this in mind it can now be seen that the gradient (3) has the usual geometric meaning [3, p. 188]: $\nabla_h U$ at a point $A(a_i)$ is normal to the level surface $U(x_i) = U(a_i)$. We use now (6) for the vector product

$$\epsilon_{iklm}\xi_k\eta_l x_m$$

which is obviously a vector perpendicular to both ξ_i and η_i. Its magnitude $\xi\eta\sin\theta$ is found by means of the contraction

$$\epsilon_{iklm}x_m\epsilon_{ist h}x_h = (\delta_{ks} - x_kx_s)(\delta_{lt} - x_lx_t) - (\delta_{kt} - x_kx_t)(\delta_{ls} - x_lx_s).$$

This fundamental contraction may be obtained by a straightforward manipulation of determinants [4, p. 32]. At this stage it would be quite simple to show that the ordinary formulas of spherical trigonometry hold for our 3-space [1, p. 44].

To conclude this paper we show that a differentiation process can be found which applied to a tensor will yield a tensor. We apply the operator ∇_h (3) to the identities in (5) and have

$$\tau_{hk} + x_i\nabla_h\tau_{ik} = 0, \qquad \tau_{ih} + x_k\nabla_h\tau_{ik} = 0.$$

Hence it can be seen that

$$\nabla_h\tau_{ik} + x_i\tau_{hk} + x_k\tau_{ih}$$

is a tensor of order three and represents a proper definition of the derivative of a typical tensor τ_{ik}. Also the differentiation of the sum or product of tensors obeys the same rules as in ordinary differentiation. Using this concept of derivative, divergence and curl may be defined in the ordinary manner. Moreover it is not hard to show that the well-known integral transformations are applicable.

For simplicity of exposition we have restricted ourselves to the spherical case. A slight modification [2, p. 164] would make our treatment valid for elliptic space. The hyperbolic case could be developed in a similar fashion, although it would be analytically cumbersome.

Finally, we wish to acknowledge with thanks the assistance of the referee in revising this paper.

References

1. L. Brand, Vector and tensor analysis, New York, 1947.
2. E. Cartan, Leçons sur la Géométrie des espaces de Riemann, Paris, 1928.
3. H. V. Craig, Vector and tensor analysis, New York and London, 1943.
4. A. Duschek, W. Mayer, Lehrbuch der Differentialgeometrie, Band I, Leipzig und Berlin, 1930.
5. L. P. Eisenhart, Riemannian Geometry, Princeton, 1949.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

RULINGS

C. S. OGILVY, Syracuse University

Unless he has become so mathematically browbeaten by a year or so of the calculus that he receives all new concepts with apathy, a student's curiosity should be stimulated and perhaps a healthy skepticism aroused by his first encounter with ruled surfaces. How can a curved surface like a hyperbolic paraboloid contain straight lines? The purpose of this note is to show, by a method well within the student's grasp at this level, not only that rulings exist on certain surfaces but that in many special cases it is an easy matter actually to find them.

If the instructor is not averse to a bit of showmanship, he can set the stage by the use of a rather striking example. The class has just found approximately by differentials the value of $\sqrt{1+(1.99)(4.02)}$. The instructor remarks that the values of the increments of x and y in the expression $\sqrt{1+xy}$ must be small relative to 2 and 4 in order for the result to be a good approximation. He then takes as a counter example increments of 1 and 1, puts them through the formula for the total differential, and finds to his apparent surprise that the result is not approximately but exactly correct:

$$dz = \frac{y}{2\sqrt{1+xy}}\,dx + \frac{x}{2\sqrt{1+xy}}\,dy\Bigg]_{x=2,\,y=4,\,dx=1,\,dy=1} = 1.$$

Thus $z=3+1=4$, the exact value of $\sqrt{1+(2+1)(4+1)}$. In feigned desperation he tries 2, 2 or 3, 3 as increments, only to have the same thing occur: every "approximation" gives the exact square root. If he now asks the class what is happening, someone should volunteer the suggestion that the instructor has been

going out not on a limb but along a ruling of the surface $z=\sqrt{1+xy}$.

The process of the example itself suggests a criterion for the existence of rulings: if, starting from an arbitrary point (x, y, z) on a surface $z=f(x, y)$ it is possible for the whole increment Δz to equal the differential element dz, for some fixed ratio $dx:dy$ depending only on the point (x, y, z) and not on the size of the increments, then $f(x, y)$ represents a ruled surface.

To say that $\Delta z=dz$ means

$$f(x+h, y+k)-f(x, y)=\frac{\partial f(x, y)}{\partial x}h+\frac{\partial f(x, y)}{\partial y}k, \tag{1}$$

where $h=\Delta x=dx$ and $k=\Delta y=dy$. (1) is a necessary and sufficient condition for the existence of rulings; it remains, however, to solve for h in terms of k, or in other words to obtain an expression for h/k, to answer the question whether the rulings are real, which they are if and only if h/k is real. Unfortunately (1) is sometimes an awkward equation to deal with; but we can exhibit a few of the many easy immediate applications.

(a) A proof that $z=Ax+By+C$ is a plane. Equation (1) becomes the identity $0=0$, which says that the criterion is fulfilled independent of h/k. Starting at any point (x, y, z) and going in *any* direction on the surface, we are travelling along a ruling. This property characterizes a plane.

(b) $z=x^2+y^2$. (1) yields $h^2+k^2=0$. This is a typical situation where there are clearly no rulings, since h, k cannot vary, nor is h/k defined, since $h=k=0$ alone satisfies.

(c) $z=x^2-y^2$. From (1) we find $h/k=\pm 1$. Since these ratios are real, there are two families of rulings, whose projections on the xy-plane are two orthogonal sets of parallel lines intersecting the axes at 45°.

Testing by means of (1) is often easier said than done. But in seeking a more workable criterion we may have to leave our second year calculus class behind as we turn to the Taylor expansion of $f(x+h, y+k)$ around the point $f(x, y)$:

$$\begin{aligned} f(x+h, y+k)=f(x, y)&+\frac{1}{1!}\left[h\frac{\partial f(x, y)}{\partial x}+k\frac{\partial f(x, y)}{\partial y}\right] \\ &+\frac{1}{2!}\left[h^2\frac{\partial^2 f(x, y)}{\partial x^2}+2hk\frac{\partial^2 f(x, y)}{\partial x\partial y}+k^2\frac{\partial^2 f(x, y)}{\partial y^2}\right] \\ &+\cdots . \end{aligned} \tag{2}$$

If (1) is to hold, the terms of the second and all higher degrees in h, k must vanish identically:

$$h^2f_{xx}+2hkf_{xy}+k^2f_{yy}=0. \tag{3}$$

For a solution real in h/k, the discriminant of (3) must be non-negative:

$$f_{xy}^2-f_{xx}f_{yy}\geqq 0. \tag{4}$$

Thus (4) is a readily available necessary condition for the existence of rulings: all points must be hyperbolic or parabolic. It is obviously not sufficient; for all higher terms of (2) must also vanish for the values of h/k which satisfy (3) if rulings are to exist. As an example, take $z=xy+x^4-y^4$. Equation (4) is satisfied:

$$f_{xy}^2 - f_{xx}f_{yy} = 1 + 144x^2y^2 > 0 \qquad \text{for all } x, y.$$

Hence h/k has two real solutions in (3). However, neither of them annuls identically the third order term of (2), and hence the surface is not ruled.

Equation (3) may be obtained in another way by starting with the assumption that rulings exist. One can go further and observe that eliminating the quantity h/k between (3) and the equation obtained by setting the third order term of (2) equal to zero results in a single partial differential equation of third order, namely the differential equation of the general ruled surface. This implies that if for a given function a solution of (3) annuls also the third order terms of (2), then the surface is ruled. This is indeed the necessary and sufficient condition. Its application is tedious, however, and certain shortcuts can be obtained; but to develop them here is beyond the scope of this note.

All treatment of this topic is omitted from most modern textbooks. De Morgan deals briefly with equation (2) *et seq.* in his 1842 calculus book (pp. 424–425), where he credits the idea to Monge. The differential equation of the general ruled surface is of course well known; (see for instance Salmon, *Analytic Geometry of Three Dimensions*, V Ed., Vol. II, p. 19). Equation (1) appears not to have been exploited, although it has the advantage of enlightening the student who is not yet ready for (2).

A NOTE ON INDETERMINATE FORMS

ROGER OSBORN, The University of Texas

Recently, in the discussion of indeterminate forms in an elementary calculus class, one of the students posed the question, "Why isn't ∞^0 equal to one?" He based this question not on a consideration of the inapplicability of the algebraic rule $a^0=1$ to the concept ∞, but on the fact that he had yet to see a limit problem of the form ∞^0 for which the answer was different from 1.

The first two examples that I chose for illustration had the same fault as those in the textbook being used by the class—the answer in each case was 1. We found that it was not difficult to construct an example of a limit problem of the form ∞^0 which had an arbitrarily chosen answer, but that very few were given in textbooks. Even though the subject did not warrant the time spent on it, we found and worked examples of this type given in fifteen standard texts on elementary calculus which were readily available in my office. These ranged from old, well accepted treatises on the subject to some of the newer books which combine calculus with analytic geometry. The books examined had from five to

no problems of the type ∞^0 apiece, and in only one of the books* was there a problem given of the form ∞^0 for which the answer was different from 1. It was the conclusion of the class (and of the instructor) that if it is worthwhile to consider such problems, the cases in which they differ from ordinary algebra should be included by the instructor since they are generally not included by the textbook.

ON LINES OF INVARIABILITY OF CONTINUOUS FUNCTIONS

B. B. MISRA, Ravenshaw College, Cuttack, India

1. It has been stated in Hobson's *Theory of Functions of a Real Variable*, vol. I (1950), that the lines of invariability of a continuous function defined over a finite interval (a, b) may form a finite set or an indefinitely great but enumerable set (page 348). An example of a continuous function having an enumerable set of lines of invariability is given below.

$F(x)$ is defined in $0 \leqq x \leqq 1$ in the following way:

$$F(x) = 1/2^n \text{ in } 1/2^{2n-1} < x \leqq 1/2^{2n-2}; \quad \text{for } n = 1, 2, 3, \cdots$$

$$F(x) = 2^{n-1}x \text{ in } 1/2^{2n} < x \leqq 1/2^{2n-1}; \quad \text{for } n = 1, 2, 3, \cdots$$

$$F(0) = 0.$$

It is easy to see that $F(x)$ is continuous throughout the interval $0 \leqq x \leqq 1$; and the lines of invariability of $F(x)$ consist of the set of intervals $1/2^{2n-1} \leqq x \leqq 1/2^{2n-2}$, for $n = 1, 2, 3, \cdots$; and hence form an enumerable set.

2. Again on page 349 of the same book it is stated that a line of invariability is said to be a maximum (or minimum) of the function, if both its end points are improper maxima (or minima). From ordinary geometrical analogies, a student is likely to think that the extremities of lines of invariability are either improper maxima or minima. An example of a function is given below which has a line of invariability, the extremities of which are neither improper maxima nor minima.

$$F(x) = K + (x + a) \sin \frac{1}{x + a}, \qquad \text{for} \quad x < -a;$$

$$F(x) = K, \qquad \text{for} \quad -a \leqq x \leqq a;$$

$$F(x) = K + (x - a) \sin \frac{1}{x - a}, \qquad \text{for} \quad x > a.$$

The above function $F(x)$ is continuous; the line of invariability of the function is the interval $(-a, a)$. But the end points $-a$ and a are neither improper maxima nor minima.

* Peterson, Thurman S., Elements of Calculus, New York, Harper and Brothers, 1950, p. 155, problem 11 is: $\operatorname{Lim}_{x \to \infty} (e^x + x)^{1/x} = e$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1031. *Proposed by A. J. Friedland, Brookhaven National Laboratory, Upton, L. I.*

A certain game is played between two participants in the following manner. A set of numbers is written down and the players alternately cross out any one of the numbers and substitute any smaller number until nothing remains but zeros. The player putting down the last zero is the winner. What method of play should one follow in order to win this game?

E 1032. *Proposed by E. W. Marchand, Eastman Kodak Company, Rochester, N. Y.*

Solve the differential equation $(dy/dx)^3-3y(dy/dx)^2+4y^3=a$.

E 1033. *Proposed by G. B. Robison, University of Connecticut*

A horizontal plank is balanced on a cylindrical surface whose elements are perpendicular to the length of the plank. What is the cross section of the cylindrical surface if the plank has neutral equilibrium?

E 1034. *Proposed by M. Narasimhamurthy, Presidency College, Madras, India*

Add to its own reverse the difference between any four digit number and its reverse. What are the possible results and what are the probabilities of getting each of these results?

E 1035. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Show that if the circle passing through the feet of the symmedians of a non-isosceles triangle of sides a, b, c is tangent to one side, then the quantities b^2+c^2, c^2+a^2, a^2+b^2, arranged in some order, are consecutive terms of a geometric progression.

SOLUTIONS

Center and Semi-Axes of a Section of an Ellipsoid

E 999 [1952, 41]. *Proposed by L. G. Johnson, General Motors Corporation, Detroit*

Derive general expressions for the coordinates of the center and for the lengths of the semi-axes of the ellipse formed by the plane $Ax+By+Cz+D=0$

cutting the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$.

Solution by Roscoe Woods, State University of Iowa. To facilitate the discussion we shall refer to the given ellipsoid as E, to the given plane as P', and to the section of E by P' as S.

In the set of planes parallel to the given plane P' there are two which are tangent to the ellipsoid E. Let T and T' be the two points of contact of these two planes. From the properties of diametral planes it follows that the centers of the sections of E by this system of parallel planes lie on the line TT', which of course passes through the origin, the center of E.

The plane $Ax+By+Cz=q$, where $q^2=a^2A^2+b^2B^2+c^2C^2$, is tangent to E at the point $T(x_0, y_0, z_0)$ when $x_0=a^2A/q$, $y_0=b^2B/q$, $z_0=c^2C/q$. The equations of the line TT' may then be written in the form $x/a^2A=y/b^2B=z/c^2C$. This line intersects the plane P' in the point

$$E'(-a^2AD/q^2, -b^2BD/q^2, -c^2CD/q^2),$$

which is the sought center of the section S.

Let $P(x, y, z)$ be any point on S. Denote the direction cosines of $E'P$ by l, m, n. Since P lies in the plane P' we must have $Al+Bm+Cn=0$. A short calculation shows that the square of the radius vector $E'P$ is given by the equation

$$\text{(a)} \qquad (E'P)^2p^2 = a^2b^2c^2(1 - D^2/q^2), \qquad p^2 = b^2c^2l^2 + a^2c^2m^2 + a^2b^2n^2.$$

From (a) it follows that the square of the radius vector r parallel to $E'P$ in the section of E by the plane $Ax+By+Cz+D'=0$ is given by the equation

$$\text{(b)} \qquad r^2p^2 = a^2b^2c^2(1 - D'^2/q^2).$$

By comparing equations (a) and (b) we may state the well known

THEOREM. *The sections of an ellipsoid by a system of parallel planes are similar and similarly placed ellipses.*

To find the semi-axes of the section S we must find the maximum and minimum values of $E'P$ in (a), remembering that $l^2+m^2+n^2=1$ and $Al+Bm+Cn=0$. It will, however, be simpler to find the semi-axes of the section S_0 of E by the plane P_0, $Ax+By+Cz=0$, and make use of the theorem above to find those of S.

The semi-axes of S_0 may be found as follows. The equation of the cone with vertex at the origin and passing through the intersection of E and the sphere $x^2+y^2+z^2=r^2$ is

$$\text{(c)} \qquad x^2(1/a^2 - 1/r^2) + y^2(1/b^2 - 1/r^2) + z^2(1/c^2 - 1/r^2) = 0.$$

The plane P_0 intersects the cone (c) in a pair of lines which pass through the intersections of the ellipse and the circle which the plane P_0 cuts from the ellipsoid E and the sphere $x^2+y^2+z^2=r^2$. It is evident that the values of r which

make these two lines coincide are the semi-axes of the section S_0. These lines will coincide when the plane P_0 is tangent to the cone (c). The condition is readily seen to be

$$A^2/(1/a^2 - 1/r^2) + B^2/(1/b^2 - 1/r^2) + C^2/(1/c^2 - 1/r^2) = 0$$

which may be written in the form

$$\text{(d)} \qquad r^4q^2 - r^2[a^2A^2(b^2 + c^2) + b^2B^2(a^2 + c^2) + c^2C^2(a^2 + b^2)] + a^2b^2c^2(A^2 + B^2 + C^2) = 0.$$

To complete the solution we proceed as follows. Solve equation (d) for r^2, denoting the two roots by r_1^2 and r_2^2. The squares of the semi-axes of the section S are then given by the expressions

$$r_1^2(1 - D^2/q^2) \quad \text{and} \quad r_2^2(1 - D^2/q^2).$$

Remark. The interpretation of the proportionality factor $1 - D^2/q^2$ is readily made. For the expressions $D^2/(A^2+B^2+C^2)$ and $q^2/(A^2+B^2+C^2)$ are respectively the squares of the perpendicular distances of the given plane P' and the parallel tangent plane from the origin.

Concyclic Points in a Triangle

E 1000 [1952, 41]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB of a triangle ABC, A', B', C' the feet of the corresponding altitudes, and G the centroid of the triangle. Show that the centroid of triangle $A'B'C'$ and the reflections A_2, B_2, C_2 of the centroids of triangles $AB'C'$, $BC'A'$, $CA'B'$ in the lines B_1C_1, C_1A_1, A_1B_1 are concyclic on a circle with center at G, and that triangles ABC and $A_2B_2C_2$ are inversely similar.

Solution by Josef Andersson, Vaxholm, Sweden. Let G' and G_A denote the centroids of triangles $A'B'C'$ and $AB'C'$, respectively, and let Q_A denote the reflection of G in B_1C_1. Then

$$\overrightarrow{G'G_A} = \overrightarrow{(A'A)}/3 = \overrightarrow{GQ_A}.$$

Also, B_1C_1 perpendicularly bisects GQ_A and A_2G_A. It follows that $GA_2G_AQ_A$ is an isosceles trapezoid and $GG'G_AQ_A$ is a parallelogram, and hence that $GA_2 = GG'$. Similarly, $GB_2 = GC_2 = GG'$, and A_2, B_2, C_2, G' lie on a circle with center at G.

Now A_2, B_2, C_2 are the reflections of G' in lines through G parallel to the sides of triangle ABC. This guarantees that triangle $A_2B_2C_2$ is inversely similar to triangle ABC.

Also solved, with the aid of complex coordinates, by Mary Stephanie.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4503. *Proposed by Harold Shapiro, Massachusetts Institute of Technology*

Given n points $P_1, \cdots, P_n$ in $-1 \leqq x \leqq 1$, let τ_k denote the product of the distances from P_k to the other points. Prove

$$\sum_{k=1}^{n} \frac{1}{\tau_k} \geqq 2^{n-2},$$

and equality is attained for suitable P_i.

4504. *Proposed by Olga Taussky, National Bureau of Standards, Washington, D. C.*

Prove that a set S which is closed under an associative composition law which satisfies the following three axioms is a group:

1. There exists an idempotent e such that $e^2 = e$.
2. Every element has at least one left inverse with respect to e.
3. Every element has at most one right inverse with respect to e.

4505. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N.Y.*

1. Show that if $\theta_n = \pi/2 \cdot 3^n$, then

$$\sum_{n=1}^{\infty} \frac{\theta_n \sin \theta_n \sin 2\theta_n}{\sin 3\theta_n} = \frac{1}{4}.$$

2. Show that if ϕ_p is the p-th positive root of $\tan x = x$ and $\theta_n = \phi_p/3^n$, then

$$\sum_{n=1}^{\infty} \frac{\theta_n \sin \theta_n \sin 2\theta_n}{\sin 3\theta_n} = 0.$$

4506. *Proposed by Paul Erdös, University College, London, England*

Let $a_1 < a_2 < \cdots$ be an infinite sequence of real numbers, $\sum 1/a_i < \infty$. Denote by $n_0 = 1 < n_1 < n_2 < \cdots$ the numbers of form $\prod a_i^{\alpha_i}$, where the α_i are non-negative integers. Assume that the number of $n_i \leqq x$ equals

$$\frac{x}{f(x)} + o\left(\frac{x}{f(x)}\right)$$

where $f(x)$ is increasing and $f(x^2)/f(x)<c$, $f(2x)/f(x)\to 1$. Show that the number of $a_i \leqq x$ equals

$$\frac{x}{f(x)\sum_{i=0}^{\infty}\frac{1}{n_i}}+o\left(\frac{x}{f(x)}\right).$$

The conditions of the theorem are satisfied if, for example, $f(x)=(\log x)^c$, $c>1$.

4507. *Proposed by Leo Moser, University of Alberta, Canada*

In L. E. Dickson, *History of the Theory of Numbers*, we find the statement: Charles de Neuvegelise proved that the products $3\cdot 4, \cdots, 8\cdot 9$ of two consecutive numbers are abundant.

Prove that there are infinitely many numbers of the form $a(a+1)$ which are deficient.

SOLUTIONS

Repeated Moduli of a Continuous Function

4433 [1951, 266]. *Proposed by J. W. Gaddum, University of Missouri*

Let $f(z)$ be a continuous function of a complex variable, finite at z_0. Then in every neighborhood of z_0 there are distinct z_1, z_2 such that $|f(z_1)|=|f(z_2)|$.

I. *Solution by Albert Wilansky, Lehigh University.* If the conclusion is false there is a disk D: $|z-z_0|\leqq r$ in which $u(z)\equiv|f(z)|$ is continuous, real, and takes on each of its values at most once. Thus u defines a continuous 1-1 map of D into the real axis. Since D is compact, u also has a continuous inverse. Thus we have the situation, proved impossible by Brouwer, of a 2-dimensional region homeomorphic with a subset of a 1-dimensional region.

II. *Solution by H. D. Block, Iowa State College.* Let D be a domain of the complex plane. If $|f(z)|$ is constant in D the desired result is obvious. Otherwise, the result is a special case of the following

THEOREM: *Let $g(z)$ be a real valued function of the complex variable z, continuous and not identically constant in D. Then, for each u such that*

$$\inf_{z\in D} g(z) < u < \sup_{z\in D} g(z),$$

there are a non-denumerable number of points $\xi_\alpha\in D$ such that $g(\xi_\alpha)=u$. Moreover (for each such u) the set C of the ξ_α divides D in the following sense. There are non-null open sets $A\subset D$, $B\subset D$, such that $A+B+C=D$ and any rectifiable arc in D from any $a\in A$ to any $b\in B$ must intersect C.

Proof. Let u be given. Then there are points ζ_1, ζ_2, such that $g(\zeta_1)>u$, $g(\zeta_2)<u$. Join ζ_1, ζ_2 by a rectifiable arc, C_1, lying in D. Then $g(z)$ is a continuous function on this arc and hence takes on the intermediate value $u=g(\xi_1)$ with ξ_1 on C_1. To obtain further ξ_α, we connect ζ_1, ζ_2 with other rectifiable arcs, C_α, lying

in D and having only the points ζ_1, ζ_2 in common. This provides a ξ_α on each C_α. Finally let A be the set of all z in D for which $g(z)>u$; B the set for which $g(z)<u$. The stated properties are then clear.

Also solved by F. Bagemihl, P. T. Bateman, V. L. Klee, Jr., M. J. Norris, C. S. Ogilvy, L. A. Ringenberg, W. G. Rouleau, W. Seidel, Peter Treuenfels, F. H. Young, and the Proposer.

A Trigonometric Sum and Integral

4434 [1951, 266]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Prove

$$\sum_{k=1}^{n-1} \frac{\sin^2 (kr\pi/n)}{\sin^2 (k\pi/2n)} = 2r(n-r). \qquad 0 \leqq r \leqq n,$$

and thus evaluate

$$\int_0^\pi \frac{\sin^2 rx}{\sin^2 \frac{1}{2}x}\, dx.$$

Solution by Jack Indritz, University of Minnesota. For $m \geqq 2$, we note

$$1 - \cos mx = 1 - \cos x + \sum_{p=1}^{m-1} [\cos px - \cos (p+1)x],$$

$$\cos px - \cos (p+1)x = 1 - \cos x - \sum_{q=1}^{p} [\cos (q-1)x - 2\cos qx + \cos (q+1)x],$$

$$\cos (q-1)x + \cos (q+1)x = 2\cos qx \cos x,$$

so that

$$1 - \cos mx = (1 - \cos x)\left[m + 2\sum_{p=1}^{m-1}\sum_{q=1}^{p} \cos qx\right]$$

$$= (1 - \cos x)\left[m + 2\sum_{j=1}^{m-1} (m-j)\cos jx\right].$$

It follows that

$$\frac{\sin^2 \frac{1}{2}mx}{\sin^2 \frac{1}{2}x} = \frac{1 - \cos mx}{1 - \cos x} = m + 2\sum_{j=1}^{m-1} (m-j)\cos jx,$$

for $0 < \frac{1}{2}x < \pi$. With $m = 2r > 0$ and $x = k\pi/n$, the proposed sum equals

$$\sum_{k=1}^{n-1}\left[2r + 2\sum_{j=1}^{2r-1}(2r-j)\cos(jk\pi/n)\right] = 2r(n-1) + 2\sum_{j=1}^{2r-1}(2r-j)\sum_{k=1}^{n-1}\cos(jk\pi/n). \tag{1}$$

From the formula

$$\cos x + \cos 2x + \cdots + \cos mx = \frac{\sin(m+\frac{1}{2})x}{2\sin(\frac{1}{2}x)} - \frac{1}{2},$$

we obtain $\sum_{k=1}^{n-1}\cos(jk\pi/n) = -\frac{1}{2}[1+(-1)^j]$. Hence, from (1), the proposed sum equals $2r(n-1)-2r(r-1)=2r(n-r)$ as required.

The case $r=0$ is trivial.

Since

$$\int_0^{\pi}\frac{\sin^2 rx}{\sin^2\frac{1}{2}x}\,dx$$

exists as a Riemann integral, it may be evaluated for integral r as

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sin^2(kr\pi/n)}{\sin^2(k\pi/2n)}\cdot\frac{\pi}{n} = \lim_{n\to\infty}\frac{\pi}{n}\cdot 2r(n+1-r) = 2\pi r.$$

Also solved by T. M. Apostol, W. Seidel, O. E. Stanaitis, and Chih-yi Wang.

Right Circular Cone Intersecting a Given Conic

4435 [1951, 266]. *Proposed by J. P. Ballantine, University of Washington, Seattle*

Find the locus of points in xyz-space from which the conic $b^2x^2+a^2y^2=a^2b^2$ in the plane $z=0$ looks like a circle.

I. *Solution by C. S. Ogilvy, Syracuse University.* The ellipse looks like a circle if and only if it is viewed from the vertex of some right circular cone of which it is a plane section. The smaller Dandelin sphere* of any such cone is tangent to the xy-plane at a focus F. The vertex P of the cone lies in the xz-plane. Let V_1 be that vertex of the ellipse on the same side of the origin as F, and let V_2 be the other vertex of the ellipse. Let PV_1 and PV_2 be tangent to the sphere at points Q_1 and Q_2 respectively. Then $PV_2-PV_1=Q_2V_2-Q_1V_1=FV_2-FV_1$, constant for all P. This is the definition of a hyperbola with foci at V_1 and V_2 and a vertex at F. Its equation is

$$\frac{x^2}{a^2-b^2} - \frac{z^2}{b^2} = 1, \qquad (y=0). \tag{1}$$

* The Dandelin spheres are tangent to the cone along a circle and also tangent to the plane of the elliptic section.

II. *Solution by H. S. M. Coxeter, University of Toronto.* The enveloping cone to any quadric of a confocal system from any point on a focal conic is right-circular (George Salmon, *Analytic Geometry of Three Dimensions*, 6th ed., 1914, p. 162). Since the focal conics are degenerate members of the system, it follows that either of the two real conics is the locus of points from which the other one looks like a circle. If the given ellipse is one of them, with $a>b$, the other is the hyperbola (1).

Also solved by R. S. Underwood and the Proposer.

Editorial Note. Norman Anning provides the following comments.

(1) The problem is found as no. 21, p. 92 of Charles Smith, *Solid Geometry*, 8th ed., London, 1901.

(2) Historical references are given in *Enciclopedia della Matematiche Elementari*, v. 2, part 2, pp. 85–87 under the heading, "Coni rotondi passanti per una data conica." Most interesting is the reference to Apollonius:

There is due to Apollonius, κωνικα, book I, propositions 52 to 58, a construction of the vertices of circular cones passing through a given conic. See the *Heiberg* edition, v. I, pp. 159–185, or the *ver Eecke* edition, pp. 97–112.

(3) A wire model exists which can be used for string constructions and should be of interest to a student mathematics club. See *Samon-Fiedler*, page 446 and near it, and also *Salmon-Rogers* at the end of volume I.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

An Introduction to Applied Mathematics. By J. C. Jaeger. Oxford, 1951. vii+445 pages. $7.00.

This is mainly a textbook on differential equations and their applications. Assuming no previous knowledge of differential equations on the part of the reader it develops their theory from the beginning. The topics covered include all of the conventional elementary linear theory together with topics like Bessel's equation, Legendre's equation, Schrödinger's equation, boundary-value problems with Green's function and Laplace Transforms. Some of the simpler partial differential equations are studied. There are chapters on vectors and numerical methods.

Applications, particularly to dynamics and electricity, are extensively discussed. Under dynamics are found harmonic oscillators, coupled systems, some

non-linear mechanics, particle and rigid body dynamics including Lagrande's equations and boundary-value problems. Under electricity are found linear circuits, vacuum-tube circuits and servomechanisms.

Among the most interesting features are the treatments of non-linear problems, such as relaxation oscillations and vacuum tube circuits, topics not often found in books of this type. The chapter on numerical methods includes a section on Southwell's relaxation methods.

The chapter headings are as follows: I. Ordinary Differential Equations and their Occurrence. II. Ordinary Linear Differential Equations with Constant Coefficients. III. Differential Equations of the First Order. IV. Dynamical Problems Leading to Ordinary Linear Differential Equations. V. Electric Circuit Theory. VI. Vectors. VII. Particle Dynamics. VIII. Rigid Dynamics. IX. The Energy Equations and Lagrange's Equations. X. Boundary Value Problems. XI. Fourier Series and Integrals. XII. Ordinary Linear Differential Equations with Variable Coefficients. XIII. Partial Differential Equations. XIV. Numerical Methods.

W. S. LOUD
University of Minnesota

Advanced Engineering Mathematics. By C. R. Wylie, Jr. McGraw-Hill Book Company, 1951. xiii+640 pages. $7.50.

The above textbook is designed, in the words of the author, to cover "those branches of mathematics which a student of engineering or applied science normally encounters in the first year or two following his introductory work in calculus." It includes topics from the following fields: elementary differential equations, Fourier series and integrals, Laplace transforms, partial differential equations, Bessel functions, functions of a complex variable, vector analysis, and numerical analysis. An unusually large number of applications of the above fields is given. There is an abundance of problems in every chapter.

Among the noteworthy features of the book the following deserve special mention. Chapter 4 includes an excellent discussion of the mathematical equivalence of mechanical and electrical systems. Chapter 6 includes an interesting original treatment of inversion of the Laplace transform of a periodic function.

The explanations and illustrative examples in this book are extensive enough so that a student can use it for self-study and reference. The reviewer would also recommend it highly to any student who is uncertain about the practical usefulness of mathematics. There are many references to mathematical literature for the interested reader in places where complete proofs are not given.

The chapter headings are as follows: 1. Ordinary Differential Equations of the First Order. 2. Linear Differential Equations with Constant Coefficients. 3. Simultaneous Linear Differential Equations. 4. Mechanical and Electrical Circuits. 5. Fourier Series and Integrals. 6. The Laplace Transformation. 7. Partial Differential Equations (includes orthogonal functions). 8. Bessel Functions. 9. Analytic Functions of a Complex Variable. 10. Integration in the

Complex Plane. 11. Infinite Series in the Complex Plane. 12. The Theory of Residues. 13. Conformal Mapping. 14. Analytic Functions and Fluid Mechanics (includes some airfoil theory). 15. Vector Analysis. 16. Numerical Analysis (includes finite differences and harmonic analysis). Appendix: Selected Topics from Algebra and Calculus.

One error which might be confusing occurs on page 147. The right member of the next to last equation should have $\frac{1}{2}$ added. This arises from the cosine term in the integral two lines above when a approaches zero. The last equation on page 147 should have $\frac{1}{2}$ in the left member instead of 1.

A bibliography of references for further reading would be useful.

On the whole this is an excellent book, clearly written and with broad content. It should serve well for both classroom and reference work.

W. S. LOUD
University of Minnesota

NEW BOOKS RECEIVED

An Introduction to the Theory of Differential Equations. By Walter Leighton. New York, McGraw-Hill Book Company, 1952. viii+174 pages. $3.50.

Theory of Matrices. By Sam Perlis. Cambridge, Mass., Addison-Wesley Press, Inc., 1952. xiv+237 pages.

Symmetry. By Herman Weyl. Princeton, Princeton University Press, 1952. 8+168 pages. $3.75.

Übungen zur Projektiven Geometrie. By H. Herrmann. Basel, Switzerland, Verlag Birkhauser, 1952. 168 pages. Bound, 17 Swiss francs; unbound, 14 Swiss francs.

Inequalities. Second Edition. By G. H. Hardy, J. E. Littlewood and G. Polya Cambridge, England, at the University Press. (American Branch, 32 West 57th St., New York 22, N. Y.), 1952. 12+324 pages. $4.75.

A New Calculus, Part III. By A. W. Siddons, K. S. Snell and J. B. Morgan. Cambridge University Press, 1952. 8+463 pages. $3.75.

Methods of Algebraic Geometry, Vol. II. By W. V. D. Hodge and D. Pedoe. Cambridge University Press, 1952. 10+394 pages. $7.50.

The Elements of Analytical Solid Geometry. By Shanti Narayan. India, Doaba House, 538 Nai Sarak, Delhi, 1952. viii+272 pages. Rs. 5.

Statistical Theory with Engineering Applications. By A. Hald. New York, John Wiley and Sons, Inc. xii+783 pages.

Statistical Tables and Formulas. By A. Hald. New York, John Wiley and Sons, Inc. 97 pages. $2.50.

Calculus of Variations. By Robert Weinstock. New York, McGraw-Hill Book Company. x+326 pages. $6.50.

Differential Calculus. Fifth Edition. By Shanti Narayan. India, Doaba House, 538 Nai Sarak, Delhi, 1951. xii+460 pages. 6 Rupees.

Calculus. By J. F. Randolph. New York, The Macmillan Company, 1952. x+483 pages.

The Nature of Some of our Physical Concepts. By P. W. Bridgman. New York, Philosophical Library, 1952. 64 pages. $2.75.

Conformal Representation. Second Edition. Cambridge Tracts in Mathematical Physics, No. 28. By C. Caratheodory. Cambridge University Press, 1952. 10+115 pages. 12s. 6d.

Elementary Analysis. By K. O. May. New York, John Wiley and Sons, 1952. xviii+635 pages. $5.00.

Commercial Algebra. By R. M. Parker. New York, American Book Company, 1952. vii+263 pages. $3.25.

Hypothesis Testing in Time Series Analysis. By Peter Whittle. Printed in Sweden. American Agents, Hafner Publishing Company, New York. 1951. 121 pages. $3.50.

Advanced Calculus. By Wilfred Kaplan. Cambridge, Mass., Addison-Wesley Press, Inc., 1952. xiii+679 pages. $8.50.

Leçons de Gêometrie Projective. Second Edition. By Lucien Godeaux and Octave Rozet. Liége, Belgium, Sciences et Lettres, Boulevard de la Constitution, April 9, 1952. 279 pages. 2.800 frs. francais.

Algebra, Its Big Ideas and Basic Skills, Book II. By D. J. Aiken and K. B. Henderson. New York, McGraw-Hill Book Company, 1952. xii+397 pages. $2.72.

An Introduction to Mathematical Physics. By R. A. Houston. Glasgow, Scotland, Blackie and Son, Ltd., 1952. 10+262 pages. 25s. net.

Grundlag for Den Projective Geometrie. By Johannes Hjelsmlev. Kobenhaven, Denmark, Gyldendalske Boghandel-Nordisk. 1952.

Building Mathematical Concepts in the Elementary School. By P. L. Spencer and Marguerite Brydegaard. New York, Henry Holt and Co., 1952. ix+372 pages. $3.75.

Electronic Analog Computers. By G. A. Korn and T. M. Korn. New York, McGraw-Hill Book Company, 1952. xv+378 pages. $7.00.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of special features, student papers, bibliographies of program topics, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

DO YOU WANT TO MAJOR IN MATHEMATICS?

The University of Georgia Mathematics Department

1. **Question:** In what does a major in mathematics consist?
   Answer: A major in mathematics consists of a minimum of two or three mathematics courses in advance of the usual college calculus, coupled with

some study of physical science.

2. **Question:** What subjects other than mathematics should a student majoring in mathematics take while in college?
   Answer: A student majoring in mathematics should have as broad a foundation in the "arts" generally as is feasible. Particularly he should take work in physics, some work in statistics, and at least one basic course in philosophy.
3. **Question:** What professional advantages are to be derived from majoring in mathematics?
   Answer: A student who majors in mathematics may become a professional mathematician or may use his mathematics in a variety of other professional ways. Among these one can mention junior college and secondary school teaching, actuarial work, statistical work, civil service employment of various kinds and industrial and business positions of considerable variety.
4. **Question:** Is a mathematics major a good foundation for professions which are not strictly mathematical?
   Answer: A mathematics major is generally regarded as one of the best preparations for those professions which require accurate work and precise reasoning. Thus mathematics is particularly important for persons desiring to enter scientific work of any description, engineering or law or any of their allied occupations.
5. **Question:** Is a person who graduates with a major in mathematics prepared to be a professional mathematician?
   Answer: No. Special pamphlets* have been prepared on the professional mathematician and his career to which the interested student is referred.
6. **Question:** Are there advantages other than financial which are to be derived from majoring in mathematics?
   Answer: Yes. Mathematical ability and training are sources of permanent satisfaction to those who have them. There is nothing which attests mental power as well as mathematics. It also is one of the oldest and largest fields of human endeavor and some knowledge of its processes and history is necessary for an understanding of civilization. Certainly no one can understand the advance of science who does not have an appreciation of the mathematical method.
7. **Question:** What type of student should major in mathematics?
   Answer: Only students who have achieved above the average in their school grades particularly in mathematics and who are interested in the subject should attempt a mathematics major.

---

* One of these, also prepared by the University of Georgia Mathematics Department, will be printed in an early issue of this MONTHLY. Another such pamphlet appeared in the January 1951 issue of this MONTHLY; reprints of this are available at 25 cents (or 10 for $1.00) from the office of the Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

8. **Question:** What salary can a teacher in a secondary school or junior college expect?
   Answer: The teacher in a secondary school or junior college can expect a salary not in excess of $5000 per year. Beginning salaries would be certainly less. Allied administrative positions sometimes pay considerably more.
9. **Question:** What salary can a teacher of university grade expect?
   Answer: For success as a university teacher some years of graduate study must follow an undergraduate major. Few university teachers of mathematics are paid as much as $15,000 per year. Salaries range from $3000 up to $15,000.
10. **Question:** What salary can an actuary expect?
    Answer: There are a number of actuaries in America who are paid a salary in excess of $20,000 per year. However, the beginner in the actuarial office does not often receive more than $200 per month.
11. **Question:** What salary can a statistician expect?
    Answer: If a person has one or two years training in statistics beyond his undergraduate work, he may well expect a salary ranging from $4000 to $10,000.
12. **Question:** What salary can a government employee expect?
    Answer: Salaries for mathematicians in government employ range from $2400 to $8000. Salaries in the upper brackets are obtained only by mathematicians of maturity and experience.
13. **Question:** What money can a person earn from writing mathematics books?
    Answer: Royalties paid to some writers of mathematics text books exceed $25,000 per year.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

FELLOWSHIPS OF THE AMERICAN ASSOCIATION OF UNIVERSITY WOMEN

Twenty-five fellowships are offered by the American Association of University Women to American women for advanced study or research during the academic year 1953–54. In general, the $1,500 fellowships are awarded to young women who have completed two years of residence work for the Ph.D. degree or who have already received the degree; the $2,000-$2,200 awards to more advanced students or those who may need to study abroad; the $3,000 awards to

more mature scholars who need a year of uninterrupted work for writing and research.

Applications and supporting materials must reach the office in Washington by December 15, 1952. For detailed information concerning these fellowships and instructions for applying, address the Secretary, Committee on Fellowship Awards, American Association of University Women, 1634 Eye Street, N.W., Washington 6, D. C.

PERSONAL ITEMS

Professor A. A. Albert of the University of Chicago has been appointed Chairman of the Division of Mathematics of the National Research Council.

Professor K. J. Arnold of Michigan State College has been elected Secretary-Treasurer of the Institute of Mathematical Statistics for the term July 1, 1952 to June 30, 1955.

The newly-elected officers of the Mathematics Division of the American Society for Engineering Education are: Chairman, Dr. C. V. Newsom, Associate Commissioner for Higher Education, University of the State of New York; Secretary-Treasurer, Professor J. H. Zant, Oklahoma Agricultural and Mechanical College; Director, Professor H. M. Gehman, University of Buffalo.

Central State College, Oklahoma, announces the following: Mrs. Mary E. Waller has returned to her position as Assistant Professor of Mathematics after a year's leave of absence; Associate Professor Truman Wester is on leave of absence during 1952–53 and is engaged in graduate study at the University of Oklahoma.

Cornell University reports the following: Dr. Jack Kiefer and Dr. A. S. Shapiro have been promoted to assistant professorships; Dr. Pierre Samuel has been appointed to an assistant professorship; Dr. Lionel Weiss has been appointed Visiting Assistant Professor; Mr. N. T. Hamilton, Dr. Hyman Kamel, and Dr. Ilse L. Novak, who has been at the Institute for Advanced Study, have been appointed to instructorships; Mr. Jean-Pierre Meyer has been granted a National Science Foundation Fellowship; Associate Professor Harry Pollard is on leave of absence during 1952–53 and is at the Institute for Advanced Study; Professor Jacob Wolfowitz is also on leave of absence and is spending the first term of 1952–53 at the University of California and the second term at the University of Illinois.

At Oberlin College: Instructor A. G. Anderson has been promoted to an assistant professorship; Mr. E. H. Crisler has been appointed to an instructorship; Dr. W. R. Orton, who has been studying during the past year in Paris on a Fulbright Fellowship, has been appointed to an instructorship; Professor C. H. Yeaton has retired with the title of Emeritus Professor; Assistant Professor Bryant Tuckerman has been granted a year's leave of absence effective July 1, 1952.

Kansas State College announces the following: Dr. L. E. Fuller who has been with the Goodyear Aircraft Corporation has been appointed to an assistant

professorship at Kansas State College; Assistant Professor Violet H. Larney is on leave of absence for the academic year 1952–53 and is at New York State College for Teachers, Albany.

Montana State College reports: Associate Professor Henry Antosiewicz has been granted a year's leave of absence to take a research position at American University; Professor J. J. Livers who has been on leave of absence during 1951–52 has resigned to accept a permanent position with the Boeing Aircraft Corporation, Seattle, Washington.

At the University of Chicago: Dr. M. H. Stone, Andrew MacLeish Distinguished Service Professor of Mathematics, has completed his second term as Chairman of the Department; Professor Saunders MacLane is now Chairman of the Department of Mathematics.

University of Colorado makes the following announcements: Visiting Professor Sarvadaman Chowla of the University of Kansas has been appointed to a professorship; Mr. Robert Osserman of Harvard University has been appointed to an instructorship.

University of Florida announces the following: Associate Professor H. A. Meyer has been promoted to a professorship; Assistant Professor David Ellis has been promoted to an associate professorship; Mr. S. T. Gormsen and Dr. J. W. Young have been promoted to assistant professorships; Professor D. E. South of the University of Kentucky has been appointed to a professorship; Mr. W. P. Morse, who has been a graduate assistant at the University, has been appointed to an instructorship.

The University of Washington reports: Associate Professor H. S. Zuckerman has been promoted to a professorship; Assistant Professor E. Paulson has been promoted to an associate professorship; Instructors R. W. Ball and R. P. Peterson have been promoted to assistant professorships; Dr. L. A. Kokoris of the University of Chicago has been appointed to an instructorship; Professor Z. W. Birnbaum and Associate Professor E. Hewitt have returned from leaves of absence; Assistant Professor F. Yagi is on sabbatical leave at Massachusetts Institute of Technology for the current academic year.

University of Wisconsin announces the promotions of Associate Professors R. H. Bing and R. H. Bruck to professorships.

Mr. A. G. Andersson, formerly a student at the University of Alabama, is now a graduate student at Uppsala University, Uppsala, Sweden.

Mr. P. N. Armstrong who has been a student at the University of Nebraska has accepted a position as Technical Engineer with International Business Machines, New York City.

Mr. P. H. Arnold has received an appointment as Staff Member with the Sandia Corporation, Albuquerque, New Mexico.

Professor J. G. Bowker, head of the Department of Mathematics at Middlebury College, has been appointed Baldwin Professor of Mathematics and Natural Philosophy.

Associate Professor Evelyn Boyd of Fisk University has accepted a position

as a mathematician with the National Bureau of Standards, Washington, D. C.

Mr. J. C. Bradford, previously a graduate student at North Texas State College, has been appointed to a graduate assistantship at the University of Oklahoma.

Professor Richard Brauer of the University of Michigan has been appointed to a professorship at Harvard University.

Professor W. H. Brothers of Talladega College has been appointed to a professorship at Dillard University.

Mr. R. W. Brown, formerly a teaching fellow at Oregon State College, has a position as a mathematician with the National Bureau of Standards, Corona, California.

Mr. G. C. Burns, previously a statistician with the Blue Cross plan, Tulsa, Oklahoma, is now with Douglas Aircraft, Tulsa.

Assistant Professor E. G. H. Comfort of Illinois Institute of Technology has been appointed to an associate professorship at Ripon College.

Dr. P. F. Conrad of the University of Illinois has been appointed to an assistant professorship at Newcomb College, Tulane University.

Mr. D. E. Deal has been appointed to an instructorship at Ball State Teachers College, Muncie, Indiana.

Professor L. E. Dix, head of the Department of Mathematics of Norwich University, has retired.

Professor William Feller of Princeton University gave a series of lectures at the Sorbonne and a lecture before the Societé Mathematique de France in June, 1952.

Professor L. R. Ford, chairman of the Department of Mathematics at Illinois Institute of Technology, has retired.

Dr. A. E. Foster of Florida State University has been appointed to an instructorship at Newark College of Engineering.

Dr. Louis Garfin of the Insurance Department of the State of Oregon has accepted a position as Associate Actuary with the Pacific Mutual Life Insurance Company, Los Angeles, California.

Dean V. L. Good of Skagit Valley Junior College is now engaged in graduate work at State College of Washington.

Mr. C. C. Grandy, formerly a student at Colorado Agricultural and Mechanical College, has a position with the Digital Computer Laboratory, Massachusetts Institute of Technology.

Mr. H. C. Griffith, who has been an assistant instructor at the University of Missouri, has been appointed to a teaching assistantship at the University of Tennessee.

Assistant Professor L. Aileen Hostinsky of Temple University has been appointed to an assistant professorship at Pennsylvania State College.

Dr. Hyman Kaufman, who has been a research geophysicist with Continental

Company, Ponca City, Oklahoma, is now Associate Professor of Mathematics at McGill University.

Professor J. R. Kline, chairman of the Department of Mathematics of the University of Pennsylvania, is on leave of absence during the fall semester of 1952–53 and is Visiting Professor at the University of Tübingen, Germany.

Assistant Professor S. R. Knox of Millsaps College has been granted a two-year leave of absence and is engaged in graduate study at the University of Michigan.

Mr. R. A. C. Lane, formerly a graduate assistant at Lehigh University, has accepted a position as Engineer with R.C.A. Victor Division, Camden, New Jersey.

Research Associate S. L. Levy of Brown University has been promoted to an assistant professorship in the Department of Applied Mathematics.

Mr. D. B. Lowdenslager, formerly a graduate student at the University of Virginia, has a position with the Institute for Cooperative Research, Johns Hopkins University.

Mr. R. A. Lufburrow, formerly a graduate assistant at the University of Wisconsin, has accepted a position as a mathematician at the Woods Hole Oceanographic Institute, Woods Hole, Massachusetts.

Mr. R. W. Marsh, who has been a graduate student at George Washington University, has a position as a mathematician with the Defense Department, Washington, D. C.

Dr. Ella Marth, Dean of Women of Harris Teachers College, has been appointed Chairman of the Division of Mathematics and Business Education of James Ormond Wilson Teachers College with the rank of Professor.

Professor E. J. Moulton of Northwestern University has retired.

Mr. J. D. Neff, previously a graduate assistant at Purdue University, is now a member of the technical staff of Bell Telephone Laboratories, New York City.

Assistant Professor Louisa G. Plummer of West Liberty State College has been appointed to an assistant professorship at Beaver College.

Mr. E. J. Polak, who has been serving as a mathematician with Roy S. Sanford and Company, Oakville, Connecticut, has accepted a position as Senior Project Engineer with Curtiss-Wright Corporation, Electronics Division, Carlstadt, New Jersey.

Mr. J. J. Schoderbek, formerly an electronics engineer with the Glen L. Martin Company, Baltimore, Maryland, has a position as Research Assistant with the Willow Run Research Center, University of Michigan, Ypsilanti, Michigan.

Mr. Seymour Schuster of Pennsylvania State College has been granted a fellowship at the University of Toronto for the academic year 1952–53.

Professor R. C. Taliaferro of Portsmouth Priory School, Rhode Island, has been appointed to a visiting associate professorship in the General Program of

Liberal Education at Notre Dame University for the academic year 1952–53.

Professor Alexander Tartler of Lafayette College has returned to Drexel Institute of Technology as Professor and Head of the Department of Mathematics.

Mr. W. G. Thornton, who has been teaching at Madison High School, Florida, is now a physicist with the United States Navy Mine Countermeasures Station, Panama City, Florida.

Dr. Harold Weintraub of Tufts College has been promoted to an assistant professorship.

Professor M. E. Wescott of Northwestern University has been appointed Professor of Applied Statistics on the faculty of Rutgers University College.

Professor Hassler Whitney of Harvard University has been appointed a permanent member of the Institute for Advanced Study.

Assistant Professor Albert Wilansky of Lehigh University has been promoted to an associate professorship.

Professor E. D. Meacham, Dean of the College of Arts and Sciences of the University of Oklahoma, died on June 28, 1952.

Associate Professor E. B. Wedel of the University of Wichita died on July 3, 1952.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following one hundred and twelve persons have been elected to membership by the Board of Governors on applications duly certified.

O. B. ADER, Ph.D.(Kentucky) Instr., New Mexico College of A. and M. A., State College, N. M.

COLETTE R. ANDRZEJCZAK, Student, Marquette University, Milwaukee, Wis.

M. G. ARSOVE, Ph.D.(Brown) Instr., University of Washington, Seattle, Wash.

R. A. ASKEY, Student, Washington University, St. Louis, Mo.

E. H. AUERBACH, Student, Columbia University, New York, N. Y.

W. E. BALL, Student, Washington University, St. Louis, Mo.

MRS. HELEN G. BASS, M.A.(N.Y.U.) Engineering Assistant, General Electric Company, Schenectady, N. Y.

IMOGENE C. BECKEMEYER, B.S. in Ed.(Southern Illinois) Asst. Instructor, Southern Illinois University, Carbondale, Ill.

ALEX BEGROWICZ, JR., Student, Ball State Teachers College, Muncie, Ind.

BERNARD GERALD, Brother, B.A. (Manhattan) Asst. Professor, Manhattan College, New York, N. Y.

OLIVIER BIBERSTEIN, Diploma(Strasbourg) Professor, University of Ottawa, Ontario, Canada

P. J. BLANK, Student, Marquette University, Milwaukee, Wis.

ADELE C. BLEIER, Student, Texas State College for Women, Denton, Tex.

FRED BRAFMAN, Ph.D.(Michigan) Instr., Wayne University, Detroit, Mich.

F. G. BRAUER, B.A.(Toronto) Grad. Student, Massachusetts Institute of Technology, Cambridge, Mass.

J. F. BURKE, Instr., Saranac Lake Study and Craft Guild, N. Y.

W. L. CARTER, M.A.(Ohio State) Instr., Western Illinois State College, Macomb, Ill.

MRS. HILDA CASE, Student, Marquette University, Milwaukee, Wis.

J. J. CHAKALIS, Student, Northeastern University, Boston, Mass.

A. R. CHANDLER, Student, Boston University, Mass.

D. R. CHILDS, Student, University of New Hampshire, Durham, N. H.

Jo ANN CIPOLLA, Student, Elmira College, N. Y.

R. L. CONLON, B.S.(Creighton) Grad. Student, University of Michigan, Ann Arbor, Mich.

W. A. COUCH, M.A.(Washington) Instr., Washington University, St. Louis, Mo.

W. R. DAVIS, Student, University of Oklahoma, Oklahoma City, Okla.

T. H. DEWEY, B.A.(California) Research Assistant, Los Alamos Scientific Laboratory, N. M.

T. L. DOWNS, Ph.D.(Harvard) Asso. Professor, Washington University, St. Louis, Mo.

PALMER DYAL, 99 22nd Avenue, Southwest, Cedar Rapids, Iowa.

LILLIAN R. ELVEBACK, M.A.(Columbia) Lecturer in Biostatistics, School of Public Health, University of Minnesota, Minneapolis, Minn.

R. V. ESPERTI, M.S.(Purdue) Mathematician, U. S. Naval Ordnance Plant, Indianapolis, Ind.

H. G. FALAHEE, M.A.(Michigan) Asst. Professor, Michigan State Normal College, Ypsilanti, Mich.

A. L. FASS, Ph.D.(Columbia) Instr., Queens College, Flushing, N. Y.

CHARLOTTE FROESE, Student, University of British Columbia, Vancouver, B. C., Canada

OTHA FULLER, JR., B.S. in Ed.(Bluefield S. C.) Instr., Claflin College, Orangeburg, S. C.

BETTY J. GASSNER, Student, Washington Square College, New York University, New York, N. Y.

E. D. GLENNEY, Tool Engineer, Piasecki Helicopter Corporation, Morton, Pa.

JOSEPH GOLOB, Analytical Statistician, U.S.A.F., Wright-Patterson A. F. Base, Dayton, Ohio

J. C. GOULD, Student, Wabash College, Crawfordsville, Ind.

J. E. GUINANE, Student, Brown University, Providence, R. I.

J. W. HAMBLEN, M.S.(Purdue) Grad. Assistant, Purdue University, West Lafayette, Ind.

C. J. HAMMOND, B.S.(A. & T. C. of N. C.) Mathematician, Army Map Service, Department of Army, Washington, D. C.

HUNTER HARDMAN, M.S.(Chicago) Asst. Professor, Marshall College, Huntington, W. Va.

GEORGIA M. HASWELL, M.A.(Ohio Wesleyan) Professor, Kansas Wesleyan University, Salina, Kan.

T. H. HAYNES, JR., B.A.(Chicago) Grad. Student, University of Chicago, Ill.

G. H. HEMBREE, B.S.(Eastern Kentucky S. C.) Grad. Student & Research Assistant, Ohio State University, Columbus, Ohio

B. H. HENRY, M.S.(Iowa) Asst. Professor, Parsons College, Fairfield, Iowa

MAX HERZBERGER, Ph.D.(Berlin) Research Scientist, Eastman Kodak Research Laboratory, Rochester, N. Y.

T. L. HILLBURN, Student, University of Oklahoma, Norman, Okla.

F. L. HOLZHAUSER, JR., M.A.(Kent State) Asst. Professor, Kent State University, Ohio

J. E. HOUSEHOLDER, SR., B.S.(Arizona) Part-time Assistant, University of Arizona, Tucson, Ariz.

R. L. HUCKINS, B.A.(Wichita) Teaching Fellow, Wichita University, Kan.

W. J. HUEBNER, JR., B.S.(St. Louis) Grad. Student, St. Louis University, Mo.

D. G. HUMM, Ph.D. (Southern California) Director, Humm Personnel Service, Los Angeles, Calif.

C. A. JACOKES, M.A. (Michigan S.) Instr., Knox College, Galesburg, Ill.

L. J. LANDER, Student, University of California at Los Angeles, Calif.

J. N. LEMON, JR., Student, Wabash College, Crawfordsville, Ind.

REV. BROTHER LEO, M.A. (St. John's) Professor, St. Francis College, Brooklyn, N. Y.

K. L. LEVERENCE, Student, Marquette University, Milwaukee, Wis.

R. E. LEWKOWICZ, B.A. (Michigan) Assistant in Research, Willow Run Research Center, Ypsilanti, Mich.

N. S. LICHTBLAU, M.S. (Chicago) Grad. Student, University of Chicago, Ill.

C. B. LINDQUIST, Ph.D. (Wisconsin) Asso. Professor, University of Minnesota, Duluth, Minn.

R. W. MACDOWELL, M.A. (Michigan) Instr., University of Rochester, N. Y.

J. N. MANGNALL, B.Sc. (Florida Southern C.) Mechanical Research Engineer, Bell Aircraft Corporation, Niagara Falls, N. Y.; Instr., Niagara University, N. Y.

CESLOVAS MASAITIS, M.A. (Kaunas) Assistant, University of Kentucky, Lexington, Ky.

W. S. MASSEY, Ph.D. ((Princeton) Asso. Professor, Brown University, Providence, R. I.

A. A. MATHIASEN, M.A. (Columbia) Research Engineer, Digital Computer Laboratory, Cambridge, Mass.

K. A. MCGOWN, B.A. (New Jersey S. T. C.) Chairman, Department of Mathematics, Wallington High School, N. J.

W. H. MEAD, JR., Student, University of Southern California, Los Angeles, Calif.

MRS. DOROTHEA MEAGHER, M.A. (Oklahoma) Chairman, Department of Mathematics, Central State College, Edmond, Okla.

MRS. ALVA P. MOORE, B.A. (Smith) Head, Department of Mathematics, Halsted School, Yonkers, N. Y.

T. D. NAGLE, Student, University of Bridgeport, Conn.

MARIA T. PAN, Student, Chestnut Hill College, Philadelphia, Pa.

L. L. PENNISI, Ph.D. (Chicago) Instr., University of Illinois, Navy Pier, Chicago, Ill.

H. E. PICKETT, B.S. (Arizona S. C.) Mathematician, White Sands Proving Ground, Las Cruces, N. M.

R. P. POLIVKA, B.A. (North Central) 3720 Harrison Avenue, Brookfield, Ill.

MELVIN POMERANTZ, Student, Polytechnic Institute of Brooklyn, N. Y.

ANTHONY RALSTON, Student, Massachusetts Institute of Technology, Cambridge, Mass.

J. R. REEDER, M.S. (Bradley) 315 N. 3rd, Box 143, Chillicothe, Ill.

B. E. RHOADES, B.A. (Ohio Northern) Rutgers University, New Brunswick, N. J.

ROBERTA E. ROBINSON, M.A. (Boston U.) Asst. Professor, Tennessee Agricultural & Industrial State University, Nashville, Tenn.

CHARLES ROTH, Ph.D. (N.Y.U.) Major and Instr., U. S. Military Academy, West Point, N. Y.

H. R. ROUSE, B.A. (Vanderbilt) Teaching Fellow, Vanderbilt University, Nashville, Tenn.

ELSIE C. RUMP, B.S. (McPherson) Teacher, Madison High School, Kan.

C. W. SAALFRANK, Ph.D. (Pennsylvania) Professor, Lafayette College, Easton, Pa.

S. C. SAUNDERS, Student, University of Oregon, Eugene, Ore.

B. D. SECKLER, M.A. (Columbia) Instr., Long Island University, Brooklyn, N. Y.

HARRY SHERMAN, B.S. (Brooklyn) Mathematician, Reeves Instrument Corporation, New York, N. Y.

L. L. SHETLER, M.S. (Indiana) Asst. Professor, Bluffton College, Ohio

L. S. J. SIMEONE, M.A. (Boston U.) Instr., Bradford Durfee Technical Institute, Fall River, Mass.

H. F. SIMMONS, B.A. (Wichita) Teaching Fellow, Wichita University, Kan.

A. F. STALCUP, Foreman, Aluminum Company of America, Bettendorf, Iowa

J. J. STEPANOWICH, M.S. (Illinois) Asst. Professor, Western Illinois State College, Macomb, Ill.

M. E. STERN, Student, Washington Square College, New York University, New York, N. Y.

L. R. STEWART, JR., B.A. (Washington & Jefferson) Salesman, International Business

Machines Corporation, Pittsburgh, Pa.

L. R. STIDHAM, M.S.(New Mexico C. of A. & M. A.) Assistant in Mathematics, New Mexico College of A. & M. A., State College, N. M.

DOROTHY J. STODOLA, M.S.(Marquette) Grad. Assistant, Marquette University, Milwaukee, Wis.

NIKOLA SVILOKOS, Student, Loyola College, Montreal, P. Q., Canada

L. E. TAYLOR, M.A.(Illinois) Supervising Teacher, Western Illinois State College, Macomb, Ill.

O. C. TRAMPE, B.A.(Carroll C., Wisconsin) Office Manager, Film Service, Milwaukee, Wis.

R. J. WAGNER, Student, University of Minnesota, Minneapolis, Minn.

S. V. S. WALKER, Student, Citadel, Charleston, S. C.

J. W. WARWICK, Ph.D.(Harvard) Scientific Consultant, Sacramento Peak Station, Harvard College Observatory, Cloudcroft, N. M.

L. A. WARWICK, M.A.(Michigan) Head, Department of Mathematics, Bob Jones University, Greenville, S. C.

M. S. WATKINS, Student, University of Toronto, Ontario, Canada

E. B. WEINBERGER, Ph.D.(Pittsburgh) Mathematician, Gulf Research & Development Company, Pittsburgh, Pa.

M. N. WELLNER, Student, Massachusetts Institute of Technology, Cambridge, Mass.

ARNOLD WENDT, Ph.D.(Wisconsin) Asst. Professor, Western Illinois State College, Macomb, Ill.

BETTY J. WHALEY, Student, Tennessee Polytechnic Institute, Cookeville, Tenn.

H. S. WILF, Student, Massachusetts Institute of Technology, Cambridge, Mass.

DEL WILLARD, M.A.(West Virginia) Asst. Professor, Baldwin-Wallace College, Berea, Ohio

E. W. WOMBLE, Student, Wofford College, Spartanburg, S. C.

L. L. WRIGHT, Student, Wabash College, Crawfordsville, Ind.

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The twenty-sixth meeting of the Allgeheny Mountain Section of the Mathematical Association of America was held at Waynesburg College, Waynesburg, Pennsylvania on May 10, 1952. Professor Morris Ostrofsky, Chairman of the Section, presided at the morning session, while Dean L. T. Moston of the host institution presided in the afternoon.

There were fifty persons present including the following thirty-five members of the Association:

Thomas Bauserman, W. H. Blum, J. O. Blumberg, A. M. Bryson, Abraham Charnes, J. G. Christiano, W. E. Cleland, A. B. Cunningham, H. A. Davis, R. C. DiPrima, F. E. Justis, J. C. Knipp, George Laush, C. E. Lemke, E. R. Michalik, David Moskovitz, L. T. Moston, B. H. Mount, Pauline C. Mount, E. F. Myers, Morris Ostrofsky, B. C. Patterson, D. D. Peters, C. N. Reynolds, B. L. Schwartz, F. H. Steen, J. K. Stewart, E. A. Sturley, W. C. Styslinger, T. T. Tanimoto, Margaret O. Taylor, Bird M. Turner, C. H. Vehse, M. L. Vest, V. S. Zora.

During the business session a resolution on the death of Professor J. B. Rosenbach was read by Professor David Moskovitz of the Carnegie Institute of Technology.

It was decided to hold the next meeting of the Section in May 1953 at the Carnegie Institute of Technology, Pittsburgh, Pennsylvania.

The following papers were presented:

1. *Analytic extension by summability methods*, by Professor George Laush, University of Pittsburgh.

An expository discussion of the regions of summability of power series by the Borel, Hausdorff,

and circle methods.

2. *Vibrations of twisted cantilever beams*, by Mr. R. C. DiPrima, and Professor G. F. Handelman, Carnegie Institute of Technology.

The problem of transverse vibrations of a twisted beam of variable cross section, clamped at one end, was studied, and the associated geometry was discussed briefly. The basic vector equations for the static deflections of a twisted beam were derived and interpreted as dynamic equations by regarding the distributed applied forces and couples as reversed effective forces and couples. Finally, the vector differential equation of the vibratory motion was derived and a minimum principle obtained for the lowest natural frequency of vibration of the beam. Orthogonality relations were found for distinct eigenfunctions, and minimum principles were derived for higher natural frequencies.

3. *The triangular resolution of matrices and its application*, by Dr. E. B. Weinberger, Gulf Research and Development Company, introduced by the Secretary.

Almost any matrix A may be resolved into two triangular matrices S and T such that $A=ST$ and S and T are lower and upper triangular, respectively. If T (or S) is specified to be a *unit* triangular matrix, then S and T are unique for a given A. If A is symmetric, then the resolution $A=SS'$ may be performed.

It was shown that two currently popular desk calculator methods for the solution of simultaneous linear equations, Crout's method and the "square root" method, are based on the triangular resolution process. A comparison was made between the efficacy of these methods and the Gauss elimination method.

A recent paper by Hayes and Vickers was discussed which describes the effective use of the triangular resolution process for the purpose of finding the least squares solution of a set of unequally-spaced data by means of orthogonal polynomials. In this technique the coefficients which multiply the polynomials are obtained incidentally to the usual solution of the least squares problem by means of a power series.

4. *Maximization of linear functionals subject to linear inequalities*, by Professor Abraham Charnes, Carnegie Institute of Technology.

The problem was introduced by examples from economics, theory of structures, and simultaneous linear equations. A geometric interpretation was provided in terms of which various features of the simplex method of solution were elucidated, and it was indicated how all singular situations could be avoided. The convenience of this method as a single digital computer for many of the classical linear equations problems was pointed out.

5. *Trend analysis in certain types of chance fluctuations*, by Professor E. R. Michalik, University of Pittsburgh.

The speaker discussed the cumulative effects in certain kinds of chance fluctuations and the use of the arc sine law in evaluating probabilities of these cumulative types of events. The classical coin tossing game was used to point out possible pitfalls in observing "obvious" trends. He also displayed an unusual property of the arc sine law when dealing with a certain kind of conditional probability, *i.e.*, when the final outcome of the game is known. Finally, the speaker discussed a theorem which replaces the classical central limit theorem when the variates do not have finite variances.

6. *On the inadequacy of the definition of "singular point" in complex function theory*, by Mr. B. L. Schwartz, Duquesne University.

There is no universally accepted definition for the term "singular point" in complex function

theory. Definitions vary with authors and are not generally equivalent. Nor do they accurately express the surprisingly consistent usage among mathematicians. The author pointed out the shortcomings of the definitions given in several well-known texts, and advanced the following suggestion for an acceptable definition: By a singular point of a function f we mean a point p at which f is not analytic, but such that any neighborhood N of p contains points at which f is analytic, and such that for any two such points of N, f can be continued analytically from one to the other.

7. *The futility of mathematics without philosophy and history*, by Professor W. H. Blum (mathematics) and Professor G. E. Moore (history), Waynesburg College.

A report issued by the National Security Resources Board expressing alarm at the shortage of creative thinkers in America set the tone for the paper which recommended a more direct emphasis on philosophy, logic, and history as a partial solution of the problem. Since mathematics is a human experience, its study should be a confluence of its history and philosophical implications. A purely mechanical and rote approach is worthless in the sense that this does not lead to a full consciousness of broad universal principles found in mathematics and common to all branches of knowledge.

The authors felt that if the student does not see the underlying esthetic patterns found in mathematical discourse as logical structure he will not be likely to create anything new. Awareness of these patterns does not lead to automatic creation, and although it may be impossible to teach people to think, they may be aided in discovering for themselves how it is done.

The interrelations between mathematics and general cultural developments were emphasized, and it was pointed out that the mathematician must be made aware of the effects of his discipline on civilization as well as to appreciate the importance of historical developments to the growth of mathematical thought.

F. H. STEEN, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-first annual meeting of the Illinois Section of the Mathematical Association of America was held at Western State College, Macomb, Illinois, on May 9 and 10, 1952. Professor S. S. Cairns, Chairman of the Section, presided at all sessions.

There were fifty-three in attendance, including the following thirty-eight members of the Association:

Beulah Armstrong, H. G. Ayre, Max Beberman, A. H. Black, A. I. Boatman, B. K. Brown, S. S. Cairns, Josephine Chanler, E. G. H. Comfort, J. E. Forbes, M. K. Fort, Jr., A. E. Gault, M. C. Hartley, Gertrude Hendrix, E. H. C. Hildebrandt, Martha Hildebrandt, F. E. Hohn, E. C. Kiefer, Rose Lariviere, Harry Levy, A. O. Lindstrum, Jr., C. T. McCormick, W. C. McDaniel, A. W. McGaughey, B. E. Meserve, M. G. Moore, Elsie C. Muller, F. S. Nowlan, Gordon Pall, J. W. Peters, W. L. Pickard, R. E. Pingrey, Haim Reingold, J. A. Schumaker, M. Anice Seybold, R. C. Stephens, Dunstan Velesz, Alice K. Wright.

At the business meeting the following officers were elected for the coming year: Chairman, Professor H. G. Ayre, Western State College; Vice-Chairman, Professor M. C. Hartley, University of Illinois, Navy Pier, Chicago; Secretary-Treasurer, Professor E. C. Kiefer, Millikin University. The annual meeting for 1953 will be held at the University of Illinois, Navy Pier, Chicago, on May 8 and 9, 1953.

The following program was presented:

1. *Some applications of matrices*, by Professor F. E. Hohn, University of Illinois.

Professor Hohn outlined the range of application of matrix methods in the physical and social sciences, and illustrated his remarks with examples from operations research and the theory of communication nets.

2. *Geometric primitive solutions of the Diophantine equation* $G^2-2V^2=\pm K^{2n}$, by Miss Rose Lariviere, University of Illinois, Navy Pier, Chicago.

If equations for the sides of primitive, integral, right triangles are written in terms of the lesser parameters of each of the two Euclidean pairs, any such triangle can be used as a root from which two infinite sequences of primitive, integral, right triangles with legs of constant difference can be developed. Geometric counterparts for all primitive solutions of the Diophantine equation $G^2-2V^2=\pm K^{2n}$, except $1^2-2(1)^2=-(1)^{2n}$, can be based upon these sequences and their root triangles by means of equations using all four Euclidean parameters.

3. *The factorization of the quadratic trinomial*, by Professor F. S. Nowlan, University of Illinois, Navy Pier, Chicago.

This paper outlines a procedure for expressing a quadratic

$$p(x) = fx^2 + gx + h,$$

where f, g, and h are relative prime integers, as a product of linear factors,

$$p(x) = (ax + b)(cx + d),$$

where also the pairs a and b, c and d, are relatively prime integers. (This, of course, assumes that such a factorization is possible.)

We choose any convenient positive integer k and consider two cases: Case 1. Let $p(k)=n=n_1n_2$, where $n_1=a(k)+b$, $n_2=c(k)+d$. Then to determine b and d, we write:

$$b = n_1 - a(k), \qquad d = n_2 - c(k). \tag{1}$$

Case 2. Let $p(-k)=n=n_1n_2$. Then,

$$b = n_1 + a(k), \qquad d = n_2 + c(k). \tag{2}$$

We note that to determine b and d, Case 1 involves subtraction and Case 2 involves addition. (This seems the simplest approach in presenting the work to a class in high school or freshman algebra.)

The factors of $p(x)$ are determined by reference to a trial matrix

$$\begin{pmatrix} a'(k) & n_1' \\ c'(k) & n_2' \end{pmatrix}$$

where $a'c'=f$ and $n_1'n_2'=n$. The appropriate choice of a', c', n_1' and n_2' is the choice which upon applying (1), or (2), gives $bd=h$.

It is desirable to choose k so that $p(k)$, or $p(-k)$, is either a prime or the product of but a few prime factors, none of which are repeated. The method provides a simple and quick means of factoring the more involved quadratics of the type that has been indicated.

4. *The determinant of a rectangular matrix*, by Professor Gordan Pall, Illinois Institute of Technology.

In a very natural way, the expression $D=(\sum\Delta^2)^{1/2}$ where Δ ranges over the abstract values of the minor determinants of order S in the matrix $A=(a_{ij})$, $(i=1, \cdots, n; j=1, \cdots, S)$; $S\leqq n$, should be called the absolute determinant of A. Indeed D, considered as a function of the S-row vectors $\rho_1, \cdots, \rho_S$ of A, has the four properties: (1) $D(\rho_1, \cdots, \rho_S)$ is unaltered by permutation

of $\rho_1, \cdots, \rho_S$; (2) $D(K\rho_1, \rho_2, \cdots, \rho_S) = |K| D(\rho_1, \rho_2, \cdots, \rho_S)$; (3) $D(\rho_1 + K\rho_2, \rho_2, \cdots, \rho_S) = D(\rho_1, \cdots, \rho_S)$; (4) if $\rho_1, \cdots, \rho_S$ are of unit length and mutually orthogonal, then $D(\rho_1, \cdots, \rho_S) = 1$. Any function $D(\rho_1, \cdots, \rho_S)$ with these properties is uniquely determined, since its value can be computed by use of these properties for any given vectors $\rho_1, \cdots, \rho_S$. Also, if $\rho_i = b_{i1}\epsilon_1 + \cdots + b_{iS}\epsilon_S$ (where ϵ_i are mutually orthogonal unit vectors) ($i = 1, \cdots, S$), then the absolute value of the determinant $|b_{ih}|$ also has these properties. Hence D coincides with this absolute value. It also follows, in the real case, that D gives the "volume" of the S-dimensional parallelepiped with the row-vectors of A as edges.

5. *A generalization of the Pellian equation and applications*, by Professor W. L. Pickard, Northern Illinois State Teachers College.

Formulas are derived that will give the Nth solution of the cubic Pellian equation $x^3 + Dy^3 + D^2z^3 - 3Dxyz = 1$, where D is a positive non-square integer, in terms of the fundamental solution. Generalized hyperbolic functions are used to obtain the corresponding equation $\phi^3(x) + \psi^3(x) + \chi^3(x) - 3\phi(x)\psi(x)\chi(x) = 1$. These functions satisfy the equation $[e^{\omega^3 x}]^N = [\phi(x) + \omega\psi(x) + \omega^2\chi(x)]^N$ where $\omega^3 = 1$. By analogy with the quadratic case, the transition relations $\phi(Nx) = X_N$; $\psi(Nx) = \sqrt[3]{D}\, Y_N$; $\chi(Nx) = \sqrt[3]{D^2}\, Z_N$ are written. By applying a generalized DeMoivre's theorem, the equation $[\phi(x) + \omega\psi(x) + \omega^2\chi(x)]^N = \phi(Nx) + \omega\psi(Nx) + \omega^2\chi(Nx)$ is obtained. By applying the multinomial theorem to the left side, equating equal powers of ω, and substituting the transition relations the required formulas are obtained:

$$X_N = \sum \frac{N!}{\alpha!\beta!\gamma!} X_1^\alpha (\sqrt[3]{D}\, Y_1)^\beta (\sqrt[3]{D^2}\, Z_1)^\gamma, \quad \text{where } \beta + 2\gamma = 0 \pmod 3,$$

$$Y_N = \sum \frac{N!}{\alpha!\beta!\gamma!} \frac{X_1^\alpha (\sqrt[3]{D}\, Y_1)^\beta (\sqrt[3]{D^2}\, Z_1)^\gamma}{\sqrt[3]{D}} \quad \text{where } \beta + 2\gamma = 1 \pmod 3,$$

$$Z_N = \sum \frac{N!}{\alpha!\beta!\gamma!} \frac{X_1^\alpha (\sqrt[3]{D}\, Y_1)^\beta (\sqrt[3]{D^2}\, Z_1)^\gamma}{\sqrt[3]{D^2}} \quad \text{where } \beta + 2\gamma = 2 \pmod 3,$$

α, β, γ and N must be positive integers, $\alpha + \beta + \gamma = N$.

6. *The place of mathematics in our Illinois schools*, by Professor E. H. C. Hildebrandt, Northwestern University.

7. *The problem of preparation in mathematics*, by Professor M. Anice Seybold, North Central College; Mr. Max Beberman, University High School, Urbana; Mr. Edgar Leach, Head of the Mathematics Department, Evanston Township High School; Dr. E. L. Welker, American Medical Association, Chicago, and Professor Gertrude Hendrix, Eastern Illinois State College.

Professor Seybold introduced the program with the statements that in the previous year's discussion the problem of poor preparation was shown to exist and that now is the time for the Illinois Section to do something about it. She read quotations from Nimitz, MacDuffee, Lindsay, and others to show that there is an awakening interest on a national scale in higher standards for elementary and secondary schools. She pointed out trends to which we should give our support, mentioning such studies as that conducted at the University of Illinois by a committee from the departments of mathematics, engineering, and education.

Mr. Beberman reported that this committee is seeking to develop a single mathematics program beginning with the ninth year on the assumptions: (1) the minimum mathematical needs of prospective engineers are essentially the same as the needs of students preparing for study in any area requiring courses in collegiate mathematics, (2) both the general educational needs of students and prerequisites for college mathematics courses can be satisfied by a one-track mathematics course in the first and possibly in the second year of high school mathematics, (3) there are many

concepts of general educational worth now found in later years of high school mathematics that should be brought down into ninth or tenth year mathematics. Eventually, through experimental work, the committee hopes to develop new and better high school mathematics courses.

Miss Seybold then mentioned the core curriculum as a teaching trend not suitable for mathematics instruction and introduced the plan for college admission in bulletin No. 9 of the Illinois Curriculum Program, the objective of which plan is to make way for experimental high school courses.

A report by Mr. Edgar Leach on his work as a member of the Steering Committee of the Illinois Curriculum Program was read.

Dr. E. L. Welker analyzed Volume IV, "*Did They Succeed in College?*," by Chamberlin and Chamberlin of the reports on the Eight Year Study, which study is often quoted as having proved that the pattern of subjects a student takes in high school has no effect upon his college success. Dr. Welker stated that in his opinion the study was not valid and that the statistical techniques were highly inadequate.

Professor Gertrude Hendrix spoke on difficulties in implementing the college admission plan in bulletin No. 9. She pointed out confusion and incompleteness in implementation of the proposed admission plan and gave the results of a study at Eastern Illinois State College. Of a total of 2002 freshman admissions, 1947–50, the number of survivors is 886 or 44%. Of seventy freshman admissions on General Educational Development tests in the same period only ten students, or 14%, survive, and some of these ten may not graduate.

Professor Seybold then called on Professor Cairns to speak informally. Dr. Cairns described the rising concern of people both inside and outside of Illinois over the problem of the poor preparation for life and for college which so many children are now receiving in school. A discussion followed during which resolutions were adopted. A committee will be appointed to implement these resolutions.

E. C. KIEFER, *Secretary*

THE MAY MEETING OF THE INDIANA SECTION

The twenty-ninth annual meeting of the Indiana Section of the Mathematical Association of America was held at Indiana University, Bloomington, Indiana, on May 3, 1952. Two sessions were held at which Professor M. W. Keller of Purdue University, Chairman of the Section, presided.

There were fifty-nine in attendance including the following thirty-nine members of the Association:

Juna L. Beal. L. G. Black, Stanley Bolks, C. F. Brumfiel, G. E. Carscallen, W. W. Chambers, H. E. Crull, R. E. Dowds, W. E. Edington, P. D. Edwards, R. E. Ekstrom, E. L. Godfrey, S. H. Gould, G. H. Graves, H. E. H. Greenleaf, J. R. Hadley, Ralph Hull, M. W. Keller, Florence Long, Saunders MacLane, D. S. Merrilees, G. T. Miller, J. A. Nickel, P. W. Overman, T. P. Palmer, J. C. Polley, D. H. Porter, Arthur Rosenthal, R. M. Ross, J. F. Schell, C. P. Sousley, Anna K. Suter, T. Y. Thomas, Jane A. Uhrhan, M. S. Webster, K. P. Williams, H. E. Wolfe, R. C. Wrede, J. W. T. Youngs.

The following officers were elected: Chairman, Professor M. A. Zorn, Indiana University; Vice-Chairman, Professor C. P. Sousley, Rose Polytechnic Institute.

The annual meeting of 1953 will be held at Ball State Teachers College, Muncie, Indiana, on May 2.

The following papers were presented:

1. *On two topics in advanced calculus*, by Professor George Whaples, Indiana

University, introduced by Professor H. E. Wolfe.

Simple methods are outlined for deriving trigonometry from the differential equation satisfied by the sine function and for proving the theorems on implicit functions by use of sequences of linear approximations.

2. *Conics among the Greeks*, by Professor S. H. Gould, Purdue University.

After a brief discussion of some examples of conics in literature and art, and of the exact meaning of the names, parabola, hyperbola, and ellipse, the speaker undertook to show that every modern definition of a conic appears, in more or less explicit form, in the works of the Greek mathematicians.

3. *An elementary derivation of the formula of Foucault*, by Professor P. D. Edwards, Ball State Teachers College.

Draw a circle in a plane tangent to the surface of the earth and sufficiently small that the radius may be considered either as a straight line or an arc of a great circle of the earth. Due to the rotation of the earth the points of the circle farthest from the pole move toward the East faster than the center. By familiar formulas of trigonometry it is shown that this difference in speed gives a relative motion about the center that is identical with the formula of Foucault. The only advantage of the method is that it can be understood by students who are studying trigonometry, whereas the development of the Foucault formula by the usual methods of dynamics involves rather difficult differential equations.

4. *Some elementary principles of circulates in different number scales*, by Professor E. L. Godfrey, Defiance College.

An experimental study of a tabulation of the circulates of the reciprocals of prime numbers expressed in successive scales of notation suggested an arrangement for each prime, p, of its circulates into $p-1$ groups. In each of these groups there is a constant difference between the corresponding digits of the circulates for radix r and $r+p$. The digits of this difference were shown to be equal to the successive powers of r modulo p. A discussion of some of the further properties of these differences resulting from experimental study included some suggestions concerning the possible use of such studies in the teaching program.

5. *Unified theories of relativity*, by Professor Vaclav Hlavaty, Indiana University, introduced by Professor H. E. Wolfe.

The speaker presented a brief outline of different attempts to obtain the unified theory of relativity (Weyl, Kaluza, Schrödinger) and the solution of Einstein's basic problems of the theory.

6. *Homological algebra*, by Professor Saunders MacLane of the University of Chicago, President of the Association. (By invitation.)

Recent developments in algebraic topology and in the homology theory of groups and of other algebraic systems indicate the existence of a new branch of algebra which may be dubbed "homological algebra." A preliminary notion of this subject, that of the kernel and the image of a homomorphism, may be illustrated even in the simple examples, such as the homomorphism $x \to e^{2\pi i x}$ of the additive group of reals into the multiplicative group of complex numbers. In this connection, the notion of an exact sequence of groups and homomorphisms is introduced, and the 5-lemma for such exact sequences is established.

The central concept of homological algebra is that of a differential group A, composed of an abelian group A and a homomorphism $\partial: A \to A$ with $\partial\partial = 0$. Each such differential group determines a homology group $H(A)$, defined as the quotient of the kernel of ∂ by the image of ∂. If A has a differential subgroup A', this determines a quotient differential group $\bar{A} = A/A'$; the asso-

ciated sequence $H(A')\to H(A)\to H(\bar{A})\to H(A')\to\cdots$ is exact. If a homomorphism f of A with subgroup A' into a second differential group B with subgroup B' induces isomorphisms on $H(A')$ and $H(\bar{A})$, the 5-lemma may be used to prove that f also induces an isomorphism of $H(A)$ to $H(B)$. It was indicated how this and similar results may be used to establish the equivalence of different definitions of homology theories for various types of mathematical systems.

J. C. POLLEY, *Secretary*

THE MAY MEETING OF THE METROPOLITAN NEW YORK SECTION

The eleventh annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Hofstra College, Hempstead, New York, on May 3, 1952. Mr. E. I. Shapiro, Vice-Chairman of the Section, presided at the morning session, and Professor James Singer, Chairman of the Section, presided at the afternoon session.

One hundred fifty-one persons attended the sessions, including the following eighty-four members of the Association:

R. G. Archibald, I. L. Battin, Brother Bernard Alfred, W. W. Bessell, Samuel Borofsky, C. B. Boyer, Benjamin Braverman, A. B. Brown, Charlotte Brown, J. H. Bushey, Jewell H. Bushey, P. J. Cocuzza, L. W. Cohen, T. F. Cope, W. H. H. Cowles, I. A. Dodes, J. N. Eastham, J. E. Eaton, W. H. Fagerstom, A. B. Farnell, J. M. Feld, Daniel Finkel, Edward Fleisher, William Forman, R. M. Foster, B. P. Gill, A. J. Goldman, I. L. Goldman, Bernard Greenspan, Harriet M. Griffin, Carl Hammer, Frank Hawthorne, G. C. Helme, H. H. Hinman, E. Marie Hove, R. J. Jaeger, Jr., D. B. Jordan, L. S. Kennison, G. A. Keyes, E. R. Kiely, H. S. Kieval, M. S. Klamkin, David Kotler, Edna Kramer-Lasser, H. C. Kranzer, C. H. Lehmann, M. E. Levenson, Walter Littman, E. R. Lorch, W. A. Lucas, J. D. Matheson, G. J. Mazzara, F. H. Miller, Morris Morduchow, A. J. Mortola, G. R. Mott, A. F. Nickl, C. J. Oberist, L. F. Ollmann, Martin Orr, J. J. Quinn, M. R. Reeks, Moses Richardson, Selby Robinson, N. J. Rose, H. D. Ruderman, J. P. Russell, J. B. Ryan, John Salerno, Charles Salkind, Arthur Sard, A. H. Sarno, Abraham Schwartz, Aaron Shapiro, E. I. Shapiro, James Singer, F. E. Smith, E. R. Stabler, Mildred M. Sullivan, R. L. Swain, L. F. Tolle, A. W. Tucker, J. A. Vollkommer, Sue R. Waldman.

The officers elected at the business meeting were: Chairman, Professor L. F. Ollmann, Hofstra College; Collegiate Vice-Chairman, Professor W. H. Fagerstrom, The College of the City of New York; High-School Vice-Chairman, Mr. H. D. Ruderman, Manhattan High School of Aviation Trades; Secretary, Dr. H. S. Kieval; Treasurer, Mr. Aaron Shapiro, Midwood High School.

At the business meeting, the following report on the activities of the Committee on Contests and Awards was given by its chairman, Professor W. H. Fagerstrom: "The Committee on Contests and Awards of the Metropolitan New York Section of the Mathematical Association of America enlarged its area of operation this year to include the state of Connecticut along with its own area which includes southern New York, Long Island, and northern New Jersey. In addition, the University of Oregon and the University of British Columbia conducted their own contests but used the questions of the Metropolitan New York Section. One hundred thirty-three invitations were sent out to the larger high schools in the other 44 states of the union, not more than four invitations going to any one state. The registration of 39 schools from 29 states was most gratifying. There were 295 schools registered for the contest and

7,867 copies of the examinations were used. Further information regarding the contest and copies of the 1951 contest questions may be obtained from the Chairman, W. H. Fagerstrom, City College, New York 31, N. Y."

The twelfth annual meeting of the section will be held in the spring, 1953.

After a welcoming address given by President J. C. Adams of Hofstra College the following papers were presented:

1. Panel discussion, *The mathematics curriculum*:

(a) *Training for industry*, by Professor R. M. Foster, Polytechnic Institute of Brooklyn.

(b) *Undergraduate mathematics for its own sake*, by Professor Arthur Sard, Queens College.

The speaker made a few general remarks, largely supporting the customary mathematical curriculum.

(c) *Modern psychology in the teaching of high school mathematics*, by Dr. I. A. Dodes, Stuyvesant High School, New York City.

The following conclusions, based on fifty years of experimentation, were presented. (1) If the teacher emphasizes a pattern, it will be reproduced as a learning pattern. If he attempts to correlate or integrate various fields, the learning which takes place will probably follow a subject—field pattern, anyhow. (2) Motivating devices do not guarantee good learning outcomes. Certain motivating devices which tend to "bring the goal nearer" are reported to be very successful. (3) No "type" of lesson will guarantee a better learning outcome than any other. (4) No specific classroom practice will guarantee good or bad learning outcomes. (5) The evidence concerning homework and tests as teaching devices is scanty. (6) There is no decisive proof that any specific "philosophy" of teaching (questioning, pupil activity, pupil-centered methods, *etc.*) guarantee good learning outcomes. (7) There is considerable proof that the teacher "as a whole" is the important factor in successful learning. The evidence does not show clearly why this is so.

2. *The theory of games*, by Professor A. W. Tucker, Princeton University.

The speaker discussed zero-sum two-person games in matrix form, giving several simple examples of such games and their solutions. He illustrated the role of bluffing in poker by a highly simplified two-person model permitting one bet and one raise. He also presented a two-person "dilemma" to show the complications that arise when a two-person game is *not* zero-sum.

3. *On the application of algebraic methods to analysis*, by Professor E. R. Lorch, Columbia University.

A discussion of the introduction into problems of analysis of methods, concepts, and points of view which belong to algebra. Three analytical problems were considered. To determine: (a) the structure of a linear operator (*e.g.*, differential or integral); (b) the structure of the ring of functions $f(x)$ continuous on the closed unit interval; (c) the structure of the ring of functions $f(x)$ integrable $(-\infty, +\infty)$ where multiplication is defined by convolution.

In discussing briefly the solution of these problems, the role of the algebraic concepts of ring, field, ideal, radical, idempotent, reducibility, and Boolean algebra was brought to the fore.

4. *Collegiate mathematics clubs*, by Professor L. F. Ollmann, Hofstra College.

Among the important advantages and opportunities offered the student members of our present day mathematical clubs over those of earlier clubs are: (1) the existence of the national honorary mathematical societies, Pi Mu Epsilon and Kappa Mu Epsilon; (2) the availabity of publica-

tions which cater to students of mathematics by publishing articles written by and for the students.

The author urges all teachers to become actively interested in existing mathematical clubs or to sponsor new mathematical organizations to meet the needs of other students. He suggests that such extra effort on the part of the teacher may be very rewarding and that the attitudes toward mathematics of many students may be changed from indifference or even hatred to keen interest in mathematics by an inspiring teacher working through the medium of a well-conducted mathematics club.

5. *Mathematics clubs for high schools*, by Mr. E. I. Shapiro, Abraham Lincoln H.S., Brooklyn, N. Y.

In recent years, a great deal of attention has been paid to the slow student but the needs of the superior student have been overlooked. The superior student can be greatly stimulated through mathematics clubs, contests, and by means of special classes. Most of the schools in the Metropolitan area have some form of club specializing in making models, using the slide rule, studying surveying, and taking up topics of a general nature. At least two schools publish excellent printed mathematics magazines. Some schools provide for the superior student by means of special classes. There are three mathematics contests conducted in the Metropolitan area. One is sponsored by the Interscholastic Mathematics League of New York City with frequent meets each term. The other two are annual contests. Of these two, one is given by the Pi Mu Epsilon chapter of New York University and the other is conducted by the Metropolitan New York Section of the Mathematical Association of America. Several hundred schools participate in these contests.

H. S. KIEVAL, *Secretary*

THE MAY MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at the College of St. Catherine in St. Paul, Minnesota, on May 10, 1952. Sessions were held in the forenoon, at luncheon and in the afternoon. Sister M. Seraphim, Professor W. R. McEwen and Professor H. D. Colson, Chairman of the Section, priesided at the respective sessions.

Sixty-nine persons attended the meeting including the following fifty-four members of the Association:

H. M. Anderson, V. W. Beck, E. J. Berger, C. R. Bonnell, R. W. Brink, L. E. Bush, W. H. Bussey, R. H. Cameron, E. J. Camp, C. S. Carlson, Elizabeth Carlson, H. D. Colson, Helen Engebretson, I. C. Fischer, G. C. Francis, W. B. Fulks, Gladys Gibbens, J. E. Hafstrom, K. L. Hankerson, W. L. Hart, A. G. Hill, J. S. Hill, G. H. Jaeger, G. K. Kalisch, Karlis Kaufmanis, W. S. Loud, H. B. MacDougal, Kenneth May, W. H. McBride, W. R. McEwen, A. G. Montgomery, E. O. Nelson, M. J. Norris, J. C. Peterson, P. A. Rognlie, P. C. Rosenbloom, Ruth Scholten, L. W. Sheridan, S. C. Simonson, Sister M. Joanne, Sister M. Mercedes, Sister M. Prudentia, F. C. Smith, R. C. Staley, O. E. Stanaitis, Irwin Stoner, P. Y. Tani, F. J. Taylor, Takashi Terami, H. P. Thielman, Ella Thorp, S. E. Warschawski, K. W. Wegner, Irene L. Wente.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor A. G. Swanson, Gustavus Adolphus College; Secretary, Professor F. C. Smith, College of St. Thomas; Members of the Executive Committee, Professor H. D. Colson, Bemidji State Teachers College, Mr. J. S. Hill, Minnesota Mutual Life Insurance Company, Professor G. K. Kalisch, University of Minnesota.

By invitation of the Executive Committee, Professor H. P. Thielman of

Iowa State College delivered an address at the morning session entitled "Types of Functions."

The following short papers were presented:

1. *The generalized Euler's summation formula*, by Professor O. E. Stanaitis, St. Olaf College.

The formula

$$\sum_{k=0}^{m}\sum_{v=0}^{n} f(v,k) = -\int_0^m\int_0^n f(x,y)dxdy + \int_0^m\int_0^n P_1(x)P_1(y)f''_{xy}(x,y)dxdy$$
$$-\frac{1}{4}[f(0,0)+f(0,m)+f(n,0)+f(n,m)]$$
$$+\frac{1}{2}\sum_{v=0}^{n}[f(v,0)+f(v,m)]+\frac{1}{2}\sum_{k=0}^{m}[f(0,k)+f(n,k)]+\sum_{k=0}^{m}\int_0^n f(x,k)dx$$
$$+\sum_{v=0}^{n}\int_0^m f(v,y)dy-\frac{1}{2}\int_0^n[f(x,0)+f(x,m)]dx-\frac{1}{2}\int_0^m[f(0,y)+f(m,y)]dy,$$

where $P_1(t)=t-[t]-\frac{1}{2}$ and $[t]$ denotes, as usual, the largest integer not greater than t, is derived. The above formula, which is believed to be new, gives a closed expression for a double sum in terms of double integrals and some additional terms. If the series $P_1(t)=-\sum_{\alpha=1}^{\infty}\dot{x}_1 \sin 2\alpha\pi t/\alpha\pi$ is substituted and the second integral on the right is integrated by parts, a more advantageous representation follows. The most favorable results are obtained when the higher derivatives of $f(x, y)$ are very small or vanish.

If $m\to\infty$, $n\to\infty$, the formula may be used also to prove the convergence of double series.

2. *Constructive equivalents of Vitali's theorem*, by Professor P. C. Rosenbloom, University of Minnesota.

If $\epsilon>0$, $N\leqq \exp(12/\epsilon^2)$, and if $f_1, f_2, \cdots, f_N$ are analytic and $|f_k(z)|\leqq 1$ for $|z|<1$, then there are integers i and j, $1\leqq i<j\leqq N$, such that $|f_i(z)-f_j(z)|<4\epsilon$ for $|z|\leqq 1-\epsilon$. This may be proved constructively in the sense that if the functions f_k are defined constructively in terms of their Taylor coefficients, then we can give a constructive process for finding f_i and f_j with the above property. This can easily be proved to be equivalent to Vitali's theorem, but the proof is non-constructive. The bound on N is far from the best possible. If the functions are defined by algorithms for computing the values at a given sequence of points $\{z_k\}$ such that $\sum|z_k|=+\infty$, then we can again obtain a constructive process for picking out of a large number of functions in the family two which differ by a small amount if $|z|\leqq 1-\epsilon$.

3. *On Zorn's lemma*, by Professor M. J. Norris, College of St. Thomas.

It was noted that in several proofs of the well-ordering theorem from the choice axioms, Zorn's lemma is almost proved as an intermediate step. It was suggested that a direct proof of Zorn's lemma might be useful for pedagogical reasons, and one such proof was sketched.

4. *Tests for completeness*, by Professor R. H. Cameron, University of Minnesota.

This paper extends the validity of a formula of Ross Graves for testing sets of orthogonal functions for completeness.

5. *New magnetic device for computers*, by Mr. C. R. Bonnell, Minneapolis-Honeywell Regulator Co.

A short survey and explanation of pulse counting by using magnetic cores with square hys-

teresis loops was given and was followed by a description of several new magnetic devices of extreme accuracy and high resolution. The latter devices are of two types, torque devices and displacement measuring devices. The torque and displacement devices are being used in torque rebalance computers by summing torques on a shaft. The computers are closed loop servo mechanisms with high frequency response. Methods of solving algebraic and transcendental equations, performing arithmetical operations, and integrating were shown in block diagrams.

6. *Problems from Archimedes' Method*, by Professor R. C. Staley, University of North Dakota.

Three problems from Archimedes' treatise, *The Method*, were presented with the proofs based on mechanics. The close approach to the method of integral calculus was pointed out. These problems were used to illustrate the value of a course in the history of mathematics when such a course is taught to show how the mathematicians obtained their results.

F. C. SMITH, *Secretary*

THE MAY MEETING OF THE MISSOURI SECTION

The annual spring meeting of the Missouri section of the Mathematical Association of America was held at Lindenwood College, St. Charles, Missouri, on May 2, 1952. Professor P. B. Burcham, Vice-Chairman of the Section, presided at the morning session and Professor Francis Regan, Chairman of the Section, presided at the afternoon session.

Sixty-eight persons attended this meeting, including the following thirty-four members of the Association:

J. J. Andrews, W. L. Ayres, P. B. Burcham, Mary Cummings, H. Margaret Elliott, H. M. Feldman, W. E. Felling, C. V. Fronabarger, W. A. Golomski, E. A. Goodhue, Franklin Haimo, Nola L. A. Haynes, F. F. Helton, I. I. Hirschman, L. W. Jacobs, III, G. H. Jamison, L. O. Jones, Paolo Lanzano, F. H. Lloyd, Jessie McLean, R. J. Michel, R. R. Middlemiss, Marie A. Moore, Zeev Nehari, Margaret Owchar, R. M. Rankin, Francis Regan, J. H. Skelton, W. L. Stamey, Choy-tak Taam, W. J. Thron, G. B. Van Schaack, Margaret F. Willerding, J. L. Zemmer.

Special guests were Mr. M. J. Gauss and Miss Virginia Gauss, who are great-grandchildren of Karl Friedrich Gauss, Mrs. M. J. Gauss, and Mr. David Gauss.

At the business meeting the following officers were elected: Chairman, Professor L. O. Jones, William Jewell College; Vice-Chairman, Reverend W. C. Doyle, Rockhurst College; Secretary-Treasurer, Professor Nola L. A. Haynes, University of Missouri. The Chairman announced the election of Dr. R. J. Michel, Southwest Missouri State College, to the Board of Governors as Sectional Governor for the period July 1, 1952 to July 1, 1955.

The next meeting will be held at William Jewell College, Liberty, in the spring of 1953.

The following papers were presented:

1. *Gauss and Göttingen*, by Professor Herman Betz, University of Missouri.

2. *Tensor aspects of the calculus of variations*, by Dr. Paolo Lanzano, St. Louis University.

Euler's stationarity conditions for the line integral, $\int_{t_0}^{t_1} F(x; \dot{x})dt$, where $\dot{x}=dx/dt$ (F being a scalar function) along a parametrized curve $x^i=x^i(t)$ of a regular V_n are analyzed for tensor be-

havior; it is proved they constitute the components of a covariant vector; its vanishing assures the unicity of stationarity conditions no matter how the comparison curves are allowed to vary. The problem of reducing these conditions to normal form (solving for the second derivatives of the parameter) is shown to be intrinsically related to the $F(x;\ \dot{x})$ and not to the coordinate system; the case in which $F(x;\ \dot{x})$ is homogeneous of the first degree in the $\dot{x}$'s (which makes the integral independent of the parametrization) is handled by the use of the infinitesimal invariant $d\tau = F(x;\ \dot{x})dt$ (affine parameter) which enables us to write the equations of the extremals in the form $d^2x^i/d\tau^2 = -\Gamma^i_{jh}(x;\ dx)(dx^j/d\tau)(dx^h/d\tau)$ where the $\Gamma^i_{jh}(x;\ dx)$'s are the coefficients of a non-positional affine symmetric connection.

Extremals thus appear to be the paths of an affinely connected manifold and many results of the geometry of paths can be extended to the calculus of variations.

3. *Mathematics and college students*, by Professor C. V. Fronabarger, Southwest Missouri State College.

This paper dealt with the relationship of the high school mathematical background and entrance tests to "pursuance of" and "success in" college mathematics at Southwest Missouri State College. The following conclusions were presented. (1) Competency in fundamental mathematical skills and processes increases with each half unit of high school mathematics. (2) When students are matched, pairwise, on the basis of the ACE Psychological Examination for College Freshmen, competency is independent of the particular pattern or combination presented. (3) There is little if any correlation between the number of units of high school mathematics and success in college mathematics. (4) The Stanford Achievement Test (Advanced Arithmetic) correlates more highly with success in college mathematics than the quantitative, linguistic, or gross score on the ACE Examination. Furthermore, this correlation is within the range usually found for the mathematics section of the Scholastic Aptitude Test. (5) Better than 75 per cent of the students who pursue the study of mathematics through at least five hours of calculus are above the mean of the quantitative score of the ACE Examination but this same group is just average with respect to the linguistic score. (6) Success in college mathematics courses correlates more highly with success in prerequisite courses, singly and combined, than with scores on entrance tests. (7) Students of relatively low mental ability and arithmetic achievement can, by means of a twelve week remedial course, achieve and retain an advance in arithmetic skills equivalent to two elementary grades.

4. *On a criterion of non-oscillation*, by Professor Choy-tak Taam, University of Missouri.

5. *Why a converse?*, by Mr. W. A. Couch, Washington University.

6. *A critical look at undergraduate mathematics*, by Dean W. L. Ayres, Purdue University.

The paper points out the rigidity of the usual undergraduate program through the calculus and questions whether this traditional sequence of the last century fits the science of today. Among other points the need for an early introduction to statistics is discussed. The time spent on trigonometry, the conics, and some other topics seems excessive and some of the drill on technique may be omitted and obtained incidentally in applications. Finally the special needs of the non-mathematics major who takes a year of college mathematics are discussed.

7. *A tribute to Karl Friedrich Gauss and a presentation of the members of the Gauss family residing in St. Charles, Missouri*, by Professor S. A. E. Betz, Department of English, Lindenwood College.

S. LOUISE BEASLEY, *Secretary*

THE MAY MEETING OF THE NEBRASKA SECTION

The twenty-eighth annual meeting of the Nebraska Section of the Mathematical Association of America was held on the Agricultural College Campus of the University of Nebraska, Lincoln, Nebraska on May 3, 1952. Professor W. G. Leavitt presided at the sessions.

Thirty persons attended the morning session including the following nineteen members of the Association:

M. A. Basoco, H. W. Becker, A. K. Bettinger, C. C. Buck, C. C. Camp, F. Marion Clarke, Morris Dansky, H. W. Doss, J. M. Earl, C. B. Gass, Edwin Halfar, A. L. Johnson, W. G. Leavitt, N. H. Mewaldt, C. R. Perisho, H. B. Ribeiro, Harry L. Rice, D. D. Rippe, Lulu L. Runge.

Officers elected at the meeting were: Chairman, Mr. H. W. Becker, Electronic Radio-Television Institute, Omaha; Vice Chairman, Professor W. G. Leavitt, University of Nebraska; Secretary, Professor Edwin Halfar, University of Nebraska.

Professor H. B. Ribeiro of the University of Nebraska was the guest speaker of the section. The afternoon session was held in conjunction with the Nebraska Section of the Council of Teachers of Mathematics.

The following papers were presented:

(1) *Circuit means*, by H. W. Becker, Electronic Radio-Television Institute, Omaha.

The arithmetic, harmonic, and geometric means of $a/c>b/d$ are special cases of the circuit mean $CM_+=Z(z)_+=(za+b)/(zc+d)$. This is the impedance of a four terminal network as seen from the source s, whose open- and short-circuit components are $Z(\infty)_+=a/c, Z(0)_+=b/d$. Looking back from the load end z, $CM_-=Z(s)_-=(sd+b)/(sc+a)$ is the circuit mean of $d/c=Z(\infty)_-$ and $b/a=Z(0)_-$. $CM_\pm=AM_\pm$ when $z=Z(\infty)_\pm$, and $CM_\pm=HM_\pm$ when $z=Z(0)_\pm$. $CM_\pm=GM_\pm$ when $z=[Z(\infty)_\pm\cdot Z(0)_\pm]^{1/2}$, and the network is said to be terminated in its image impedances. The iterative mean of a/c and b/d is $IM=Z(z)=z=[a-d+\{(d-a)^2+4bc\}^{1/2}]/2c$, where the network is said to be terminated in its iterative impedance.

(2) *On certain solutions of a non-linear differential equation*, by Professor George Seifert, University of Nebraska, introduced by the Secretary.

The equation $\theta+\alpha\dot{\theta}+\sin\theta=\beta$, where α and β are positive constants, is the equation describing the motion of a rigid pendulum driven by a constant torque. This equation also arises in descriptions of synchronous motors.

It was shown by F. Tricomi (1933) that if $0<\beta<1$ and

$$\alpha \leqq \frac{\beta}{\pi}\left(\frac{\beta}{\pi}+\cos\theta_1\right)^{-1/2},$$

where $\theta=\pi-\arcsin\beta$, then there exists a solution $\bar{\theta}(t)$ such that if $d\bar{\theta}/dt=\bar{y}$, then $\bar{y}(\bar{\theta})=\bar{y}(\bar{\theta}+2\pi)$ for all t, while if $\beta\geqq 1$, such a solution exists for all $\alpha>0$. By a somewhat different approach, it is possible to obtain a somewhat larger bound on α; more precisely, if

$$\alpha \leqq \frac{\beta}{\theta_1}\left(\frac{\beta}{\theta_1}+\cos\theta_1\right)^{-1/2}$$

then such a solution $\bar{\theta}(t)$ will exist.

(3) *Phragmén-Lindelöf theorems for generalized subharmonic functions*, by Professor L. K. Jackson, University of Nebraska, introduced by the Secretary.

The notion of subharmonic function is generalized by replacing the dominating family of harmonic functions by a more general family of functions. It is shown that an analogue of the principle of the maximum modulus holds for these generalized subfunctions. This property is used to prove theorems of the Phragmén-Lindelöf type. Solutions of certain types of elliptic differential equations are shown to be examples of such functions.

(4) *Ideals and the prime factorization theorem*, by Miss F. Marion Clarke, University of Nebraska.

The author demonstrated the failure of the fundamental theorem of arithmetic in certain algebraic fields, showed the validity of an analogous theorem with respect to ideals instead of integers, sketched the generalization of this theorem for transcendental extensions and indicated the rule of irreducibility and maximality in characterizing the property of being prime.

(5) *A development of the identities for* $\sin(\alpha+\beta)$ *and* $\cos(\alpha+\beta)$, by Professor A. K. Bettinger, Creighton University.

The new feature of this development is a variation in the geometric construction which greatly simplifies the proof. Application was also made to the rotation formulas in plane analytic geometry.

(6) *Some recent developments of the analysis of the logical foundations and methods of algebra*, by Professor H. B. Ribeiro, University of Nebraska.

Included was a discussion of metamathematical proofs in algebra followed by an introduction to Tarski's mathematical theory of arithmetical classes of algebraic systems and its applications.

The afternoon session was devoted to a discussion of the problems of the teaching of secondary school mathematics. A panel composed of persons representing the university, the small college, the high school, and the administrators, led the discussion. Members of the Association on the panel were Professor W. G. Leavitt, University of Nebraska, and Professor C. B. Gass, Nebraska Wesleyan University.

EDWIN HALFAR, *Secretary*

THE MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The thirty-fifth annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Western State College of Colorado, Gunnison, Colorado, on May 23 and 24, 1952. Professor C. H. Cook, Western State College of Colorado, presided.

Forty-nine persons registered including the following thirty-seven members of the Association:

C. F. Barr, J. R. Britton, R. G. Buschman, F. M. Carpenter, H. W. Charlesworth, C. H. Cook, G. S. Cook, Mary C. Doremus, Albert Edrei, F. N. Fisch, H. T. Guard, R. R. Gutzman, J. R. Hanna, I. L. Hebel, Anna S. Henriques, C. A. Hutchinson, B. W. Jones, M. W. Jones, A. J. Kempner, Claribel Kendall, F. A. Kros, J. S. Leech, M. L. Madison, Greta Neubauer, D. O. Patterson, Lily B. Powell, Nathan Schwid, S. R. Smith, W. N. Smith, L. C. Snively, K. H. Stahl, P. O. Steen, Wilmont Toalson, E. P. Tovani, E. L. Vanderburgh, V. J. Varineau, W. W. Varner.

At the business meeting the following officers were elected: Chairman, Professor B. W. Jones, University of Colorado; Vice-Chairman, Professor C. F. Barr, University of Wyoming; Secretary-Treasurer, Professor J. R. Britton, University of Colorado. It was decided to hold the next annual meeting at the University of Colorado, the date to be announced later. Upon a motion by H. T. Guard it was voted to donate the sum of $25 to the Wald Memorial Fund.

The following papers were presented:

1. *The spirit of discovery in mathematics*, by Professor B. W. Jones, University of Colorado.

Attention was called to some of the similarities between our international situation and our educational difficulties. In both we are apt to blame sinister, all-powerful, and all-wise forces outside our control. Actually what danger there is at present is chiefly from within. It is up to us to adopt an aggressive rather than a defensive attitude in both cases and with due modesty to realize that we have already made some impressive improvement over the past. Not the least of our opportunities in the field of education is to cultivate in our students from kindergarten upward a spirit of discovery. We should not do all the exploration for them but should rather encourage them in their natural curiosity so that our pioneering heritage may be perpetuated in the realm of the mind.

2. *Mathematics applied to the calculation of rocket trajectories*, by Mr. F. A. Kros, University of Colorado.

The following method can be used to determine the position of a rocket at any instant using data from three camera stations. Let the lines of sight from the camera to the rocket be written in the form $x=x_0+\lambda r_1$, $y=y_0+\mu r_1$, $z=z_0+\nu r_1$, where $(x_0, y_0, z_0$ are the coordinates of the camera, λ, μ, ν are the direction cosines of the line of sight and r_1 is the distance from the camera to any point (x, y, z) on this line. Consider a function $D=f(r_1, r_2, r_3)$ such that D is the sum of the squares of the distances between points on each of the three lines. Minimizing this function gives us three linear equations in the r's, the solution of which determines three points on the lines of sight. The average of these three points gives the centroid of the triangle determined by them. This centroid is taken as the best approximation of the rocket's position.

3. *Forms for the solution of spherical triangles*, by Mr. E. L. Vanderburgh, Pueblo College.

A booklet was distributed which the author uses in teaching spherical trigonometry in seven to ten lessons. Three forms, each with the formulas needed and with solution outlines, are used to solve the six cases of oblique spherical triangles. The number of formulas is kept to a minimum and each one is derived in a form students of trigonometry can easily follow. By using the forms, it was pointed out, students can spend their time on the numerical work and easily check results on the form in the place provided. Also, since the complete plan is on the form, one can solve similar problems many years later by use of the forms and a very little review. Six students, over a period of four years, helped prepare the booklet as a part of their trigonometry courses.

4. *A minimum problem in Banach spaces*, by Professor J. S. Leech, Colorado College.

In Monatshefte für Mathematik, XXXII Band, 1922, pp. 204–218, Georg Pick considers the following problem: Let $f(z)$ be a function analytic in the unit circle, $|z|<1$. Among all such functions satisfying the conditions (1) $f(z_k)=\alpha_k$, $k=1, 2, \cdots, n$, where $z_1, z_2, \cdots, z_n$ and $\alpha_1, \alpha_2, \cdots, \alpha_n$ are given complex numbers such that $|z_k|<1$, what function makes the integral $I=\int_0^{2\pi}|f(e^{i\theta})|^2 d\theta$ a minimum? Pick shows that this problem always has a unique solution.

A generalization of this problem to arbitrary Banach spaces may be stated as follows: If $f_1, f_2, \cdots, f_n$ are n given functionals defined on a Banach space X, what element $x \epsilon X$ satisfying (2) $f_k(x) = \alpha_k$, $k = 1, 2, \cdots, n$, has the least norm? We will refer to this as the minimum problem.

The principal result obtained is: In a Banach space X, in order that every minimum problem have a unique solution, it is necessary and sufficient that X be reflexive and strictly convex. In particular, L_2 is both reflexive and strictly convex; hence every minimum problem in L_2 has a unique solution. In this case, it is shown how to construct this minimum solution.

5. *The roots of a certain exponential equation*, by Professor W. N. Smith, University of Wyoming.

The equation considered is of the form

$$A_1(\lambda, \nu) - A_2(\lambda, \nu)e^{\lambda R_1} + A_3(\lambda, \nu)e^{\lambda(R_1+R_2)} - A_4(\lambda, \nu)e^{\lambda R_2} = 0,$$

where R_1 and R_2 are complex constants, and the $A_i(\lambda, \nu)$ are asymptotically representable by power series in $1/\lambda$ with coefficients which are at most quadratic in ν. It is desired to solve for λ as a function of the complex variable ν, with $|\lambda|$ taken to be large and $|\nu|$ small. The equation is studied by means of approximation equations. Two of these are explicitly solvable for λ and are independent of ν. The other two may be regarded, after certain conditions have been imposed upon ν, as defining λ as an infinity of distinct single-valued functions of ν. As ν approaches zero along a suitable path these functions are found to lie in subregions of the λ-plane. Finally it is shown that the roots of the approximation equations actually represent the roots of the given equation asymptotically.

6. *A singular integral equation*, by Professor Albert Edrei, University of Colorado.

J. R. BRITTON, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Hobart and William Smith Colleges, Geneva, New York, on May 10, 1952. The Chairman of the Section, Professor C. W. Munshower of Colgate University, presided at the morning session; the Vice-Chairman, Professor J. F. Randolph of the University of Rochester, presided at the afternoon session. At the conclusion of the afternoon session a tea was served to members and guests.

Ninety-seven persons attended the meeting, including the following seventy-two members of the Association:

H. T. R. Aude, H. W. Baeumler, Frances E. Baker, M. R. Bates, W. R. Baum, R. A. Beaver, R. L. Beinert, Dorothy L. Bernstein, W. W. Bessell, H. F. Bligh, F. J. H. Burkett, K. A. Bush, E. A. Butler, W. B. Carver, Nancy Cole, Geraldine A. Coon, A. E. Danese, W. A. Dolid, E. J. Downie, Walter H. Durfee, William H. Durfee, G. V. Emerson, H. W. Eves, Jean B. Feidner, A. D. Fleshler, C. W. Foard, A. H. Fox, J. E. Freund, H. M. Gehman, B. H. Gere, J. C. Gibson, Lillian Gough, N. G. Gunderson, H. K. Holt, Anna M. Howe, J. R. F. Kent, D. E. Kibbey, F. W. Lane, R. D. Larsson, Caroline A. Lester, R. C. Luippold, R. W. MacDowell, Dis Maly, E. W. Marchand, Harriet F. Montague, Mabel D. Montgomery, L. J. Montzingo, D. S. Morse, Abigail M. Mosey, C. W. Munshower, W. V. Nevins, III, F. D. Parker, W. B. Pitt, Theresa L. Podmele, L. R. Polan, J. F. Randolph, C. E. Rhodes, M. F. Rosskopf, P. T. Schaefer, Edith R. Schneckenburger, W. A. Small, S. T. Smith, Ruth W. Stokes, Mary C. Suffa, Nura D. Turner, G. W. Walker, R. J. Walker, F. C. Warner, A. E. Whitford, Mary E. Williams, A. G. Wootton, Frances M. Wright.

At the business meeting the following officers were elected: Chairman, Professor J. F. Randolph, University of Rochester; Vice-Chairman, Professor Harriet F. Montague, University of Buffalo; Secretary, Professor N. G. Gunderson, University of Rochester. Mr. L. F. Scholl reported for the Committee on Mathematics Contests for High School Students that a contest would be held in the high schools of Buffalo on May 24. Professor M. F. Rosskopf reported for the Committee on the Relationship of Secondary School and College Mathematics that the work of the Committee was completed. Motions were passed that this committee be dissolved, and that the Executive Committee be authorized to cooperate with other groups in the state to form a Joint Committee for the Advancement of Mathematics in New York State. The 1953 meeting will be held at the United States Military Academy at West Point, New York.

The following papers were presented:

1. *Parallel plate optics*, by Professor F. D. Parker, St. Lawrence University.

The problem of designing a rapid scanning antenna which conducts the energy between two parallel curved plates and directs the energy in a single predetermined direction is shown to be equivalent to requiring the geodesics on the mean surface between the parallel plates to have certain properties. Any surface of revolution has some of these properties, but it is shown that under rather weak hypotheses, that no such surface has all the required properties. If the geodesics are altered by the introduction of a dielectric into the surface, so that the "time geodesics" are to be considered rather than the "length geodesics," then still no complete solution is possible. The analysis is carried through from the standpoint of the optic theory rather than the corpuscular theory.

2. *The problem section of the Monthly*, by Professor H. W. Eves, Champlain College, State University of New York.

The speaker has often been asked to indicate certain broad types of problems that have been proposed in the MONTHLY Problem Department. In this paper a crude classification is offered, and the categories of the classification are illustrated by selected problems from the MONTHLY. Although there certainly is overlapping in the classification, it does, to some extent, suggest the origin of many problems. Following is the classification: (1) Challenge problems, (2) Problems presenting new results perhaps not important enough for a paper or a note, (3) Analogies and generalizations suggested by previous problems, (4) Problems originating as by-products of more extended investigations, (5) Problems involving pedagogical matters, (6) Problems specifically proposed for possible classroom use, (7) Entertaining problems, such as puzzles, dissections, cryptarithms, and occasional recurring teasers, (8) Paradoxes and fallacies, (9) Problems having historical connections, (10) Problems, including occasional textbook problems, proposed in an effort to obtain simpler solutions, (11) Problems whose solutions are needed in order to complete an investigation or piece of research.

3. *High speed computing and its effect on teaching*, by Dr. William H. Durfee, National Bureau of Standards.

The development of high speed computing machines has revolutionized computing. Many problems, heretofore incapable of solution, can now be solved with their aid. This has stimulated research in applied mathematics and at the same time in numerical analysis, that branch of mathematics which is concerned with the science and techniques of computing. Mathematicians at all levels of training are in demand in considerable numbers to prepare problems to be solved on computers and this situation promises to continue for some time. The speaker gave a brief description

of electronic digital computers, their operation and the difficulties one encounters in the preparation and solution of problems on them. He then went on to outline a program showing how through the regular calculus courses colleges could give their students some work in numerical analysis which would be desirable for all mathematics majors to have and at the same time provide training for those who plan to make a career of computing.

4. *A freshman looks at mathematics*, by Mr. W. B. Pitt, University of Rochester.

The speaker argued that particularly for his fellow students who are not mathematics majors a more intuitive approach to freshman mathematics would seem desirable.

5. *The mean value theorem and inequalities*, by Professor K. A. Bush, Champlain College, State University of New York.

A number of inequalities are direct consequences of the following lemma which in turn follows immediately from the mean value theorem: If $a \leqq b \leqq c$, if $f'(x)$ is monotone increasing $a < x < c$, then $(b-a)f(c)+(c-b)f(a) \geqq (c-a)f(b)$. The inequality is reversed if $f'(x)$ is monotone decreasing.

The lemma was used to establish such familiar inequalities as

$$|x+a|^{1+s} \leqq 2^s(|x|^{1+s}+|a|^{1+s})$$

and the Liapounoff moment inequality. Generally substantial simplification of the standard proofs is secured by finding the proper convex function and using the lemma.

6. *Functions of subdivisions and their use in advanced undergraduate mathematics*, by Professor Dorothy L. Bernstein, University of Rochester.

The concepts of a subdivision function and of the limit of a subdivision function are presented in a form which can be readily understood by mathematics majors in a course in advanced calculus or introduction to analysis. After the basic limit theorems have been developed, one can apply them to present the properties of the Riemann integral in a simple and coherent manner. A second application of subdivision functions is then made to functions of bounded variation, a topic which is often neglected in such courses; the development can be made without using positive and negative variation, but only total variation, and applies equally well to complex-valued functions. On the basis of these two applications the important topic of Riemann-Stieltjes integration can be developed for the simple cases which are used in most applications. The integral of a function of a complex variable along a path, as it is given in most first courses in functions of a complex variable, is just such a Riemann-Stieltjes integral; one need not, therefore, present a new integration theory to students whose previous work had included the above topics. Furthermore, some of the fundamental inequalities concerning such integrals are made clearer by using the variation function of the path.

7. *The Leibniz characteristic triangle*, by Professor Frances E. Baker, Vassar College.

Two concepts that led to Leibniz' earliest mathematical successes became main streams of thought until they merged in his infinitesimal calculus. These were the operations of summing and differencing series, and the method of moments for determining areas and surfaces. For the second concept he maintained his initial indebtedness to Pascal, while denying that he owed his methods to Barrow. Evidence from Leibniz' manuscripts indicates his steady progress from a treatment of tangents following Descartes and Sluse, until the differential triangle of Pascal had become the characteristic triangle employed by Leibniz. Likewise his calculus of differences developed into a true differential calculus, and the problem of quadratures, at first solved by moments, became generalized from finite summation to a calculus of integration, recognized as the

inverse of differentiation. There is some evidence that clues for this final metamorphosis were obtained from Barrow's *Geometrical Lectures*, but no indication of indebtedness to Newton. The influence of Cavalieri, Sluse and Hudde was considerable, as was acknowledged by Leibniz. The latter, however, used his characteristic triangle primarily as a means to attain his purpose of generalization and abstraction freed from geometry. Leibniz was fully cognizant of the advantages gained by his symbolism.

N. G. GUNDERSON, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The twentieth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the University of Wisconsin in Milwaukee, Wisconsin, on May 10, 1952. Professor Morris Marden presided at the morning session; Professor R. H. Bing presided at the afternoon session.

About one hundred ten persons were present, including the following fifty-two members of the Association:

K. J. Arnold, R. H. Bardell, Leon Battig, L. J. Berner, W. W. Bigelow, R. H. Bing, C. C. Braunschweiger, Leonard Bristow, R. H. Bruck, R. C. Buck, G. L. Bullis, K. L. Clark, W. R. Cowell, W. F. Eberlein, W. S. Ericksen, H. P. Evans, Harold Glander, R. O. Grosnick, E. G. Harrell, J. S. Hokanson, Fannie Hopkins, L. B. Houk, Rev. M. L. Jautz, Frances W. Jones, Dora E. Kearney, J. F. Kenney, S. C. Kleene, R. E. Langer, W. E. Lawrence, A. P. Loomer, R. A. Lufburrow, C. C. MacDuffee, Saunders MacLane, Morris Marden, A. E. May, Genevieve T. Meyer, Marianne S. Otto, G. A. Parkinson, H. P. Pettit, J. G. Renno, Jr., D. A. Rux, R. C. Seber, Sister M. Elizabeth, Sister Mary Felice, Sister Mary Petronia, Elizabeth S. Sokolnikoff, Abraham Spitzbart, J. V. Talacko, R. D. Wagner, L. F. Wahlstrom, N. M. Watermolen, Louise A. Wolf.

At the business meeting the following officers were elected for the coming year: Chairman, Professor R. H. Bing, University of Wisconsin; Secretary, Professor Louise A. Wolf, University of Wisconsin in Milwaukee; Program Committee Chairman, Mr. A. C. Moeller, Marquette University. The 1953 May Meeting is to be held at Mount Mary College, Milwaukee.

The program consisted of the following papers:

1. *Mathematical models for statistical quality control and industrial statistics*, by Professor J. V. Talacko, Marquette University.

The mathematical base of the statistical quality control charts, sampling inspection and industrial statistics are several distribution functions. Four probability models are most important: the normal distribution, the binomial, the Poisson and the hypergeometric distribution. In presenting statistical quality control principles, ideas, the operation of a chance-cause system, the meaning of plus or minus three sigmas, the central limit theorem *etc.*, to non-mathematicians, the special models, devices may be used: bags of chips with model populations and a box of 2,000, white and colored, wooden beads, with paddles to scoop—100, 50 or 20 beads.

2. *Definitions of trigonometric functions*, by Professor W. F. Eberlein, University of Wisconsin.

3. *Homotopy*, by Professor Saunders MacLane, University of Chicago.

Two continuous mappings f_0 and f_1 of a space K into a space X are homotopic if it is possible

to continuously deform the first map f_0 into the second in the space X. This concept, in its precise formulation, is basic in algebraic topology and its applications. For example, to any map f of a circle into a circle we may assign an integer n as degree, in such fashion that f is of degree n if and only if it is homotopic to the map $\phi \to n\phi$ which wraps the first circle n times around the second. This result is involved in the usual topological proof of the fundamental theorem of algebra. After other examples, the talk turned to the definition of the fundamental group and of the higher homotopy groups of a space X. These groups, whose elements are essentially the classes of homotopic maps of a sphere into the space X, are difficult to compute explicitly. An essential prerequisite to the study of other homotopy problems is the determination of the homotopy groups of an n-sphere. On this question there was cited the recent substantial progress due to Serre, Whitehead, Blakers, Massey, Hilton, Pontrjagin and others.

4. *The student talks back*, by Professor Elizabeth S. Sokolnikoff, University of Wisconsin.

The speaker mentioned some of the common, and frequently contradictory, criticisms made by students in complaining about instruction in elementary college courses in mathematics. Then it was pointed out that the problem of the weak and poorly prepared students has received much attention, but comparatively little has been done for the exceptionally good student who may be wasting his time taking the regular sequence. There should be more provision for these students and an adequate amount of publicity, so that students would be informed, in advance, of these opportunities.

5. *What we are doing in geometry*, by Professor L. F. Wahlstrom, Wisconsin State College, Eau Claire.

During the last thirty years or more, various authorities have agreed that one of the objectives of teaching demonstrative geometry in the high school is that of developing the ability to think clearly in non-mathematical as well as mathematical situations. Even at the present time, however, many high school teachers admit that they are not prepared to teach geometry for critical thinking. The fault lies with their training, at least in part. Before high school teachers can teach the course in geometry as it should be taught, the colleges and universities must revise their own courses in geometry in order that the prospective high school teacher may have experiences in critical thinking both in mathematical and non-mathematical situations. If there is any merit to the statement that teachers tend to teach as they have been taught, perhaps some adaptation on the college level of the procedures used by Dr. H. P. Fawcett at Ohio State University as described in the Thirteenth Yearbook of the National Council of Teachers of Mathematics may be the answer.

6. *Inservice training program in the Milwaukee public schools*, by Mr. H. W. Aker, Milwaukee Public Schools, introduced by the Secretary.

Inservice training in Milwaukeee is tied to an incentive type salary schedule. Teachers, administrators, and supervisors may move to higher salary divisions on the basis of advanced academic study or its "equivalent." Equivalencies have been set up in the fields of travel, work-experience, professional committee service, study in non-accredited schools and with private instructors, *etc.* With this flexible program, additional opportunities for inservice training are being provided in cooperation with the University of Wisconsin Extension Division, Marquette University, and Wisconsin State College at Milwaukee. Classes are set up each semester on an 8-week, 1-credit basis in four areas: (1) teaching techniques, (2) community study, (3) general or cultural interest, (4) follow-up of professional committee work. Acceptance of the program, which is on an entirely volunteer basis is indicated by enrollments which, apart from the dozens of persons who enroll in regular college courses for degree work, have increased from 300 to 400 to 450 in the past three semesters.

7. *What membership in the Wisconsin Mathematics Council can do for you*, by Sister Mary Felice, Mount Mary College.

The Wisconsin Mathematics Council is the mathematics section of the Wisconsin Educational Association. Its object is the improvement of the teaching of mathematics on all levels. It does this by means of its journal, *The Wisconsin Mathematics Teacher*, published three times a year. Its research committee is responsible for the gathering of material for this journal. The journal offers the members suggestions for the overcoming of various difficulties in the teaching of mathematics, reviews of books that are recommended for the school library or for the teacher's professional library. It carries reports of studies made by teachers of the state, suggests problems for which solutions are needed, and invites the members of the council to share their successful devices with their fellow teachers.

The chief advantage of membership in the council comes from the mutual help derived by teachers at all the levels, elementary, high school, and college, from the study and discussion of their problems together. The successful solution of a teaching problem on one level suggests a solution to some corresponding problem on another level.

LOUISE A. WOLF, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The sixth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at the University of Oregon, Eugene, Oregon, on June 20, 1952, in conjunction with the four hundred eighty-second meeting of the American Mathematical Society. The meetings were a part of the Seventy-Fifth Anniversary celebration of the University of Oregon. In the absence of Professor R. D. James, Chairman of the Section, Professor A. S. Merrill, Vice-Chairman, shared with Professor C. M. Cramlet and Professor M. S. Knebelman the duties of presiding at the afternoon session.

Eighty persons were in attendance, including the following sixty members of the Association:

C. B. Allendoerfer, T. M. Apostol, R. W. Ball, J. P. Ballantine, R. A. Beaumont, R. F. Bell, A. I. Benson, J. S. Biggerstaff, Z. W. Birnbaum, J. L. Brenner, D. G. Chapman, Harold Chatland, H. E. Chrestenson, Paul Civin, K. L. Cooke, D. F. Coulter, Jr., C. M. Cramlet, E. L. Crow, D. B. Dekker, Douglas Derry, K. S. Ghent, E. G. Goman, F. L. Griffin, S. G. Hacker, Mary E. Haller, A. J. Hoffman, V. E. Hoggatt, Jr., H. H. Irwin, Mark Kac, J. M. Kingston, M. S. Knebelman, D. H. Lehmer, R. B. Leipnik, A. E. Livingston, J. E. Maxfield, H. C. Mayer, Jr., A. S. Merrill, W. E. Milne, A. F. Moursund, Ivan Niven, Andrewa R. Noble, T. G. Ostrom, T. S. Peterson, A. R. Poole, H. F. Price, J. F. Price, P. V. Reichelderfer, A E. Richmond, Louise J. Rosenbaum, R. A. Rosenbaum, W. G. Scobert, W. L. Shepherd, Andrew Sobczyk, R. D. Stalley, Gabor Szegö, G. E. Uhrich, J. R. Vatnsdal, J. G. Wendel, F. E. Wood, F. H. Young.

A business meeting was held in the evening at which the following officers were elected: Chairman, Professor R. A. Rosenbaum, Reed College; Vice-Chairman, Professor Harold Chatland, Montana State University; Secretary-Treasurer, Professor J. M. Kingston, University of Washington.

The Section accepted the invitation extended by Professor A. S. Merrill to hold its next meeting at Montana State University, Missoula, Montana, in June, 1953, and the hope was expressed that it might be a joint meeting with the Society in continuation of this successful custom. A Program Committee for the 1953 meeting was appointed consisting of Professor Harold Chatland,

Chairman, and Professors Douglas Derry, D. R. Ryan and G. E. Uhrich.

The afternoon session consisted of the following invited hour address, three fifteen minute papers and a symposium by a panel of four speakers:

1. *Continuous vector fields*, by Professor C. B. Allendoerfer, University of Washington. (By invitation.)

A continuous vector field in Euclidean n-space is a set of n continuous functions $f^i(x)$. The field is said to be singular at a point at which $f^i(x)=0$ for all values of i. With each isolated singularity there can be attached an integer whose value is obtained by means of an appropriate integral on the surface of an $n-1$ sphere inclosing the point. This paper discussed the applications of this theory to various branches of analysis, topology and differential geometry. A set of $r(\leqq n)$ vector fields has a singularity at a point at which these fields become linearly dependent. These singularities will not in general be isolated points, but indices for them can be defined and can be calculated by suitable integrals.

2. *The NOTS REAC*, by Dr. F. H. Young, Naval Ordnance Test Station, China Lake, California.

A description is given of the Reeves Electronic Analog Computer installation at the Naval Ordnance Test Station at China Lake, California. The operation of the computer is explained and a comparison is made between digital and analog computation. An indication is given of the means employed to introduce empirical functions into the computation. The "bootstrap" technique employed in the solution of ordinary linear differential equations is illustrated.

3. *A generalization of analyticity*, by Dr. M. G. Arsove, University of Washington.

Let Ω be a fixed plane region and $\mathcal{A}$ the class of complex-valued functions ϕ on Ω such that (i) $\Re(\phi)$ and $\Im(\phi)$ are C'' and (ii) $h(\xi, \eta)$ harmonic on $\phi[\Omega]$ and $h_{\xi\eta}\equiv 0$ imply $h[\phi(z)]$ harmonic on Ω. Since analytic mappings preserve harmonicity, $\mathcal{A}$ includes all functions analytic on Ω. However, $\mathcal{A}$ generalizes analyticity also from the point of view of the Cauchy-Riemann equations: a non-constant function $\phi=\xi+i\eta$ is in $\mathcal{A}$ if and only if ξ and η are harmonic and grad η is related to grad ξ by a constant rotation (whose angle we denote by rot ϕ). It follows that each ϕ can be represented linearly in terms of the real and imaginary parts of a corresponding analytic function, and that functions of $\mathcal{A}$ satisfy the maximum modulus principle, Liouville's theorem, Rouché's theorem, and closure under uniform convergence. When locally one-one, ϕ is quasi-conformal with constant dilatation quotient:

$$[1+|\cos(\text{rot }\phi)|\,]^{1/2}/[1-|\cos(\text{rot }\phi)|\,]^{1/2}.$$

However, $\mathcal{A}$ is not closed under addition or multiplication.

4. *Note concerning an improper integral*, by Professor R. F. Bell, Eastern Washington College of Education.

In attempting to evaluate the contour integral $\oint_C(\sqrt{4-z}+\sqrt{1+z})^{-1/2}dz$, where the contour of integration C runs along the lower real axis from $z=0$ to $z=4+0$ and back along the upper side with indentations around $z=3/2$ and $z=4$, one encounters the improper real integral $\int_0^4(4-x)^{1/2}(2x-3)^{-1}dx$. The above problem may be found in a recent text on applied mathematics Schelkunoff's *Applied Mathematics for Engineers and Scientists* (page 321, problem 8), yet the answer given there is incorrect. The difficulty arises from the infinite discontinuity of the integrand at $x=3/2$. If one formally substitutes the limits into the indefinite integral the text answer may be obtained. However application of the theory of limits will give the Cauchy principal value of the improper integral to be $-2-\frac{1}{4}\sqrt{10}\,\ln\frac{1}{2}(13-4\sqrt{10})$. This is an excellent problem for a class in integral calculus.

5. *Symposium on calculus teaching*, moderated by Professor M. S. Knebelman, State College of Washington.

Panel: (i) Professor T. S. Peterson, University of Oregon.

Historically, the arrangement of topics in a course on calculus has followed, and still follows, the classical presentation of differentiation followed by integration. The different varieties of calculus courses which exist at the present time are for the most part the result of including in the course varying amounts of non-calculus material. In this country two current trends in the teaching of calculus can be noted: (1) the unification of analytic geometry with calculus into a single course and (2) the repetitive teaching of calculus, as it applies firstly to polynomials and then to general functions. The first of these trends results from an apparent curricular necessity and the second has as its motivation certain pedagogical values.

(ii) Professor R. A. Rosenbaum, Reed College.

The question as to how rigor may be introduced into calculus courses is a vexing one because of student immaturity and lack of motivation and because of the heterogeneity of the average class. It was suggested that rigor may be insinuated into the course by means of problems, beginning with extremely easy ones, the solutions of which are expected of all class members, and proceeding through more difficult ones to sophisticated problems which are proposed on an "optional" basis. How such a sequence might be used to introduce the Mean Value Theorem was illustrated.

(iii) Professor R. B. Leipnik, University of Washington.

Time limitations seem to force calculus instructors to rush through sketchy proofs of sleazy theorems. One way out is to omit the proofs. This may be acceptable if clear-cut basic concepts are developed, theorems are accurately stated and counter-examples used to show what may be hoped for. If this is done, an economy of time and thought is also possible in later courses since introduction of a whole new set of concepts can be avoided. For example, the binary relation (set of ordered pairs) can be introduced before function, to put inverse function, one-one function, *etc.*, in better perspective. The different kinds of limits can all be handled by the directed limit (E. J. McShane, *Partial Orderings and Moore-Smith Limits*, this MONTHLY, vol. 59, No. 1), which is no more complicated than the limit of a sequence. The notion of a linear functional can be used to clarify the maltreated differential and total differential. The Riemann-Darboux integral gives an intuitively reasonable and mathematically acceptable treatment of work, pressure and arc-length without the intervention of differentials. The notion of convex set can be used to advantage in multiple integration.

(iv) Professor Douglas Derry, University of British Columbia.

This paper discussed material used in outside subjects especially in relation to the requirements of the applied science students. Much of the outside material is chiefly useful to create and sustain interest but can, if properly handled, clarify the mathematical principles involved. However from the point of view of the outside subjects themselves this material can only make a fragmentary contribution. It should therefore not take up too much of the available time. The mathematical needs of an applied science student remain primarily a precise and clear understanding of the fundamental principles of the calculus. Outside material should be used only as an aid to attain this goal.

In his concluding remarks the moderator mentioned several pedagogical problems which are ever present in the teaching of calculus and there was considerable discussion from the floor. No unanimous conclusion was reached.

J. M. KINGSTON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS

KENTUCKY, University of Louisville, Spring, 1953.

LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Howard University, Washington, D. C., December 6, 1952.

METROPOLITAN NEW YORK, Spring, 1953.

MICHIGAN, Western Michigan College of Education, Kalamazoo, April 11, 1953.

MINNESOTA, College of St. Scholastica, Duluth, October 11, 1952.

MISSOURI, William Jewell College, Liberty, Spring, 1953.

NEBRASKA

NORTHERN CALIFORNIA

OHIO

OKLAHOMA, Oklahoma City, October 31, 1952.

PACIFIC NORTHWEST, Montana State University, Missoula, June, 1953.

PHILADELPHIA, University of Delaware, Newark, November 29, 1952.

ROCKY MOUNTAIN, University of Colorado, Boulder, April, 1953.

SOUTHEASTERN, Alabama Polytechnic Institute, Auburn, March 13–14, 1953.

SOUTHERN CALIFORNIA, Los Angeles City College, March 14, 1953.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanic Arts, State College, January 2–3, 1953.

TEXAS

UPPER NEW YORK STATE, United States Military Acadeny, West Point, Spring, 1953.

WISCONSIN, Mount Mary College, Milwaukee, May, 1953.

THE CARUS MATHEMATICAL MONOGRAPHS

These Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields. One copy of each Monograph may be purchased by members of the Association for $1.75 each. Additional copies and copies for non-members of Monographs 1–8 are priced at $3.00 each, and must be purchased from the Open Court Publishing Co., LaSalle, Illinois. In the case of Monographs 9 and 10, additional copies and copies for non-members must be purchased at $3.00 from John Wiley and Sons, 440 Fourth Ave., New York 16, N. Y. These numbers have been issued to date:

No. 1. *Calculus of Variations* by G. A. Bliss, xii+189 pages.

No. 2. *Analytic Functions of a Complex Variable* by D. R. Curtiss, ix+173 pages.

No. 3. *Mathematical Statistics* by H. L. Rietz, ix+181 pages.

No. 4. *Projective Geometry* by J. W. Young, ix +185 pages.

No. 5. *History of Mathematics in America before 1900* by D. E. Smith and Jekuthiel Ginsburg, viii+210 pages.

No. 6. *Fourier Series and Orthogonal Polynomials* by Dunham Jackson, xiv+234 pages.

No. 7. *Vectors and Matrices* by C. C. MacDuffee, xi+192 pages.

No. 8. *Rings and Ideals* by N. H. McCoy, xii +216 pages.

No. 9. *The Theory of Algebraic Numbers* by Harry Pollard, xii+143 pages.

No. 10. *The Arithmetic Theory of Quadratic Forms* by B. W. Jones, x+212 pages.

for your college courses

texts by
PAUL R. RIDER

ANALYTIC GEOMETRY

Understandable language, a generous supply of carefully chosen exercises, and sound mathematical treatment combine to make this text a valuable contribution to the field. Included are discussions which are helpful to every type of student need. *1947—$3.50*

COLLEGE ALGEBRA

This text contains an exceptionally thorough review of high school algebra with numerous problem exercises, carefully selected and well graded. The problems in the Alternate Edition are all new. *1940 edition, $3.40; Alternate Edition, $3.40*

FIRST-YEAR MATHEMATICS FOR COLLEGES

The author has covered in a single volume, the basic topics for the first course in mathematics. The topics are logically arranged and grouped about the function concept, yet the individual chapters are sufficiently independent for the teacher to adapt the book to his own needs. *1949—$5.00*

INTERMEDIATE ALGEBRA FOR COLLEGES

Presented on the college level of mental maturity, this text is intended for use by students having only one year of high school algebra. A clear explanation of the fundamentals is achieved through the extensive illustrative examples all of which are carefully graded and new. *1940—$3.00*

PLANE AND SPHERICAL TRIGONOMETRY

Here is a complete course in the subject with the first problems made simple from a numerical standpoint in order to enable the student to grasp principles and to learn methods without becoming lost in a maze of computations. *$3.20.* with tables: *$3.75*

—The Macmillan Company, 60 Fifth Ave., N.Y.

Two Outstanding New Texts

COMMERCIAL ALGEBRA

Robert M. Parker, Texas Technological College

"It is evident throughout the manuscript that the author has done a competent and meticulous job. The problems appear to be adequate and well-stated." Robert K. Stranathan, The City College of New York

THIS practical text shows the student how algebra is a useful tool in a variety of business situations. At the same time, it provides the student with the prerequisite background for entering courses in mathematics of finance and business statistics.

MATHEMATICS OF FINANCE

Albert E. May, Racine Extension Center, University of Wisconsin

"The simplified and concise treatment of the material is an advantage to the student. Of special importance is the use of frequent and varied illustrations." Jacob O. Kamm, Baldwin-Wallace College

BECAUSE the treatment is simple and direct, the proofs satisfying, and the applications rewarding, this book appeals to teachers and students alike. It builds the fundamentals of investment theory on just eight simple important formulas.

Three Widely Used Texts

by Frank M. Morgan

COLLEGE ALGEBRA

"I consider it the best text I have used in my twenty-five years of teaching. The exposition is clear and concise, and the exercises are uniformly excellent." G. W. Spenceley, Miami University

PLANE AND SPHERICAL TRIGONOMETRY

". . . the many examples which are included to teach the student the 'know how' of trigonometry are given in considerable detail. Checking is included as a regular part of the example and not as an extra." From a review in *School Science and Mathematics*

DIFFERENTIAL AND INTEGRAL CALCULUS

"A very teachable text for a first course in calculus. The pupil has been kept in mind at all times in the presentation of the material." As reviewed in the *Peabody Journal of Education*



American Book Company

55 Fifth Avenue, New York 3, N. Y.
300 Pike Street, Cincinnati 2, Ohio
351 East Ohio St., Chicago 11, Illinois

JUST PUBLISHED!

DIFFERENTIAL EQUATIONS

Comprehensiveness and emphasis on applications make this text suitable for engineering students as well as for students of pure mathematics. This new presentation is more inclusive than the usual text in the field. *309 pages. $3.25*

By ALFRED L. NELSON, KARL W. FOLLEY, MAX CORAL, *Wayne University*

WILLIAM L. HART'S

COLLEGE TRIGONOMETRY

A substantial treatment of plane and spherical trigonometry, of moderate length with distinctly collegiate viewpoint. Emphasizes analytic trigonometry, oriented for application in later mathematics. Extensive tables, including haversines. Text pages: *Plane Trigonometry, 151; Complex Numbers and Appendix, 21; Spherical Trigonometry, 35; Tables, 130.* *$3.50*

D. C. HEATH
AND COMPANY

SALES OFFICES: NEW YORK CHICAGO SAN FRANCISCO ATLANTA DALLAS

HOME OFFICE: BOSTON

Two Leaders

Smith's The Mathematics of Finance

Offering to the student of business administration an introductory course in the mathematics of finance, this text emphasizes basic principles rather than special formulas. Teaching aids include 27 diagrams, an abundance and variety of practical exercises, necessary tables, and an index. By Franklin C. Smith, College of St. Thomas. 356 pp., $4.00

Smail's Calculus

Placing importance on the meaning of concepts and methods as well as developing techniques of problem solving, this text is suitable for Arts, Science, or Engineering students. Early integration, involving both indefinite and definite integrals, is an important feature of the book. By Lloyd L. Smail, Lehigh University. 592 pp., $4.50

Appleton-Century-Crofts

35 West 32nd Street
New York 1, N.Y.

Now Available:

HOW TO DRAW A STRAIGHT LINE

By A. B. Kempe

The classic work on linkages first published in 1877 and reprinted from the Spring 1952 PENTAGON.

36 pages, paper covers

50¢ for single copies; 35¢ for orders of ten or more

THE PENTAGON, A Mathematics Magazine for Students, is published semi-annually by Kappa Mu Epsilon. Subscriptions $2.00 for two years. Subscribe now and receive the Spring 1952 number.

Send orders to: **THE PENTAGON**
310 Burr Oak St.
Albion, Michigan

TEACHING MATHEMATICS IN THE SECONDARY SCHOOL

By KINNEY & PURDY. A complete coverage of the curriculum and instruction in junior high school, senior high school, and junior college general mathematics. It provides a careful description of applications of present-day methods and materials, including visual aids as derived from actual classroom experience. *381 pp. $5.00*

MATHEMATICS OF INVESTMENT

By RIDER & FISCHER. A complete analysis which treats the whole field of mathematics of investment at a fairly high level without sacrificing clarity or going beyond the capability of the average student with an adequate background. Includes 160 pages of tables. *359 pp. $5.00*

time to look into . . .

FUNDAMENTALS OF SYMBOLIC LOGIC

By AMBROSE & LAZEROWITZ. "Anybody who works through it carefully will have a thorough knowledge of the elements of the propositional calculus and a useful introduction to the notation of propositional functions and classes, with applications to syllogistic theory. These topics are handled with outstanding thoroughness and precision . . . a first-rate text book which deserves to be widely used."—*Max Black* in PHILOSOPHICAL REVIEW *310 pp. $5.00*

FRESHMAN MATHEMATICS

C. V. Newsom ..... *559 pp. $5.00*

ALGEBRA FOR COLLEGE STUDENTS

Britton & Snively ..... *529 pp. $4.00*

COLLEGE ALGEBRA

Reagan, Ott, and Sigley ..... *447 pp. $4.00*

RINEHART MATHEMATICAL TABLES, FORMULAS & CURVES

Harold Larsen ..... *264 pp. $1.75*

RINEHART & COMPANY

232 madison avenue · new york 16

Prentice-Hall
COLLEGE TEXTS

CALCULUS—Rev. Edition

By GEORGE E. F. SHERWOOD and ANGUS E. TAYLOR, University of California (Los Angeles)

A rigorously accurate text placing greater emphasis on the understanding of theories than their application.

Entire chapter devoted to inverse of differentiation.

Theorems on limits are presented in the opening chapter to develop from the start the student's understanding of the limiting process.

Worked-out examples coordinate theory-development and exercises.

568 pages 6" x 9" Published 1946

MATHEMATICS OF FINANCE—Third Edition

by T. M. SIMPSON, Professor of Mathematics, Emeritus, University of Florida; Z. M. PIRENIAN, University of Florida; and B. H. CRENSHAW

Contains entirely new sets of problems throughout the book; total number of problems is 3200—an increase of 30 per cent over previous edition.

The following topics have been expanded and clarified: (a) use of subscripts in mathematics, (b) determining premiums for combination policies, (c) surrender options of life insurance policies.

512 pages 6" x 9" Published 1951

COLLEGE ALGEBRA

by MOSES RICHARDSON, Brooklyn College

To instill a feeling for mathematical rigor in the beginning student, strict proofs are used whenever possible. Clear explanations of the principles on which these are based make them really understandable, making sheer memorization unnecessary.

When it is clearly beyond the bounds of a first course to present a formal proof, an informal discussion is used. But it is stressed that this does not constitute real proof, so that nothing must be unlearned in more advanced courses, and no incorrect mathematics seeps into the student's habits.

472 pages 6" x 9" Published 1947

Send for your copies TODAY



Outstanding McGRAW-HILL Books

INTERMEDIATE ALGEBRA

By PAUL K. REES, Louisiana State University, and FRED W. SPARKS, Texas Technological College. 328 pages, $3.50

An exceptionally clear and teachable presentation of intermediate algebra which brings the student from a limited background in algebra through simultaneous quadratic equations; ratio, proportion, and variation; logarithms; etc. Topics of particular difficulty are considered at length and in detail. 2500 problems, of progressive difficulty, are a particular feature of the text.

ADVANCED MATHEMATICS IN PHYSICS AND ENGINEERING

By ARTHUR BRONWELL, Northwestern University. Ready in October

Presents a mathematical foundation in the principal branches of advanced mathematics used in physics and engineering. The fundamental laws of the more important areas are first expressed in very general form. These then become the springboard for the development of vast areas of applications. The text shows how the fundamental formulations simplify to special cases which usually form the starting point in solutions of problems.

CHARTING STATISTICS

By MARY ELEANOR SPEAR, Visual Information Specialist, U. S. Department of Labor; Lecturer, The American University. 253 pages, $4.50

A treatment of the practical graphic presentation of statistical data, depicting not only types of charts and their appropriate use, but also subject matter of economic interest. It is intended to help persons interested in putting a message in visual form before either a specialized audience or the general public. It is also intended to promote a closer relationship and better understanding between the analyst and the drafting room or artist.

GENERAL COLLEGE MATHEMATICS

By W. L. AYRES, C. G. FRY, and H. F. S. JONAH, Purdue University. 283 pages, $3.75

A new approach to freshman mathematics, designed to help the student in his application of mathematical topics to the biological and social sciences. Keyed primarily for those students who will take only one year of college mathematics, this text examines such subjects as ratios and percentages, linear and quadratic equations, trigonometry, interest and its application to installment buying, laws of growth, statistics, etc.

Send for copies on approval



McGRAW-HILL BOOK COMPANY, Inc.

330 WEST 42nd STREET • NEW YORK 18, N. Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 59 NUMBER 9

CONTENTS

Autobiographical Notes . . . . . G. A. Bliss 595

Unique Arrangements of Points on a Sphere . . . L. L. Whyte 606

Algebraic Properties of N-Valued Propositional Calculi . . . . . J. D. Swift 612

Mathematical Notes . . . . . H. P. McKean, Jr., H. F. Sandham, David McLean 621

Classroom Notes . . . . . J. W. Green, L. Fuchs and T. Szele, M. R. Spiegel 626

Elementary Problems and Solutions 633

Advanced Problems and Solutions 640

Recent Publications 650

Clubs and Allied Activities 655

News and Notices 659

The Mathematical Association of America 664

Thirty-third Summer Meeting 664

Calendar of Future Meetings 668

NOVEMBER 1952

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	J. W. GIVENS	GEORGE SPRINGER
PATRICK DU VAL	L. M. KELLY	E. P. STARKE
HOWARD EVES	H. D. LARSEN	G. B. THOMAS
F. A. FICKEN	IVAN NIVEN	E. P. VANCE

EDITH R. SCHNECKENBURGER

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 8, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of College Mathematics)

OFFICERS OF THE ASSOCIATION

President, SAUNDERS MACLANE, University of Chicago
Honorary President, W. D. CAIRNS, Oberlin College
First Vice-President, F. L. GRIFFIN, Wesleyan University
Second Vice-President, JEWELL H. BUSHEY, Hunter College
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo
Editor, C. B. ALLENDOERFER, University of Washington

Additional Members of the Board of Governors: I. A. BARNETT, C. F. BARR, T. A. BICKERSTAFF, J. C. BRIXEY, B. H. BROWN, R. H. CAMERON, G. R. CLEMENTS, T. F. COPE, H. S. M. COXETER, W. M. DAVIS, J. M. EARL, L. R. FORD, J. S. FRAME, A. E. GAULT, R. F. GRAESSER, E. H. HANSON, M. R. HESTENES, A. S. HOWARD, C. G. JAEGER, R. D. JAMES, B. W. JONES, R. E. LANGER, D. H. LEHMER, F. A. LEWIS, C. C. MACDUFFEE, R. J. MICHEL, W. E. MILNE, C. W. MUNSHOWER, H. P. PETTIT, J. C. POLLEY, G. B. PRICE, G. E. RAYNOR, E. B. ROESSLER, F. H. STEEN.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

AUTOBIOGRAPHICAL NOTES

G. A. BLISS, University of Chicago

Editorial Note—These notes are excerpts from a manuscript composed by Professor Bliss early in 1938. His purpose in setting them down was principally to save trouble for his biographers. The parts printed here include some remarks on mathematics and mathematical education, mathematicians and universities, which seem likely to be interesting to readers of the MONTHLY. A biographical note, with a bibliography and comments on Bliss's scientific work, by L. M. Graves, has appeared in the Bulletin of the American Mathematical Society, vol. 58, 1952, pp. 251–264.

Bliss was born in Chicago and attended the Chicago public schools. His interests were quite general until his second year at the University of Chicago when he studied the calculus; his serious interest in mathematics began at this time and he decided to make it his principal subject.

1. Graduate work at Chicago. In the graduate school at the University of Chicago my work at first was mostly in mathematical astronomy under F. R. Moulton. It was fascinating to me, and I owe to Professor Moulton a very great debt for his inspiring teaching, and especially for his encouragement in the writing of my first published paper. It was entitled "The motion of a heavenly body in a resisting medium" and had an influence on myself quite out of proportion to the value of the paper itself. At the end of my first graduate year I applied for a fellowship in astronomy which was not granted, though I understood that Professor Moulton recommended it. My failure to secure this appointment was a great disappointment to me, in view of my somewhat difficult financial circumstances, but the effect on me was undoubtedly good because it caused me to take stock of my own interests and plans for the future. After some careful consideration I concluded that it was really the mathematics which had the great attraction for me, and I decided to try for a Ph.D. in that subject rather than in astronomy. I have never regretted the decision, though it seems clear to me that one could find great satisfaction in any domain of applied mathematics. I must have decided to make mathematics my principal subject before the end of my first graduate year, because I received the master's degree in mathematics, apparently in the summer quarter of 1898.

The three leading men in mathematics at the University of Chicago at that time (1898) were E. H. Moore, O. Bolza, and H. Maschke. They supplemented each other beautifully. Moore was the fiery enthusiast, keenly interested in the popular mathematical research movements of his day, one after another, at that time especially in group and field theory. Bolza was a product of the meticulous German school of analysis led by Weierstrass. He was an able, aggressive, and widely read research scholar. Maschke was a geometer, much more easy-going than the other two, but brilliant in his research, and the best lecturer on geometry to whom I have ever listened. Under the leadership of these three men Chicago was unsurpassed at that time in America as an institution for the study of higher mathematics. With such representatives of the best mathematical scholarship near at hand one did not need to go abroad to study. Like many other members of our present American mathematical community I am greatly indebted to Moore and Maschke and Bolza for the example of their distinguished

research and scholarship and their continuously friendly and encouraging interest.

My last two years of graduate study passed rapidly, full of the inspiration and hard work well known to every one who has been a candidate for the doctor's degree. Bolza had a beautifully written record, made by himself, of the now famous course in the calculus of variations which Weierstrass had given in 1879, and let me make a copy of it for my own use. It is a course which contains most important results, quite new in 1879, and which has had an unusually wide influence, though in 1898 it was relatively little known. A record of it over Weierstrass' own name was first published in 1927 in volume VII of his Mathematische Werke. The possession of this masterpiece, and Bolza's fascinating lectures, gave me a dominant interest in the calculus of variations, though I was exposed to many other mathematical theories and thoroughly enjoyed them. I wrote my doctor's thesis under Bolza's direction on "The geodesic lines on the anchor ring." It was a study of the forms of these curves on the anchor ring and a classification of them according to their possession or non-possession of conjugate points. I have found the paper a useful one and from my correspondence concerning it at various times believe that others have had the same experience.

2. Early teaching experience. My first teaching position was in 1900 at the University of Minnesota. I enjoyed the students and the teaching from the start. Kneser's *Vorlesungen über Variationsrechnung* had just appeared. It was the only treatise on the subject which contained an account of the ideas of Weierstrass, and was not any too complete in that respect either. I determined to make myself as far as possible an expert in the theory of the calculus of variations, and began by reading Kneser. It was the kind of a program which I still recommend strongly to my young graduate students. If one expects to do mathematical research he must first be a master of the literature of some field, or some portion of a field. When he has become such a master he will find plenty of research questions pressing upon him for answer. Many students never reach this stage, either because they are early distracted by other interests, or because they are too anxious to do research and try to do it without having the fundamental scholarship which justifies it. In my case I found the next two years of absorbing scientific interest. I read Kneser and much other literature of the calculus of variations, and wrote my first completely independent research paper "The second variation when one end-point is variable." It contained an analysis of the movement of a focal point of a curve transversal to an extremal when the curvature of the transversal curve varies. I believe that the first paper a student writes after he has received his Ph.D., or at any time as a result entirely of his own interest and enthusiasm, is the most important one for him. In my case the paper just mentioned above was the one which first gave me some confidence that I might in time become a productive scholar.

3. Post-graduate work abroad. After two years at Minnesota the future there did not look very bright for me, though the years themselves had been

most interesting and profitable ones. My father had died in 1900, but I had saved some money out of my very moderate salary, and my mother offered to supplement this out of her own slender resources in order that I might have the benefit of a year of study abroad. It was something which she never should have done, and which can never be repaid except by helping to provide another such opportunity for another such student. I spent all of my year in Göttingen except for brief visits to Berlin, Italy, Paris, and England. The leaders in Göttingen at that time were Klein, Hilbert, and Minkowski, a notable group. Among the younger men were Zermelo, E. Schmidt, Abraham, Fejer, and Caratheodory, certainly inspiring comrades. As a visitor with his Ph.D. already behind him I was invited to attend the Mathematische Gesellschaft which met every other week. Klein was the dominant figure in the Gesellschaft, a marvel of wide acquaintance with mathematics and mathematicians, and ready with many interesting comments on their books and personalities. After the meeting was over Hilbert and Minkowski would lead a party of a dozen or so to a neighboring cafe where the evening would be spent in lively mathematical discussions.

My year at Göttingen was a most profitable one. I listened to lecture courses by Hilbert, Minkowski, and Klein, and took an active part as a student in the seminar of Hilbert and Minkowski. Hilbert was lecturing on Mechanik der Kontinua, Minkowski on Minimalflächen, and Klein on his Encyklopädie. My reports in the seminar were on some problems in conformal representation in which Hilbert was interested. None of these was closely associated with my chief interest, the calculus of variations, but I found in the excellent library of the mathematical reading room most interesting records of courses in the calculus of variations by Weierstrass, Hilbert, Sommerfeldt, and Zermelo. These alone would have justified my trip to Europe. The profit becomes much greater if one adds to them the inspiration of distinguished professors, the benefit of acquaintance with the group of younger men many of whom afterward became famous, and the great pleasure of lifelong friendship with Max Mason and O. D. Kellogg, then young American candidates for the doctor's degree at Göttingen. A paper which emerged from my studies of that year was a discussion of necessary and sufficient conditions for a minimum for a problem of the calculus of variations in the plane with both end-points variable.

In their seminar Hilbert and Minkowski first assigned reports to students in pairs. My colleague was an assistant in a neighboring department of the University who had a quite remarkable acquaintance with mathematical books. I was much discouraged at my own ignorance after our first conference, but I soon found that he had very moderate command of mathematical techniques. I have thought sometimes that our relatively informal exercises and reports in our graduate courses in America tend to develop control of technique earlier than the more formal lecture courses of the German universities. At any rate after our first report I was called upon to make all of the rest of them. It was rather rough going because of my poor German. One day Hilbert interrupted

me suddenly with the remark "Ach Herr Bliss, dies Mal haben sie sich gierrt," whereupon both he and Minkowski rushed to the board, as was their custom, and began writing and talking excitedly with no regard for each other, Hilbert with his face very close to the board because he was so nearsighted, all to the great delight of the students. Presently Hilbert turned to me, as much pleased as I was to find that there had been no mistake, and said that I must have meant "eine eingeschnittene Ellipse." An ellipse must be cut in from the ends of the major axes to the foci in order to be represented conformally on a rectangle by the method I was trying to explain. The word "eingeschnitten" was not in my vocabulary at that time, but I venture to say that it is now permanently a part of it.

Klein was a dominant figure in German mathematics, and indeed in the German educational world in general. As a younger man he did beautiful research in many domains of pure mathematics. When he grew older he turned almost entirely to applied mathematics, in which his research has seemed to me much less significant, and became rather critical, so it seemed to me, of research in pure mathematics. When I was in Göttingen he looked really ill, and though he lived many years after that I think that he had always to be very careful of his health. One day I was asked to give a paper at the Gesellschaft and presented the paper mentioned above. When I had finished Klein rather took my breath away by asking "But, Herr Bliss, who would be interested in this?" That was for the moment a poser to me, but I tried to explain my belief that one who had been especially a student of the calculus of variations might find something of interest in the results of the paper. It seems that I mixed the words "man" and "sie" so that my remark had rather the meaning that "If you knew anything about the calculus of variations you would be interested." I still remember the breathless moment which followed my remark, incomprehensible to me at that time but comprehensible enough several years later when Max Abraham retold the incident to some of us with, I hope, much exaggeration. Certainly I could never intentionally be disrespectful to one who has wielded the uniquely distinguished influence of Klein. As for my paper I must say in self defense that I have had many inquiries for the reprints which unfortunately were lost somewhere between Germany and the United States. As I look at the paper again from time to time it seems to me a rather good one, the first relatively complete treatment of a problem of the calculus of variations with two variable end-points.

4. Later teaching positions. In the autumn of 1903 a position at Minnesota would still have been open to me, but Professor E. H. Moore, head of the mathematical department of the University of Chicago, wrote to me that I could have an associateship there for one year if I was interested. It seemed to me an excellent opportunity to spend another year in an inspiring mathematical environment, and I accepted his suggestion with much alacrity. My anticipation proved to be well justified. My sojourn abroad had not weakened, but had strengthened, my respect for the Chicago department of mathematics, and for my

professors there, and I enjoyed and profited much from my associations with them as a member of the faculty during the year 1903–4.

In the autumn of 1904 I went as assistant professor to the University of Missouri where E. R. Hedrick was head of the department of mathematics. It was my first experience with life in a small town and I enjoyed my surroundings immensely. Hedrick was ideally congenial as a leader. While I was there he organized an association of Missouri teachers of mathematics. The first meeting was held at the University in Columbia, Missouri, and we all helped as much as we could to make it a success. Hedrick had, however, a knack for making things go irrespective of our assistance. I had hardly become a part of the University circle in Columbia when in the spring of 1905 I had an invitation to go to Princeton. It was with great regret that I left, carrying with me very permanent memories of a most happy year and of my friendly associations with Hedrick. In one of my classes W. A. Hurwitz, now professor at Cornell University, was a student. He was extraordinarily precocious and able.

In the autumn of 1905 the then new preceptorial system at Princeton was greatly extended. About fifty new "preceptors" were added to the faculty. I was one of them. Some of us did not like the word "preceptor" very much, because it suggested tutor or assistant and had no established meaning in the educational world. We were assured that we were the equals of assistant professors elsewhere and that we might think of ourselves as such. I always did so and have always listed myself as such to a world which has now pretty much forgotten the preceptorial system. Princeton is an exceedingly friendly but somewhat conservative place, and I have doubts if the fifty preceptors really became an integral part of the community during the three years that I was there. Those of us in mathematics had a most interesting time, however, with our science and our students.

H. B. Fine was Dean and also head of the department of mathematics during my sojourn in Princeton. He was much beloved in his community, not aggressive in mathematical research, but most sympathetic and encouraging to the younger men in his department. We regarded him with great affection. Frequently on Saturday nights the younger men in the department met in the rooms of one or another of them for refreshments and mathematical discussions. The discussions were the liveliest and in many way the most interesting ones in which I have ever taken part. Eisenhart, Veblen, J. W. Young and R. L. Moore were among the leaders and my friendship for them and admiration of their scholarship have remained with me permanently. During one of my years at Princeton Jeans was a visiting professor. He was about our own age but very unapproachable. We tried to give him a good hearing in his lectures on Dynamical Theory of Gases, but anything like free discussion proved to be impossible. I learned more of him later by reading his book on electro-magnetic theory, but have never been enthusiastic about his writing as viewed from the standpoint of pure as contrasted to applied mathematics.

5. Teaching at Chicago. About the time I was in Princeton I spent most of my summers teaching in universities other than Princeton. In 1907 a summer term at Wisconsin, where I renewed my associations with Mason and his family, gave me much pleasure. He was at Yale while I was at Princeton and I saw him there from time to time also. But most of my summer teaching was in the Department of Mathematics at Chicago. In the early part of 1908 Maschke died, and during the summer of that year, while teaching at Chicago, I was offered an associate professorship there. Maschke's loss to his Department was a very great one, in fact irreparable, and I was sure that they would have preferred an older man to replace him. But the challenge to me was one which I could not refuse and I was anxious to accept. Professor Fine at Princeton was most generous in his willingness to facilitate my move, and after teaching some weeks at Princeton in the autumn of 1908, while he made readjustments in his staff and schedule, I joined the Department at Chicago and have remained there ever since.

The next two years were rather strenuous ones for me. I undertook to give some of the advanced courses in geometry which Maschke had always given before, metric differential geometry, analytic projective geometry, higher plane curves. It was most interesting work and I have always appreciated the wider experience in mathematics which this teaching gave me though it took much time. In 1909 Kasner and I were the lecturers at the Princeton Colloquium of the American Mathematical Society. My part of the colloquium was on fundamental existence theorems for implicit functions and ordinary and partial differential equations in the theory of functions of real variables. I have always hoped to find time to pursue much further the theory of singular points of real transformations of the plane which I developed in those lectures.

In 1910 Bolza withdrew from the department of mathematics at Chicago and went back to Germany to live in Freiburg in Baden. His mother there was quite old and was not expected to live long, though as a matter of fact she did live for many years. Bolza's departure was another great loss to Chicago. Wilczynski, then of Illinois, had made a name for himself in projective differential geometry, and we were fortunate in securing him in the autumn of 1910 as a member of our mathematical staff. His arrival and Bolza's departure brought my teaching back again into my chosen field of analysis.

In 1913 I was made a full professor at the University of Chicago. Slaught as the leader, with Dickson, Moore, and of course many others, were engaged about that time in strengthening the AMERICAN MATHEMATICAL MONTHLY and in the organization of the Mathematical Association of America. In the summer of 1918, after the United States had been drawn into the war, we undertook in our Department to give courses in navigation to men who had been accepted and were waiting in Chicago to enter the naval school there. We had a number of sections. Mine had about one hundred students in it. To my surprise when the course was over the men in it presented me with a very fine watch. I have never felt that acceptance of it was appropriate, since they were the ones who were

making the sacrifices and preparing themselves for service. But I must say that I have valued their gift much, especially as a reminder of their seriously attentive and successful interest.

6. Aberdeen Proving Ground. In late summer of 1918 my old friend Oswald Veblen began writing me from Aberdeen Proving Ground saying that he needed mathematicians. He was major in charge of the Range Firing Section there, with the duty of supervising the preparation of range tables. Our work at Chicago with the prospective naval students was by that time well organized, and it seemed to me the most effective work which I could then do. But I had confidence in Veblen's judgment and after repeated letters decided to join him at Aberdeen as what they called a "scientific expert." Veblen was right, of course, about the need of mathematics in ballistics. The old Siacci theory broke down early in the war because it contained an approximation not justifiable for gun elevations greater than twenty degrees. F. R. Moulton, Veblen, and their associates had devised effective methods of successive approximation for solving the differential equations of trajectories in all cases, but the theory and application of the so-called "differential corrections" of trajectories was still very imperfect. These are corrections made to the normal range to account for the effects of wind, variations from normal in the density of the air and the weights of the projectile and powder charge, and in the case of long ranges for the rotation of the earth. This was my particular field of study, and I was astonished to find that the most advanced mathematics which I knew was needed and effective. The wind correction, for example, is a function of the wind curve when wind velocity is plotted as a function of altitude. It is what the mathematicians sometimes call a functional or a function of a line. The calculation of the differentials of such functionals is a principal problem of my special field, the calculus of variations. I found that the methods of that calculus were effective, and that in many cases with suitable adjustments they reduced by three-fourths the time necessary for the computation of range corrections. The saving for anti-aircraft trajectories was not so great, but the method is applicable there also, and I have left a detailed description of it for that case in a blue print in the archives at the Proving Ground. These results and savings were important for the Range Firing Section because of the large number of computers engaged in the construction of range tables and the constant flow of data into the Section concerning new experimental firings and new types of guns. I am happy to say that there are many applications other than in ballistics of the method which I devised. Some of them I have pursued to my own satisfaction.

7. Post-war years. In 1921 I was elected president of the American Mathematical Society. It is a position of great honor, and also of great responsibility. The Society was having financial difficulties, a condition which seems to be normally associated with healthy progress. E. R. Hedrick suggested a campaign to increase membership. I must confess that I doubted the possibility of success sufficient to be profitable, and that I undertook the labor of the campaign

with great reluctance because of the time it would take from my scientific research interests. Once started, however, both Hedrick and I devoted ourselves to it heartily. The results well justified our efforts, especially as this first campaign of the Society encouraged later and more successful ones which have developed into a continuous membership campaign under the guidance of our skillful secretary, R. G. D. Richardson.

At Chicago, as at other universities, the years following the war were exceedingly crowded ones. There seemed no limit to the number of students who were seeking college educations and higher degrees. Our situation was aggravated by the serious illness and consequent retirement of Wilczynski from active work in 1923. Fortunately we were able to secure E. P. Lane, his very able successor in the field of projective differential geometry, to carry on his work. Our climax was reached in 1927. Advanced graduate courses in one summer quarter sometimes attracted over eighty students, and it seemed almost impossible to take care of the large number of candidates seeking higher degrees in mathematics. Fortunately the head of our Department, E. H. Moore, was conservative about increasing our permanently appointed personnel, for after 1927 there was at first a gradual reduction in the demands upon us, and then a rapid decrease during the depression years. We were fortunate in not having expanded our Department unduly, and in being able to retain our regularly appointed staff intact without salary reductions.

In 1927 Professor Moore had passed the normal retiring age and was anxious to lay aside his administrative responsibilities as head of the Department. The Trustees appointed me chairman, at first without public announcement, and Mr. Moore was retained as nominal head.

In 1927 the University established a number of special professorships, one of which is the Eliakim Hastings Moore Distinguished Service Professorship. L. E. Dickson is the first incumbent. He was one of Moore's earliest students, and his career in mathematical research has been remarkable. I can think of no more appropriate appointment. One of the purposes of our Department has been to provide Dickson with a most favorable environment for the pursuit of his teaching and research interests, and I believe that we have succeeded. In 1933 I was appointed to a similar professorship bearing the name of Martin A. Ryerson. The appointment was a surprise to me and gave me one of the greatest of the pleasurable thrills of my life.

8. Editorial and committee work. In the preceding paragraphs I have not mentioned my editorial work. I was associate editor of the Annals of Mathematics from 1906 to 1908, and of the Transactions of the American Mathematical Society from 1909 to 1916. E. H. Moore, T. S. Fiske, and E. W. Brown, the first chief editors of the Transactions, actually taught many members of our then youthful research community to write in good mathematical style. Others of us who assisted in those earlier days attempted to do the same. I spent days at a time trying to master unfamiliar fields and preparing sometimes

unwelcome advice to authors. The result was that I developed a very great distaste for editorial work. The papers which especially irked me were theses from other universities than my own which I thought should have been more carefully edited at home. In 1916 a chief editorship of the Transactions was offered to me and I declined it. My decision was probably a mistake, since many of our best mathematicians have unselfishly devoted much time to the Transactions, and since the work is undoubtedly exceedingly important for our mathematical community. I did not escape permanently, however, for I was doing the equivalent of editorial criticism on the National Research Fellowship Board from 1924 to 1936, and have been an editor of the Carus Mathematical Monographs from 1924 on, and chairman of the editorial committee of the University of Chicago Science Series from 1929.

The Fellowship Board was a fine group of men, consisting, when I first became a member, of Simon Flexner, Chairman, K. T. Compton and Mendenhall and Millikan in physics, Johnston and Keyes and Kohler in chemistry, Birkhoff and Bliss and Veblen in mathematics. Johnston and Mendenhall later withdrew and were replaced by others. I often wondered how some of these men found the time for the conscientious work which they did on the Board. Our choices were criticized severely in particular cases, sometimes justly, sometimes not. At one time the three groups examined independently their list of past appointees, marking those who had distinguished themselves as Fellows A, those who would again be appointed B, and the failures C. When they compared notes the number of C's was in each field about ten percent of the total in that field. This seems to me an excellent record, and I wish that each of our critics might have an opportunity to try to equal or surpass it. But the effort to classify people according to their ability, to say that one is better than another, is distasteful to me, and in editorial work it seems to me a much greater pleasure to try to find the strong points of a paper than to criticize its deficiencies. In general I feel sure that our mathematical community will be more effective and much happier if we cultivate whenever possible an appreciative and cooperative point of view rather than a critically comparative one.

9. Philosophy of graduate work. When Mason was president of the University of Chicago our Department was under great pressure. We had too many students. He repeatedly urged us to restrict enrollment by setting higher standards for admission. I have always felt that this is a mistaken policy. Experience has shown that it is impossible in many cases to foretell an entering student's future. Some of the inexperienced ones develop real power, and some of the brilliant ones fade away. Many who are not qualified to take a higher degree get a real thrill and a new impetus for their teaching work out of the advanced courses. I therefore favor keeping graduate lecture courses open to all who have a reasonable justification for their desire to listen in. The time when distinctions should be made is when the student applies for candidacy for a higher degree, after he has given his instructors some indications of what he can be expected to

do. The candidates for higher degrees are the ones who take the time, not the listeners in lecture courses.

At the present time there are strong influences at work to cheapen higher degrees in mathematics, as also in other subjects. I hope that our standards can be preserved. In our Department we have been unwilling that a candidate for the master's degree should divide his requirement between mathematics and some other subject, because if he does so he can present for his degree only the most elementary graduate material in either field. We have insisted on a thesis containing if possible some new result of an elementary sort. The purpose of this is to give the student his first introduction to methods of research and to train him in mathematical writing. I advise students to take master's degrees on the way to their doctorates because of the great value to them of this preliminary training. We keep a record of the performance of a student in attaining his master's degree and use it as one of the aids to guide us in determining whether or not he should be recommended for candidacy for the doctor's degree. Thus we can avoid the necessity of preliminary examinations for the doctor's degree. I am much opposed to such preliminary examinations because they take much time and tend to place the emphasis on courses rather than research for both students and faculty. The sooner a candidate for the doctor's degree can start his research the better it will be for everybody.

It seems to me that there is widespread misunderstanding of the significance of doctor's degrees in mathematics. The comment is often made that the purpose of such a degree is to train students for research in mathematics, and that the success of the degree is doubtful because most of those who attain it do not afterward do mathematical research. My own feeling about our higher degrees is quite different. The real purpose of graduate work in mathematics, or any other subject, is to train the student to recognize what men call the truth, and to give him what is usually his first experience in searching out the truth in some special field and recording his impressions. Such a training is invaluable for teaching, or business, or whatever activity may claim the student's future interest.

Not long ago a committee of the Mathematical Association of America published a report in the AMERICAN MATHEMATICAL MONTHLY advocating the inauguration of a higher degree for teachers not emphasizing training in research. I should hesitate to believe that research experience is not as valuable for teaching as for other types of activity. Of the ten members of the committee nine have doctor's degrees in mathematics, and five are doctors of our own Department. Some have been active in mathematical research, and others have devoted themselves to other equally important duties in their respective environments. I would venture the opinion that the training for the doctor's degrees held by these committee members has in each case been an important factor in establishing them in the positions of usefulness and influence which they now hold. The suggestion of a higher degree for prospective college teachers, comparable with the doctor's degree but not requiring a research thesis, is a very old one,

as old as my experience in university work. I think that some one who has confidence in such degrees should try to establish them. But they should not be called doctor's degrees, since this title has long implied research.

In a recent report R. G. D. Richardson has tabulated the research output of doctors of philosophy in mathematics in the United States, and has given comparative statistics for various American universities. His figures show among other things that many of our doctors do no research after they receive the doctorate; that a considerable number continue to publish in limited amount; and that a relatively small number continue to be active in mathematical research. The report itself is purely statistical, but it has been the basis for criticisms of the training of doctors in mathematics in this country which seem to me unjustified. The fact that many doctors do no mathematical research after receiving their degrees is quite in accord with the purpose of such degrees as described in a preceding paragraph above, provided that the doctors apply in whatever may be their chosen activity the spirit of the investigator. It is not easy to measure success in this respect, but it is clear that our doctors as a group have been given relatively very great and very important educational responsibilities in the universities of the country. Those of us who have devoted our lives to the intimate study of mathematics are of course greatly interested in the relatively small number of doctors who turn out to be continuously interested and successful in research. The percentage of doctors in this group will be small in mathematics, as in any other subject. To insure adequate renewals in the ranks of such scholars it is necessary that our population of graduate students of mathematics shall always be as large as is reasonable in view of the demand for trained mathematicians in our educational and other institutions. Furthermore for the continuation of the activity and interest of this relatively small group of enthusiasts the presence of the much larger number of mathematicians who do some research themselves and who have high appreciation of it in others is essential. No great mathematical school of the past, with its heroes to whom we often ascribe too exclusively its achievements, ever flourished without its public of well-trained and appreciative listeners. Thus it seems to me that doctors of all the types described in Richardson's statistics are essential to the well being of our American mathematical school, and that criticisms of our doctorate based upon the figures given in his paper are not well founded.

I would make one final comment on Richardson's report relative to the doctors of our own Department. Their number is larger than that of any other mathematical Department, but the average of their annual publication output is lower there than in a number of other cases. The average of their total number of published pages per year is also the largest. If one is thinking only of mathematical research produced it is evident that magnitude of production can be attained either by a considerable number of workers each producing a limited amount, or by a smaller number of scholars each of whom is very active. At Chicago we have turned out men of both types and though our total productivity is large our average per doctor is relatively smaller. The discussion of

these questions impresses one, however, with the futility of attempts at measurement of the relative effectiveness of different departments. What is needed in every case is a group of men devoted to mathematics and ambitious to make the most effective use of their own personalities and interests in research and teaching. In every environment it is important to stimulate the strong graduate student by the challenge of his research thesis, but it is still more important to impress upon him the significance of a first independently conceived research paper following the doctorate. The preparation and publication of such a paper is the first conclusive evidence of true interest in scholarship.

It is clear from the preceding paragraphs that different individuals and different groups have different interests and different ideas about the training of students, and that different methods in research and teaching may achieve comparable successes. For those who have demonstrated their own effectiveness the most important things which a university can offer are freedom and encouraging recognition. There is nothing more discouraging in university work than to be continually told that methods and ideals of the past have for the most part been seriously wrong, even if one knows from his own experience that in his own department of learning the charge is not well justified. Unfortunately the psychology of our most enthusiastic students of the theory of education at the present time seems to be that of the pessimist and critic, as one may verify by an examination of the addresses on almost any recent program of speakers on educational theory.

UNIQUE ARRANGEMENTS OF POINTS ON A SPHERE

L. L. WHYTE, London

It is over seventy years since Sohncke established the theory of the 65 (infinite) *regular rectilinear point systems* in three-space, while the theory of the (finite) *unique spherical point systems*, apart from the special cases of the regular and semi-regular polyhedral arrangements, is still at an early stage.

The reasons for the relative neglect of spherical arrangements are clear. Until this century there was no adequate stimulus to explore this field beyond these special cases, and the other unique arrangements cannot be treated by established methods such as group theory.

But once an unexplored territory is known to exist, the challenge sooner or later becomes irresistible, and it seems that this is now happening. The new field of enquiry may be defined as *the theory of spherical point systems, i.e., of those arrangements of a finite number of points on a sphere in three-space which are distinguished by some extremal property*. Since 1904 more than a dozen workers

[1] have examined problems lying within this area, all but two during the last ten years.

The present paper reviews this work in order to call attention to a group of unsolved problems which may have applications to physics. Until recently the angular relations of sets of particles have been relatively neglected in theories of the structure of matter. But a systematic analysis of the angular relations of unique systems of N points (perhaps representing the motions and stable equilibrium patterns of nucleons) may guide the development of physical theory in the future, just as the study of the linear relations of pairs of points (representing the motions of atoms or massive bodies) did in the past. In particular, the analysis of the angular structure of unique spherical shells of point particles may provide a first step towards the theory of the atomic nucleus, just as the study of regular linear arrays of identical atomic groups led towards the complete theory of crystal structure.

The spherical arrangements of points determined by the vertices of polyhedra having all their vertices equivalent have been studied from about 500 B.C. onwards. But it seems that no consideration [2] was given to unique spherical arrangements of N points determined by extremal values of some function of the arrangement until 1904 A.D., when J. J. Thomson [3] sought to determine the stable equilibrium patterns of N classical electrons ($1<N<100$) constrained to move on the surface of a sphere while repelling each other by the inverse square law (here called the Thomson problem). Thomson found this problem intractable for spherical arrangements, and used approximate methods to minimize the total electrostatic energy for plane rotating rings of electrons. His papers are of historical interest, for he showed how the existence of unique stable rings and shells might follow from simple assumptions, but his results were restricted to plane rings. This is not surprising, for the solutions of the Thomson problem even for relatively low values (*e.g.* $N=9$, 11, *etc.*) are still unknown, though an experiment simulating repelling particles might be carried out to determine the answer.

In 1912 L. Föppl [4], at Hilbert's suggestion, made a rigorous examination (on slightly different assumptions) of the stability of Thomson's arrangements, and published the earliest known study of spherical point systems, based on the model of a uniform sphere of positive electricity with N electrons free to move through it. Föppl obtained stable solutions for all values of N up to 8, and for 10 12, and 14, in which the electrons all lie on one spherical surface, but as these solutions result from rather special assumptions we shall not list them. From inspection of eight solutions Föppl suggested the following tentative rules:— (i) the presence of electrons at poles and of planar rings perpendicular to the polar axis; (ii) the twisting of the rings, so that each electron is opposite the centre of a space in neighbouring rings (see below); and (iii) the discrimination of special values of N (in his problem: 4, 6, 8, 12, 20, 22, 46, 94, 118, with periods of 24 or 48) permitting uniquely stable arrangements, as in the periodic table

and the nuclear series. These three suggested rules may constitute prototypes of general principles valid for a wider class of spherical point systems than those derivable from Föppl's assumptions.

The Thomson problem and its variants have recently attracted little interest owing to the success of theories of atomic structure (Bohr, Schrödinger, Pauli Exclusion Principle) in which the stability of the atom is no longer determined by minimizing the electrostatic potential energy of mutually repelling point particles. Moreover the view that dynamical, higher-dimensional, and statistical representations are necessary for both atomic and nuclear structure has drawn attention away from the angular structure of shells of point particles in three-space. Thus Föppl's work appears to be unknown to the workers who have recently obtained similar, but not identical, arrangements as solutions of a distinct mathematical problem.

In 1942 Fejes Tóth [5] (then Fejes) began to investigate the arrangements of N points on a unit sphere which maximize the least distance, d, between any pair (here called the Fejes problem). Fejes later described this problem as the densest packing of equal spherical caps on a sphere, and formulated it thus: What is the maximum number of points one can place on a unit sphere so that the distance between any two shall be greater than a fixed amount ϵ? (ϵ plays a role similar to that of a universal minimum length separating all nucleons. Another formulation which avoids the redundant reference to two lengths, ϵ (or d) and the radius of the sphere, is: How large can the angle of N equal non-overlapping cones be, if their vertices coincide? Or: How many cones of given angle can be packed around a point?)

Fejes proved (1943) that there always exist two points whose distance is not greater than $\{4-\csc^2 \frac{1}{6}\pi N/(N-2)\}^{1/2}$ and that for $N=3$, 4, 6, and 12 the limit is exact, the last three being the cases in which the faces of the regular polyhedron are equilateral triangles. (The Fejes limit is also approached as $N\to\infty$.) This inequality, which expresses the fact that the area of the unit sphere is always at least that of $2(N-2)$ equilateral spherical triangles of side d, was obtained independently by H. Hadwiger [6] (1942/3) and by Habicht and van der Waerden (1951, see below) using different methods. Possible solutions for $N=8$ and 20 were described by H. Rutishauser [7] (1942/3, published 1944/5) using the Fejes inequality.

A point of interest is that Föppl, Fejes, and Rutishauser all independently proved that the five regular polyhedra fall into two groups: the triangular polyhedra with 4, 6, and 12 vertices, which provide solutions of their respective extremal problems; and the polyhedra with three edges at each of 8 and 20 vertices, which do not. For example, the Fejes problem for $N=8$ is solved by the vertices of a *twisted cube or square antiprism with 16 equal edges* (two squares twisted so that the transverse joins are equal to the sides of the squares). This is an application to Fejes' problem of Föppl's second rule, as shown in the table on page 610.

We must now consider a physical principle which led to interest in spherical point systems from a more general point of view. Thomson had been concerned with the stability of spherical arrangements in classical electron theory, but a similar problem arises in *any* physical theory of the stability of a finite number of identical entities of *any* kind grouped around a centre, provided that these entities can to a first approximation be represented by points on a sphere. In an extensive class of systems (*e.g.* bubbles, certain symmetrical molecules, crystalline structures, *etc.*) stability is associated with geometrical symmetry, *i.e.*, the state of minimum potential energy possesses an additional symmetry element, and the corresponding type of asymmetry decreases as the system moves towards its stable equilibrium state [8]. But the stable equilibrium form of a system composed of N identical entities arranged around a centre cannot be *uniquely* distinguished by the presence of symmetry elements for values of N allowing no spherical arrangement in which all the points are equivalent. The criterion of symmetry must therefore be replaced by some more comprehensive extremal property, if all values of N are to be treated.

If the potential function is unknown, or if the use of a potential function is inappropriate, one can nevertheless assume that the equilibrium states of spherical arrangements of identical point particles are determined by *some* extremal property. We are thus led to ask, avoiding any premature restrictive assumptions: *What spherical arrangements possess extremal properties, of any kind*? A search undertaken in 1949/50 [9] showed that no comprehensive study had been made of such spherical point systems and that even for the special Thomson and Fejes problems no general solution had been found. Solutions up to $N=7$ were available for both (for Fejes' problem in a private Swiss communication), but these had been obtained by laborious trial and error methods.

Meanwhile a more powerful attack on Fejes' problem had begun without knowledge of any of the work cited, but confirming and extending the results previously established. In May 1950 B. L. van der Waerden lectured in Göttingen on results obtained jointly by himself and W. Habicht [10] on distributions of points on a sphere, including Fejes' upper limit for d, and the fact (then known in Switzerland, but unpublished) that the maximum for d is the same for $N=5$ and 6. They asked: Of what size must a sphere be to provide room for N points of minimal distance unity? Use was made of the graphs which arise by connecting all pairs which have exactly the distance unity.

K. Schütte was led by this lecture to solve the case $N=7$, and he and B. L. van der Waerden [11] published a paper strengthening the Fejes inequality by the use of concave functions, and giving solutions up to $N=9$, suspected solutions for eight higher values, and a table and graph of limits for the radius up to $N=32$. A new concept was introduced: the density of packing of the best arrangement measured against the density given by the limit of the Fejes inequality, reached only for $N=3$, 4, 6, and 12, which may be regarded as the "most perfect" arrangements.

The known arrangements yielding solutions of Fejes' problem are:

N	*Föppl's notation**	*Description*
4	(1, 3) or (2, 2)	Regular tetrahedron.
5	(1, 3, 1) or (3, 2)	No unique arrangement. Two at poles, three on equator with distances at least $\pi/2$.
6	(1, 4, 1) or (3, 3)	Regular octahedron.
7	(1, 3, 3)	4 equilateral spherical triangles have angles of 80°.
8	(4, 4)	Square antiprism, with 16 equal edges.
9	(3, 3, 3)	8 equilateral spherical triangles have angles of $\cos^{-1}(\frac{1}{4})$.
12	(1, 5, 5, 1) or (3, 3, 3, 3)	Regular icosahedron.
∞	—	∞ equilateral plane triangles: close packing of equal circles in a plane.

These results show that the Thomson and Fejes problems are not in general solved by identical arrangements. For example, for $N=7$ Föppl obtains (1, 5, 1) for his version of the Thomson problem.

Some other work may be mentioned. The Fejes' problem for $N=5$ was recently published [12] as an unsolved problem, and the solution given by C. S. Ogilvy and by L. Moser. Spherical configurations of particles which are in equilibrium under any law of force have been examined by J. Leech (to be published). Considerable work has been done on similar problems in higher dimensional space and on the surface and volume extremal properties of polyhedra inscribed in or about a sphere, but without further results relevant here.

Unsolved Problems

The following unsolved problems arise from this survey:

A. *Thomson's Problem.* General solution, or further special solutions.

B. *Thomson's Problem.* For more general, or arbitrary potential function.

C. *Fejes' Problem.* General solution, further special solutions, or stronger inequalities. Can Föppl's rules be formulated so that they are valid for all solutions of Fejes' problem?

D. *Other Extremals.* Determine the arrangements yielding extremals of other functions, *e.g.* maximizing the product of the distances of all pairs.

E. *General Theory of Spherical Point Systems in Three-Space.*

(i) What spherical arrangements possess extremal properties of any kind? What extremal problems lead to identical solutions?

(ii) What general rules, such as Föppl's, are valid for the extremals of any non-trivial function of spherical arrangements?

(iii) Can more general extremal functions (*e.g.* of arrangements on concentric spheres or of appropriately weighted points) be found which discriminate the two sets of numbers determining nuclear and chemical stability respectively (magic numbers, and periods in periodic table), as yielding the *lowest*

* Showing number of points at poles and in twisted rings with points opposite centres of spaces in neighbouring rings.

extremal values? (These lowest extremal values may represent highest density of packing and most stable arrangements.)

References

1. Thomson, Föppl, Fejes, Hadwiger, Rutishauser, B. L. van der Waerden, Habicht, Schütte, Ogilvy, Moser (references below); in private communications: Erdös, Leech, and others.

2. If early tentative studies are excluded, such as R. G. Boscovich's theory (1758) of the stability of point systems, including a tetrahedral arrangement. "A Theory of Natural Philosophy" (Latin-English), 1912, p. 187.

3. J. J. Thomson. 1904. Phil. Mag. 7, p. 237.
1907. Corpuscular Theory of Matter, Ch. VI.
1921. Phil. Mag. 41, p. 510.
1923. Electron in Chemistry (reprint of papers in J. Frank. Inst. 1923), Franklin Institute, Ch. I.

4. L. Föppl. 1912. J. Reine Angew. Math., 141. p. 251.
1912. Proc. 5th Int. Congr. Math., Cambridge, C.U.P., Vol. II, p. 188.
1912. "Stabile Anordnungen von Elektronen im Atom." Inaugural Dissertation. Göttingen. (Printed: Reiner, Berlin.)

5. L. Fejes (or L. Fejes Tóth). 1942. In Hungarian. (See Math. Revs. 1948, p. 302.)
1943. Jber. D. Math. Ver., 53, p. 66. (Math. Revs. 1947, p. 167.)
1944. In Hungarian (Math. Revs. 1947, p. 481.)
1948. Bull. Am. Math. Soc., 54, pp. 139, 431.
1948. Duke Math. Journ., 15, p. 817.
1949. This MONTHLY, 56, p. 330.
1951. Math. Nachr., 5, p. 253.
1952. Forthcoming book: "Lagerung in der Ebene, auf der Kugel, und im Raum." Springer.

6. H. Hadwiger. 1942/3. See Fejes (1949), p. 331.

7. H. Rutishauser. 1944/5. Comm. Math. Helv., 17, p. 327. (Work done 1942/3 under H. Hadwiger.)

8. L. L. Whyte. 1931. Critique of Physics, p. 140.
1949. Nature, 163, p. 762.

9. I wish to thank all who have helped me, *e.g.* by discussions in the U.S.A., in 1949, especially L. V. Ahlfors, H. Davenport, P. Erdös, N. Goldberg, J. Leech, and H. Rademacher.

10. W. Habicht and B. L. van der Waerden. 1951. Math. Annal, 123, p. 223.

11. K. Schütte and B. L. van der Waerden. 1951. Math. Annal., 123, p. 96. See also H. S. M. Coxeter's review of these two papers, Math. Rev., 13, 1952, pp. 61, 155.

12. C. S. Ogilvy and L. Moser, 1951. This MONTHLY, 58, p. 592. (E. 946).

ALGEBRAIC PROPERTIES OF N-VALUED PROPOSITIONAL CALCULI

J. D. SWIFT, University of California, Los Angeles

1. Introduction. This paper undertakes to investigate n-valued propositional calculi or composition algebras of functions over a discrete range to the range from several algebraic standpoints. First, the general algebra is considered in terms of its basic structure. Secondly, the binary algebra is studied with particular reference to the rings it may form under special choice of a pair of functions defined as sum and product. Thirdly, a partial ordering of the range being introduced, the lattice properties of the algebra and of the ideals in the ring structure are briefly dealt with. The concluding section is concerned with independent generating functions, analogues of the Sheffer stroke in the two-valued case, and includes as its principal result, the listing of all such commutative analogues for the three-valued case.

In each of the first three topics, a portion of the material is expository in nature. An attempt is made to unify and to some extent generalize the viewpoints of various investigators in this field, particularly B. A. Bernstein, Webb, Rosenbloom, and E. T. Bell. The discussion is algebraic throughout, terms peculiar to mathematical logic are used, when possible, only as alternatives to their algebraic counterparts. The algebraic terminology employed is restricted to that which is common to standard works in abstract algebra such as [3] [4] and [10] and definitions of concepts employed may be found in these texts. The standard terminology of mathematical logic is employed where practicable [8]. Where extended or new definitions are required they are given in the context.

2. The discrete composition algebra. An m-ary composition algebra consists of a set of functions of m variables defined over a range to the range and closed under the basic operation of composition, *i.e.*, if $f, f_1, f_2, \cdots, f_m$ are members of the set, then there is a (unique) member: $f(f_1, f_2, \cdots, f_m)$.*

If the range is finite and discrete, the total number of functions from the range to the range will in turn be finite and this totality will clearly form a composition algebra. If the range consists of n marks, this latter algebra may be termed an n-valued m-ary propositional calculus. It is defined up to naming and ordering of the marks, and for fixed m and n any two such algebras are isomorphic.

In the n-valued case any function may be given by its list of values corresponding to all possible combinations of the variables. A convenient choice for the mark of the range is the set of integers, $0, 1, \cdots, n-1$. In this case the table may be formed by writing in a column the numbers from 0 to n^m-1 to the base n adjoining sufficient zeros on the left of each number to give a total

* It will be noted that $f(\cdots\cdots)$ here denotes the *function* rather than its value at some point of the range. This notation is adopted in the belief that it will prove less confusing than the standard semicolon usage in the case of multiple composition. When a value of a function is used in the sequel, the change in meaning will be apparent from the context.

of m digits; the result will be all possible combinations of the variables, the corresponding values of the function may then be adjoined. Each function will thus be represented by one of the numbers from 0 to $n^{n^m}-1$, expressed to the base n with adjunction of zeros where necessary to give n^m digits and set beside the table of the variables. We may also consider the functions as representing the possible m-ary multiplication tables for all closed operations on n elements. The first concept is the common logical one of "truth tables"; the second is of considerable use in abstract algebraical considerations.

Composition is an $m+1$-ary operation, ordering a single element to each ordered set of $m+1$ elements. The algebra can also be described as containing n^{n^m} m-ary operations, one corresponding to each element. Among these there will be some whose result is independent of the functions appearing in certain of the positions of the ordered m-uple. Thus a constant function will be independent of all its arguments; a singulary function will depend on just (a particular) one of the arguments, *etc.* It has been shown that any function may be represented as a composition of functions which are binary or simpler in character. Thus for the most part we may confine our attention to the case $m=2$. Here we tread on familiar ground for the typical algebraic properties have long been named and studied.

We may, for example, speak of commutative and associative functions: $f(f_1, f_2)=f(f_2, f_1)$; $f(f_1, f(f_2, f_3))=f(f(f_1, f_2), f_3)$; f_1, f_2, f_3 all arbitrary.

Finally, it will be desirable to specify the "range variables," x and y, which may be thought of as the individual columns of the range table or their associated singulary functions: $x(f_1, f_2)=f_1$; $y(f_1, f_2)=f_2$ for all f_i, or again the (necessarily unique) pair of functions such that $f(x, y)=f$ for all f. In the m-ary case there will be m such functions: $x_1, x_2, \cdots, x_m$.

3. Algebraic properties:

a. Isomorphism and automorphism; conjugate functions: It is obvious from the definition that any biunique mapping on the set of marks to another set will introduce an isomorphism or abstract identity between the resulting complete composition algebras. In particular, a permutation of the marks will produce an automorphism. The group of these automorphisms is the symmetric group on n marks. Let $f_1, f_2, \cdots, f_k$ be the set of functions on which the automorphisms map f; then k divides $n!$ and there is a subgroup H_f of order $n!/k$ which leaves each of the f_i invariant. We say that f belongs to H_f. It is easy to construct functions belonging to any designated subgroup of the symmetric group. In particular, the range variables belong to the full group; the constant functions to the symmetric subgroup of order $(n-1)!$, *etc.* We shall be particularly interested in those functions belonging to the identity, $k=n!$. In the two-valued case the functions associated by the automorphisms are called duals. In the m-valued case, they will be called conjugates. To sum up:

THEOREM 1. *Each function possesses k conjugates where k divides $n!$ and belongs to a subgroup of order $n!/k$. There are functions in the algebra belonging to*

each subgroup of the symmetric group including the identity and the full group.

b. The algebra as a direct sum. It has been remarked that each function could be considered as a vector with components against the basic array. This vector may, in turn, be considered as an element in a direct sum algebra where each of the summands is the set of the marks. Any f orders a particular mark to any ordered m-tuple of the marks. Thus $f(f_1, f_2, \cdots, f_m)$ may be described as the result of the operation f on the elements $f_1, f_2, \cdots, f_m$ of the direct sum. This concept is of particular value in finding functions, $f, g, \cdots$ which, considered as operations, exhibit the algebra as possessing a particular algebraic structure.

For example, we may ask, "Under what operations is the binary two-valued algebra a ring?" The answer is: Under those operations derived by the direct sum from those under which two marks form a ring. There are, up to isomorphism, two such rings, one isomorphic to the ring of integers modulo 2, and a nil-product ring. By direct sum, the former gives the ring of Stone and Bernstein and its dual.

It will be convenient to use as an alternate to the term "finite complete composition algebra," the words "propositional calculus," and to designate the m-ary, n-valued case by $PC(n, m)$.

THEOREM 2. *Every binary propositional calculus contains at least one pair of functions, f^+ and $f^\times$, with respect to which the calculus is a ring.*

THEOREM 3. *Under no set of operations is any propositional calculus a division ring or, a fortiori, a field.*

Proofs. Theorem 2: The integers $0, 1, \cdots, n-1$ form a ring modulo n.

Theorem 3: The direct sums are not simple.

The strongest structure which can be expected is a semi-simple ring; this exists for any value of n. It is merely necessary to pick basic operations under which the marks form a semi-simple ring. If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, choose basic operations under which the marks themselves are the direct sums of the Galois Fields with $p_i^{a_i}$ elements. The calculus is thus the direct sum of direct sums of fields, hence semi-simple.

THEOREM 4. *For any n, at least two operations exist under which the calculus is a semi-simple ring.*

c. Polynomial representations. In certain cases the functions of the algebra can be represented as polynomial functions in the range variables. Bernstein [2] has done this when n is a prime; there is a simple generalization to the case, $n = p^a$.

THEOREM 5. *If $n = p^a$, the $PC(m, n)$ is isomorphic to the set of polynomial functions with coefficients from the Galois field of n elements, $GF(n)$.*

Proof: Select, by the direct sum process, two operations, f^+ and $f^\times$ under which the marks form the $GF(n)$; using these basic operations form the set of polynomial functions on the range variables with the constant functions as coefficients. This set contains n^{n^m} unequal elements since $x_i^n = x_i$, and the theorems concerning the number of zeros of a polynomial with coefficients in a field are valid here. The one-one mapping to the functions of the statement of the theorem is evident and the isomorphism follows from the fact that the process of composition is precisely that of substituting functions for the range variables in the polynomials.

The last theorem furnishes, for applicable cases, an extremely simple proof of a general principle previously mentioned:

COROLLARY. *If $n = p^a$, any m-ary function can be represented as a composition of binary functions.*

Proof: The binary functions are the f^+ and $f^\times$ of the preceding theorem.

To illustrate the application of the polynomial representation, let us determine the possible implications in the three-valued case which satisfy the following requirements:

1) $p \cdot p \supset q \cdot \supset q$ where conjunction is to have its commonly accepted meaning of "minimum truth-value."

2) The implication should reduce to the standard two-valued function if the "intermediate" mark is dropped.

Let the marks in ascending order of truth value be 0, 1, and 2, and designate as sum and product the corresponding operations of the $GF(3)$. Conjugation is then represented by $C(x, y) = 2xy(xy+x+y+2)$. We desire $I(x, y) = ax^2y^2 + bx^2y + cxy^2 + dx^2 + ey^2 + fxy + gx + hy + j$ such that $I(C(x, I(x, y)), y) = 2$. Also the values of $I(x, y)$ must be such that $I(0, a) = 2$, $a = 0, 2$; $I(2, 0) = 0$, $I(2, 2) = 2$. Substituting and comparing coefficients, 20 possible functions result. The values for the cases not already determined may be tabulated:

x	y	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2
1	0	0	0	2	0	2	2	2	2	2	0	0	2	0	0	0	2	2	0	0	2
1	1	2	2	0	0	2	2	0	0	2	0	0	2	2	2	0	2	2	2	2	0
1	2	2	0	0	0	2	0	2	0	0	2	0	2	0	2	2	2	0	2	0	2
2	1	2	0	2	2	2	0	2	0	1	2	0	0	1	0	0	1	2	1	2	0

If further:

3) The value of $I(x, y) \neq 2$ if the truth value of x exceeds that of y, there remain: 2, 11, 13, 14, 15, 18. Of these, 13 does not satisfy:

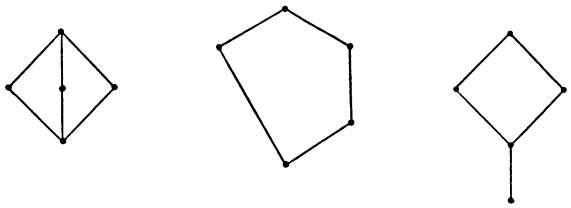
4) Implication is transitive. Finally, 14, 15, and 18 also satisfy:

5) The value of $I(x, y) = 2$ when the truth value of x is less than that of y. On intuitive grounds, a satisfactory case for the adoption of any of these three (or perhaps some of the others) might be made, depending on the individual interpretation of the mark 1. *Cf.* Webb [13].

d. Partial ordering on the marks: For any application of the propositional calculus to logic, there must exist a partial ordering relation on the marks, and, in general, a "true" or maximal mark and a "false" or minimal mark. This idea has already been used in the example. We now assume that the marks form a lattice. This is at once the simplest case with any actual generality and the most interesting one. In this case the propositional calculus also is a lattice, a direct sum of the lattices of the marks. We may define an implication relation, $\supset$, between elements of the calculus such that for two vectors, $V_1=(v_{11}, v_{12}, \cdots, v_{1n^m})$ and $V_2=(v_{21}, v_{22}, \cdots, v_{2n^m})$, $V_1 \supset V_2$ if and only if $v_{1j} \leqq v_{2j}$ for all j. Then in terms of the lattice representation, one element implies another if it is contained therein. The meet of two elements is the maximal element implying both and the join is the minimal element which both imply.

These lattices will be on n^m base lattices and will be n^m base lattices in length. Given any n, it is possible to exhibit the structure of all possible $PC(n, m)$ in a simple manner. There will always be the case where the marks are simply ordered and form a chain. The result is a direct sum of chains, a distributive lattice. This case has been studied in detail by P. Rosenbloom [9]. Their lattice properties are simple and readily noted. *E.g.*, they are complemented only for $n=2$ in which case they are Boolean Algebras.

For $n=2$ and 3, the chains are the only base lattices available. If $n=4$, the simple diamond is also possible; this form produces a Boolean algebra, a possibility when and only when n is a power of 2. For $n=5$, there are three possibilities besides the chain:



The first is modular but not distributive, complementation is not unique; the second is non-modular and non-uniquely complemented; the third is distributive but not complemented. The loss of the Boolean properties in the higher order calculi increases the difficulty of working with them and makes difficult or impossible the extension of theorems of the two-valued case. A strong case might be made for the diamond-based four valued logic as the "most reasonable" extension as in this case the well-known example is available as an endomorphic sub-lattice.

e. Ideals:

THEOREM 6. *The ideals of the $PC(n, m)$, written as a semi-simple ring, form a Boolean algebra on kn^m points, where k is the number of Galois Fields of which the ring of the marks is a direct sum. If $n=p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$, $r \leqq k \leqq \sum a_i$.*

This is essentially a specialization of a well-known algebraic result, and the proof is immediate when the full ring is regarded as the direct sum of kn^m Galois Fields.

The lattice will have 2^{kn^m} elements compared to the n^{n^m} elements of the calculus. By definition, $2^k \leqq n$, the equality holding only when $n=2^s$ and $k=s$.

If $k=1$, $n=p^a$, a simple logical interpretation may be given to the ideals. Let 0 and α be the minimal and maximal marks with respect to truth value. All ideals in this ring are principal and generated by elements having the minimum number of zeros in their vector representation. One of these generators consists entirely of α's and zeros. (Let V be any generator, and consider αV^{n-1}). The ideal then consists of all elements implied by the generator.

4. Independent generating functions. In 1913, Sheffer [10] showed that there was a binary operation from which all other elements could be generated by composition from the basic elements. In 1925, Zylinski [15] showed that this operator and its dual were the only such binary operations for the two-valued case. In 1921, Post [7] gave an argument which produces such an operation for the n-valued case. Explicit derivations and definitions of other such elements were given by Webb [12], [14] and Martin [6].

Some general considerations concerning such elements follow:

a. If an independent generator is known, all its conjugates are independent generators.

b. Any independent generator has $n!$ conjugates. If a certain element is generated by a particular composition sequence, conjugation must yield the conjugate elements. Thus, if the generator did not possess its full complement of conjugates, it cannot generate those elements with $n!$ conjugates.

c. Any independent generator must generate all the singulary functions.

d. No independent generator can be commutative and associative. If $f(f(x, y), z)=f(x, f(y, z))$ and $f(x, y)=f(y, x)$, the sequence of composition could be represented as powers of x and y and any function would be equivalent to a form x^iy^k. Further $x^n=x$ since each row in the function table must have a period with respect to exponentiation which is a divisor of n. Thus only n^2 elements can be generated by such a function.

e. The value of a generator, f, a "double mark," say $x=y=b$, must not be b. If it were, the composition process would never produce a mark other than b in this place in the function table.

f. The function must be non-degenerately binary. (The last two remarks suffice to establish Zylinski's result that the Sheffer stroke and its dual are the only possible generators for the two-valued case. Of the four elements which pass requirement e), two are the negatives of the range variables.)

g. Further, no mark may be "trapped" within a set of marks. *E.g.*, if for $n>2$, the values of $f(x, y)$ corresponding to the pairs (a, a), (a, b), (b, a), (b, b) consist only of a's and b's, no other mark could be produced for the pairs (a, a) and (b, b).

With these remarks we will proceed to obtain all commutative single generators for the three-valued propositional calculus. There are, in all $3^6=729$ commutative functions. Applying requirement e), we have $2^3 3^3=216$ remaining, which reduce to 108 when g) is applied. Of these, six are eliminated by b) as they are conjugate two by two of three separate classes. It is interesting to note that one such class is the only one of the 20 which satisfies the requirements of solvability and is thus a commutative loop operation. *I.e.*, for these functions, f, and for all functions, a and b, there exists a unique function, c, such that $f(a, c)=b$.

Using the marks, 0, 1, 2, and abbreviating the basic table by reason of the commutativity, the last rejected elements are:

		α_1	α_2	β_1	β_2	γ_1	γ_2
0	0	1	2	1	2	1	2
1	1	2	0	2	0	2	0
2	2	0	1	0	1	0	1
0	1	0	1	1	0	2	2
0	2	2	0	0	2	1	1
1	2	1	2	2	1	0	0

Here α_l are the loop operations. The rejection of α, β, γ provides a counter example to a plausible conjecture that all elements whose values on the "double lines," (a, a), represented a permutation of the marks such that no element was unaffected, are generators. All of these elements are of this type. All previously known generators were of this type.

The permutations performed on base elements and vectors to obtain the conjugates will be designated: 1. Identity; 2. Interchange 1 and 2; 3. Add 1 modulo 3; 4. Interchange 0 and 1; 5. Add 2 modulo 3; 6. Interchange 0 and 2. The 102 remaining elements constitute 17 classes. For brevity, we list one example of each class. In the sequel the elements will be distinguished by the number in the following list and a subscript indicating which permutation produces the function in question.

		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	1	0	0	0	0	0	0	0	0	0	2	2	2	2	2	2	2	2
2	2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	2	2	2	2	2	2	2	2	2	0	0	0	0	0	0	1	1
0	2	0	0	0	1	1	1	2	2	2	0	0	0	1	1	1	0	1
1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2	0	0

For example, 14_2 is:

		0	2	1	0	0	2,	or:		0	1	2	0	0	1.			
		0	2	1	2	1	1			0	1	2	1	2	2			
		2	1	0	0	2	2			2	0	1	2	0	2			

Further notation: S will denote an arbitrary element chosen from 10_1 to 17_1, inclusive; P, one of 1_1 to 9_1. The letters x and y will denote the first and second basic elements, $\bar{x}$ and $\bar{\bar{x}}$ denote the first and second "negations" of x, the elements obtained from x by adding 1 and 2 respectively, modulo 3 to the values of x. In general, $S(x, x) = \bar{x}$; $S(\bar{x}, \bar{x}) = \bar{\bar{x}}$. Thus, all the S operations generate both negations. Let $S(\bar{x}, S) = S_x$, $S(\bar{\bar{x}}, S_x) = S_{x^2}$, $S(x, S_{x^2}) = S_{x^3}$.

We shall establish the following:

THEOREM 7. *Classes* 1, 2, 3, 5, 6, *and* 8–17 *inclusive are generators and these* 90 *operations are the sole commutative generators for the three-valued case.*

In the proof, Webb's generators will furnish the initial information, and other functions will successively be established as generating functions which have previously been proved to be generators. It will then be shown that 4 and 7 do not meet requirement c).

In following any of the proofs which do not depend simply on negation, it will be necessary to rewrite the functions in terms of the full truth table rather than employing the abbreviated table used in the definition. This is necessary because while S is commutative, S_x is in general not commutative.

1) 10_1 and 17_1 are Webb's generators.

2) $\overline{16_1} = 17_6$; thus the 16 class consists of generators.

3) $\overline{15_1(15_{1x}, 15_{1y})} = 11_3$.
$15_1(15_{1x}, 15_{1y^2}) = 12_4$, $\overline{12_4} = 13_1$.
$11_3(11_3, 13_1) = 10_4$; and 15 is a generator.

4) $\overline{13_1(13_{1x}, 13_{1y})} = 15_5$; therefore 13 is a generator.

5) $\overline{12_1} = 13_4$; and 12 is a generator.

6) $\overline{11_1(11_{1x}, 11_{1y})} = 13_3$; thus 11 is a generator.

7) $\overline{14_1} = 11_6$.

8) Let $\alpha(x) = \overline{P(x, x)}$; $x_p^n = P(x, x_p^{n-1})$, $x_p^0 = x$. The subscript will be omitted when no ambiguity will result. Define $(x_p^n)^m$ as $P(x_p^n, (x_p^n)^{m-1})$, and $(x_p^n)^0 = x^n$. This notation is essentially that of Webb. Let $P = 1_1$, $x^4 = \bar{x}$ and $\alpha(x)$ is thus generated by P. Then: $1_1(1_1, \alpha(1_1(\bar{x}, \bar{y}))) = 17_4$. Therefore 1 is a generator.

9) Let P be 2_1, denote $(x^1)^2$ by $\beta(x)$ and $2_1(\beta, \beta)$ by $\gamma(x)$. Then $\bar{x} = 2_1(x, \gamma)$ so 2_1 generates $\alpha(x)$. Now $2_1(2_1, \alpha(2_1(\bar{x}, \bar{y}))) = 12_2$ and 2 is a generator.

10) Let $P = 3_1$; $x^4 = \bar{x}$; $3(3_1, \alpha(3_1(\bar{x}, \bar{y}))) = 10_2$ and 3 is a generator.

11) Let $P = 5_1$; $x^4 = \bar{\bar{x}}$; $5_1(\overline{5_1}, 5_1(\bar{x}, \bar{y})) = 15_3$ and 5 is a generator.

12) Let $P = 6_1$; $x^7 = \bar{x}$; $\overline{6_1}(6_1, x^1(6(\bar{\bar{x}}, \bar{\bar{y}}))) = 16_5$ and 6 is a generator.

13) Let $P = 8_1$; $8_1((x^1)^1, x^6) = \bar{\bar{x}}$; $\overline{8_1}(8_1, x^1(8_1(\bar{\bar{x}}, \bar{\bar{y}}))) = 10_1$ therefore 8 is a generator.

14) Let $P = 9_1$; $9_1(x, 9_1(x^2, x^3)) = \bar{x}$; $\overline{9_1}(9_1, x^1(9_1(\bar{\bar{x}}, \bar{\bar{y}}))) = 10_1$ and 9 is a generator.

15) To prove that 4 is not a generator, we shall show that no possible form beginning with the basic form (012) will result in (001).

a. To generate (001) by 4_1 requires at least one of: (110), (120), (210), or (220).

b. To generate one of the forms in a) requires either forms already in a) or one of the following: (002), (201), (202), (101), (102).

c. To generate any of the preceding, requires either one of those already listed or one of: (010), (020).

d. To generate (010) or (020) requires one of the forms already listed; the chain is then closed and (001) cannot be produced.

16) Similarly 7 is not a generator. We may vary the argument and demonstrate that from the basic form (012), 7_1 will generate only (000), (011), (012), (022), (100), (111), (122), (200), (211), and (222); and the chain is closed. Thus (001) will not be generated.

Some general conclusions may be mentioned in addition to the hypothesis discussed following the rejection of α, β and γ. First, there are in general, independent generators whose values on the double lines are not simple permutations of the marks; 42 such have just been produced. Second, there is no simple rule by which independent generators may be distinguished by their table of values; a comparison of 4 and 7 with some of the others will make this apparent. Third, the number of generators is not, at least absolutely speaking, small. From the two cases available no intelligent conjecture as to the number of such generators for the n-valued case is possible but it is interesting to note that one-fourth of the commutative functions for the two-valued case satisfy the requirements while roughly one-eighth of those in the three-valued case are generators. Recently N. M. Martin has undertaken the task of continuing the results given in this section by determining the non-commutative Sheffer strokes also for the three-valued case.

Bibliography

1. Bell, E. T., Polynomials on a finite discrete range, Duke Math. J. 10, 1943, pp. 33–47.

2. Bernstein, B. A., Modular representations of finite algebras, Proc. Int. Math. Congress, Toronto 1924; Vol. 1, pp. 207–216.

3. Birkhoff, Garrett, Lattice Theory, Am. Math. Soc. Colloquium Pub. Vol. XXV, Revised 1948.

4. ——— and MacLane, S., Survey of Modern Algebra, Macmillan, 1948.

5. Lewis and Langford, Symbolic Logic, Century Co., 1932.

6. Martin, N. M., Some analogues of the Sheffer stroke function in n-valued logic, Indagationes Math. 1950, Vol. 12, Fasc. 4, pp. 393–400.

7. Post E. L., Introduction to a general theory of propositions, Amer. J. Math. 43, 1921, pp. 163–185.

8. Quine, W. V. O., Mathematical Logic, Norton, 1940.

9. Rosenbloom, P. C., Post algebras, Amer. J. Math., 64, 1942, pp. 167–188.

10. Sheffer, H. M., A set of five independent postulates for Boolean algebras with application to logical constants, Trans. Amer. Math. Soc., 14, 1913, pp. 481–488.

11. Van der Waerden, Modern Algebra, F. Ungar, 1950.

12. Webb, D. L., Generation of any n-valued logic by one binary operation, Proc. Nat. Acad. Sci. U.S.A., 21, 1935, pp. 252–254.

13. ———, The Algebra of N-valued logic, C. R. Soc. Sci. Varsovie, 29 Classe III, 1936, pp. 1–16.

14. ———, Definition of Post's generalized negative and maximum in terms of one binary operation, Amer. J. Math., 58, 1936, pp. 193–194.

15. Zylinski, E., Some remarks concerning the theory of deduction, Fund. Math. 7, 1925, pp. 203–209.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

A NEW PROOF OF THE COMPLETENESS OF THE HERMITE FUNCTIONS

H. P. MCKEAN, JR., Dartmouth College

The purpose of this note is to give a short proof of the completeness in L_2 of the Hermite functions.

We define the nth Hermite function to be

$$\psi_n(x) = (-)^n e^{\frac{1}{2}x^2}(2^n n!\sqrt{\pi})^{-\frac{1}{2}} \times \frac{d^n}{dx^n} e^{-x^2}, \qquad n = 0, 1, 2, \cdots .$$

As is well known, these functions form an orthonormal set over the infinite line, and

$$(1) \qquad e^{\frac{1}{2}x^2}e^{-(x-\lambda)^2} = \sum_{n=0}^{\infty} \lambda^n \left(\frac{2^n\sqrt{\pi}}{n!}\right)^{\frac{1}{2}} \psi_n(x), \qquad (-\infty < \lambda < \infty, -\infty < x < \infty).$$

With the aid of (1) and the Riesz-Fischer theorem, it is easy to show that

$$(2) \qquad e^{\frac{1}{2}x^2}e^{-(x-\lambda)^2} \sim \sum_{n=0}^{\infty} \lambda^n \left(\frac{2^n\sqrt{\pi}}{n!}\right)^{\frac{1}{2}} \psi_n(x), \qquad (-\infty < \lambda < \infty, -\infty < x < \infty).$$

We now set $2\lambda = \mu$ and $\beta_n = e^{3\lambda^2}\lambda^n(2^n\sqrt{\pi}/n!)^{\frac{1}{2}}$ and rewrite (2) in the following form:

$$e^{-\frac{1}{4}(x-\mu)^2} \sim \sum_{n=0}^{\infty} \beta_n \psi_n(x), \qquad (-\infty < \mu < \infty, -\infty < x < +\infty).$$

Thus if $f(x)$ is any member of L_2 which can be represented in the form

$$f(x) \sim \sum_{0}^{m} A_n e^{-\frac{1}{4}(x-\mu_n)^2}, \qquad m < +\infty,$$

then we can also write

$$f(x) \sim \sum_0^\infty B_n \psi_n(x).$$

To prove the completeness of the Hermite functions it is then sufficient to prove that the set of all translations of $e^{-\frac{1}{2}x^2}$ is complete in L_2, and this is easily done by means of the following theorem, whose proof does not depend on the completeness of the Hermite functions.

THEOREM (Wiener*) *Let $f(x)$ be a member of L_2. The set of all its translations will be complete in L_2 if and only if the real zeros of*

$$G(u) = \text{l.i.m.}_{\beta\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\beta}^{\beta} e^{iux} f(x)dx$$

form a set of measure zero.

For if we take $f = e^{-\frac{1}{2}x^2}$ then $G = e^{-\frac{1}{2}u^2}$, Wiener's theorem holds, and the proof is complete.

THE APPROXIMATION OF RADICALS BY RATIONAL MEANS

H. F. SANDHAM, Trinity College, Dublin, Ireland

1. **Introduction.** In this paper we explain a method of approximating radicals of any order. Since a general statement is unavoidably complicated, the particular case of fourth roots, written out in full, serves to illustrate the general argument.

If a_0, b_0, c_0, d_0 are positive, and

$$a_{n+1} = \frac{a_n + b_n + c_n + d_n}{4},$$

$$b_{n+1} = \frac{\dfrac{a_nb_n + a_nc_n + a_nd_n + b_nc_n + b_nd_n + c_nd_n}{6}}{\dfrac{a_n + b_n + c_n + d_n}{4}},$$

$$c_{n+1} = \frac{\dfrac{b_nc_nd_n + a_nc_nd_n + a_nb_nd_n + a_nb_nc_n}{4}}{\dfrac{a_nb_n + a_nc_n + a_nd_n + b_nc_n + b_nd_n + c_nd_n}{6}},$$

$$d_{n+1} = \frac{a_nb_nc_nd_n}{\dfrac{b_nc_nd_n + a_nc_nd_n + a_nb_nd_n + a_nb_nc_n}{4}}, \qquad n = 0, 1, 2, \cdots,$$

* Wiener, N., Tauberian theorems, Annals of Math., 1932.

we prove that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = \lim_{n\to\infty} d_n = (a_0b_0c_0d_0)^{1/4}.$$

2. **Proof.** The idea of the proof is this: Obviously

$$a_{n+1}b_{n+1}c_{n+1}d_{n+1} = a_nb_nc_nd_n,$$

whence

$$a_nb_nc_nd_n = a_0b_0c_0d_0, \qquad n = 1, 2, \cdots.$$

It is now necessary to prove that a_n, b_n, c_n, d_n tend to the same limit. We do this, first by proving [1, 2],

$$a_{n+1} \geqq b_{n+1} \geqq c_{n+1} \geqq d_{n+1}, \qquad n = 0, 1, 2, \cdots;$$

whence it is easily deduced that

$$a_{n+1} \leqq a_n, \qquad d_{n+1} \geqq d_n, \qquad n = 1, 2, \cdots.$$

Then we prove that $\lim_{n\to\infty}(a_n - d_n) = 0$. The result then follows.

First since

$$a_{n+1} - b_{n+1} = \frac{\sum (b_n + c_n + d_n - a_n - c_n - d_n)^2}{12\sum a_n},$$

$$b_{n+1} - c_{n+1} = \frac{\sum (b_nc_n + b_nd_n + c_nd_n - a_nc_n - a_nd_n - c_nd_n)^2}{6(\sum a_n)(\sum a_nb_n)},$$

$$c_{n+1} - d_{n+1} = \frac{\sum (b_nc_nd_n - a_nc_nd_n)^2}{2(\sum a_nb_n)(\sum a_nb_nc_n)},$$

therefore $a_{n+1} \geqq b_{n+1}$, $b_{n+1} \geqq c_{n+1}$, $c_{n+1} \geqq d_{n+1}$; that is

$$a_n \geqq b_n \geqq c_n \geqq d_n, \qquad n = 1, 2, \cdots.$$

Hence

$$a_{n+1} = \frac{a_n + b_n + c_n + d_n}{4} \leqq a_n,$$

$$d_{n+1} = \frac{4}{\frac{1}{a_n} + \frac{1}{b_n} + \frac{1}{c_n} + \frac{1}{d_n}} \geqq d_n, \qquad n = 1, 2, \cdots.$$

Again, since

$$a_{n+1} - d_{n+1} = \frac{\sum a_nb_n(c_n - d_n)^2}{4\sum a_nb_nc_n},$$

it follows that

$$a_{n+1} - d_{n+1} \leqq \frac{\sum a_n b_n \mid c_n - d_n \mid}{4\sum a_n b_n c_n} \cdot (a_n - d_n)$$

$$= \frac{-3b_n c_n d_n - a_n c_n d_n + a_n b_n d_n + 3a_n b_n c_n}{4(b_n c_n d_n + a_n c_n d_n + a_n b_n d_n + a_n b_n c_n)} \cdot (a_n - d_n). \tag{1}$$

For fifth roots, these statements would be

$$a_{n+1} - e_{n+1} = \frac{\sum a_n b_n c_n (d_n - e_n)^2}{5\sum a_n b_n c_n d_n},$$

$$a_{n+1} - e_{n+1} \leqq \frac{-4b_n c_n d_n e_n - 2a_n c_n d_n e_n - 0a_n b_n d_n e_n + 2a_n b_n c_n e_n + 4a_n b_n c_n d_n}{5(b_n c_n d_n e_n + a_n c_n d_n e_n + a_n b_n d_n e_n + a_n b_n c_n e_n + a_n b_n c_n d_n} \cdot (a_n - e_n).$$

From (1), the crude inequality

$$a_{n+1} - d_{n+1} \leqq \tfrac{3}{4}(a_n - d_n)$$

is sufficient for our purpose.

References

1. These inequalities were stated by Newton.

2. For other proofs see G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities, Cambridge, 1934, pp. 52, 104.

CUBIC EQUATIONS IN GROUPS

DAVID MCLEAN, Detroit, Michigan

In groups, an analogue of the nth degree equation is the equation, $UA_{n-1}UA_{n-2} \cdots A_2UA_1U = I$. This paper concerns the general solution to the above equation when $n = 3$. By a general solution, I mean: an expression containing only the coefficients A_1 to A_{n-1} and the operations of multiplication, division, and root-extraction (assuming that the latter has meaning), which, when substituted for U in the equation, reduces the equation to an identity. The quadratic equation $UA_1U = I$ has the general solution $U = \sqrt{A_1^{-1}}$.

I shall now prove that the cubic equation $UA_2UA_1U = I$ has no general solution in the sense described above. Suppose that there were an expression that would reduce the cubic equation to an identity when substituted for U. Since, in this expression, there must be only a finite number of root extractions, there must be, at one point, a highest root, say the Nth, extracted. It follows that in any group, in which all root extractions of degree equal to or less than the Nth have meaning, all cubic equations in the group have solutions. I will now show that for any N there is a group in which all root extractions of degree equal to or less than N have meaning in the group, but such that there is a cubic equation in it with no solution and hence the absurdity of a general solution.

If P is the smallest divisor ($\neq 1$) of the order h of the group H, then the equation $U^k = q$ has a solution for all q in H if k is less than P. In fact U is in the cyclic subgroup generated by q.

Now I define a group* by the following relations between its generators α and β, with $t^r \equiv 1 \pmod{s}$:

$$(1)\ \alpha^r = I \qquad (2)\ \beta^s = I \qquad (3)\ \beta\alpha = \alpha\beta^t.$$

From relation (3) we obtain the relation $b^m\alpha^n = \alpha^n\beta^{mt^n}$. The reader may easily verify that any product of the α's and β's can be represented uniquely by the form $\alpha^m\beta^n$, and that these forms constitute a group of order rs.

In the above defined group, the cubic equation becomes

$$U\alpha^c\beta^d U\alpha^e\beta^f U = I,$$

where $U = \alpha^x\beta^y$ is the unknown. By multiplying out, we obtain

$$\alpha^{3x+c+e}\beta^{y(t^{2x+c+e}+dt^{2x+e}+yt^{x+e}+ft^x+y}} = I.$$

This is equivalent to the set of congruences

(i) $$3x + c + e \equiv 0 \pmod{r}.$$

(ii) $$y(t^{2x+e+c} + t^{x+e} + 1) + dt^{2x+e} + ft^x \equiv 0 \pmod{s}.$$

Now I shall prove the theorem by showing that, for any N, I can find numbers r, s, t, c, d, e, and f, defining the aforementioned group, such that the smallest divisor, greater than one, of rs is also greater than N and further that the congruences (i) and (ii) will have no simultaneous solution.

Let s be a prime of the form $12n-1$ and let h be a primitive root of s. Also let $t \equiv h^2 \pmod{s}$; $r = (s-1)/2 = 6n-1$, and let $d = 0$ and $f = 1$.

If the congruence $t^A + t^B + 1 \equiv 0 \pmod{s}$ holds for some integers A and B, then the congruences (i) and (ii) have no simultaneous solution for some integer values of c and e.

Proof: Let $A + B \equiv e \pmod{r}$ and $2A - B \equiv c \pmod{r}$, so $3A \equiv c + e \pmod{r}$. And if congruence (i) is to hold $-3x \equiv 3A \pmod{r}$, or $-x \equiv A \pmod{r}$; but $2x + c + e \equiv -x \equiv A \pmod{r}$ by congruence (i), and $x + e \equiv A + B - A \equiv B \bmod r$, so $t^A + t^B + 1 \equiv t^{2x+e+c} + t^{x+e} + 1 \equiv 0 \pmod{s}$, and so the congruence (ii) has no solution. The congruence $t^A + t^B + 1 \equiv 0 \pmod{s}$ has a solution if the congruence $h^Q + h^W + 1 \equiv 0 \pmod{s}$ has a solution where both Q and W are even.

In the solutions of the congruence $h^Q + h^W + 1 \equiv 0 \pmod{s}$ such that $Q \geqq W$ (where Q and W need not be even) all the reduced residues modulo s occur exactly once as either h^Q and/or h^W except the residue -1 since $h^W \equiv -1 \pmod{s}$ would imply that $h^Q \equiv 0 \pmod{s}$ which is absurd. Since h is a primitive root of s all of the reduced residues modulo s would appear once as either Q and/or W except $(s-1)/2$ since $h^{(s-1)/2} \equiv -1 \pmod{s}$.

* An interesting generalization of this group can be found in Zassenhaus "The Theory of Groups"; Chelsea Pub. Co., 1949, p. 98.

Now consider the set E of numbers 1, 2, 3, $4 \cdots s-1$ with the number $(s-1)/2$ removed. Since $(s-1)/2$ is odd, E has more even numbers in it than odd numbers. Also each solution of $h^Q+h^W+1\equiv 0 \pmod{s}$ is a pair of numbers (Q, W) taken from the set E, and no two pairs have the same number in common. Since E contains more even numbers than odd numbers, it follows that some solution pair (Q, W) is a pair of even numbers, and so the congruence $t^A+r^B+1\equiv 0 \pmod{s}$ has a solution.

It remains to be proved that the smallest divisor $(\neq 1)$ of the order of the group I have defined can be made greater than any preassigned number.

By a well known theorem the number of primes in any reduced residue class is infinite. And so, if we make $s\equiv -1 \pmod{3\cdot 4\cdot 5\cdot 7\cdot 11\cdot 13 \cdots K}$ where K is an arbitrarily large prime, then the smallest divisor of the order $s(s-1)/2$ of the group is greater than K, and the theorem is proved.

It is interesting to notice that in the group defined by the relations,

$$(1)\quad \alpha^7 = I, \qquad (2)\quad \beta^{43} = I, \qquad (3)\quad \beta\alpha = \alpha\beta^4,$$

the cubic equation $U\alpha U\alpha^4\beta U = I$ has no solution, but it can be verified by trial and error that all *fourth* degree equations in this group have solutions.

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

ON THE ENVELOPE OF CURVES GIVEN IN PARAMETRIC FORM

J. W. GREEN, University of California, Los Angeles

In a recent calculus class, the writer asked for the envelope of the family of ellipses in standard position, the sum of whose major and minor axes is 2. One student argued as follows. The parametric equations of the ellipse are $x=a\cos\theta$, $y=b\sin\theta$ or equivalently, $a=x/\cos\theta$, $b=y/\sin\theta$. It is required that $a+b=1$; hence $x/\cos\theta+y/\sin\theta=1$. If this last equation is differentiated with respect to θ and θ eliminated, there results $x^{2/3}+y^{2/3}=1$, which is correct.

At first it appeared that the student had had a striking piece of good fortune, since what he had actually done was to find the envelope of the family of lines on which the axes cut off a segment of unit length. The two families have the same envelope. An explanation, however, of his good luck is to be found in a fact well known to many persons, especially to those mathematicians who work with the calculus of variations, where the subject comes up naturally, but apparently not widely familiar to mathematicians and teachers in general. The

writer has not seen more than a bare mention of it in any of the textbooks currently used for instruction in the calculus, and it is his opinion that such an excellent and instructive application of a number of the notions and processes of calculus is worth knowing.

What the student did is as follows. He had, replacing b by $(1-a)$, the parametric equations

$$x = a\cos\theta, \qquad y = (1-a)\sin\theta, \tag{1}$$

of a family of curves, where a is the parameter which distinguishes one curve of the family from the other, and θ is the "running parameter" of the individual curves. He interchanged the roles of these two parameters and found the envelope of a different family of curves without affecting the result.

The key to the situation lies in the condition for the envelope when the curves are given in parametric form:

$$x = f(\theta, a), \qquad y = g(\theta, a). \tag{1'}$$

Suppose that we eliminate the running parameter θ from equations (1') by solving for it from the first and substituting the result in the second. We have then for the equation of our family of curves in nonparametric form,

$$y = g(\theta(x, a), a), \tag{2}$$

where

$$x \equiv f(\theta(x, a), a). \tag{3}$$

We now obtain the condition for the envelope in the usual way by differentiating (2) with respect to the parameter a:

$$0 = g_1(\theta(x, a), a)\frac{\partial\theta}{\partial a} + g_2(\theta(x, a), a).$$

To obtain $\partial\theta/\partial a$ we differentiate (3) implicitly:

$$0 \equiv f_1(\theta(x, a), a)\frac{\partial\theta}{\partial a} + f_2(\theta(x, a), a).$$

Eliminating $\partial\theta/\partial a$ between the last two equations, we have the condition

$$\begin{vmatrix} f_1 & f_2 \\ g_1 & g_2 \end{vmatrix} = \frac{\partial(f, g)}{\partial(\theta, a)} = 0 \tag{4}$$

for the envelope, together with the equations (1'). In the case of the equations (1), $\partial(f, g)/\partial(\theta, a) = a - \cos^2\theta$, and substituting $a = \cos^2\theta$ in (1) gives $x = \cos^3\theta$, $y = \sin^3\theta$, the parametric equations of the four cusped hypocycloid.

The condition (4), being merely the vanishing of the Jacobian, remains unchanged if we reverse the roles of the two parameters and so the envelope of the families (1) is the same, whichever parameter plays which role. The point of view

taken here is, of course, the formal one. Whether either of the families has an envelope is in the general case quite another matter. As a matter of fact, in most of the more obvious examples, one of the families will have an envelope and the other not, as for instance the families $x=a\cos\theta$, $y=(\sin\theta)/a$ of ellipses or hyperbolas.

The derivation of the condition (4) for the envelope is a very good illustration of the use of the implicit methods and composite function methods of partial differentiation, and if rigor is sought, it is an excellent exercise in the use of the implicit function theorem. Equation (4) appears in the literature in numerous places, but we have not seen the dual role of the parameters mentioned. In Courant's *Differential and Integral Calculus*, (4) is stated without proof on page 174 of volume II. In fact, both of the examples contained in (1) are given as illustrations of envelopes, but the connection between them is not pointed out. Equation (4) is derived in de la Vallée Poussin's *Cours d'Analyse* on page 398 of volume II, and the equivalent condition for curves in three space in Picard's *Traité d'Analyse* on page 444 of volume I (1942 edition). Also on page 430 of volume I of Goursat-Hedrick's *Course in Mathematical Analysis* there is a discussion leading to the Jacobian from a similar starting point.

Actually we might well have anticipated that (4) would be the criterion for the envelope. For a typical family of curves, for instance the ellipses considered above, the envelope is the boundary of the region in the x, y plane into which the transformation (1) carries the a, θ plane. That is, the envelope is a locus of points x, y where equations (1) do not define implicitly a, θ as single valued functions. Thus it is not surprising that the Jacobian vanishes on the envelope.

INTRODUCTION OF COMPLEX NUMBERS AS VECTORS OF THE PLANE

L. Fuchs, Budapest, and T. Szele, Debrecen, Hungary

It is a well-known elementary fact that the complex number field is the only possible field extension of degree 2 of the field of the real numbers. This statement may be sharpened to the assertion that *if a (not necessarily commutative) associative algebra of order 2 over the real number field has no divisors of zero, then it is isomorphic to the field of complex numbers.* A vector space of dimension 2 over the real numbers has a simple geometric interpretation as the set of the vectors of the Euclidean plane, this fact being used several times for introducing the complex numbers. Hence it seems to be natural to raise the problem of proving the cited theorem in a purely elementary geometric way. Such a proof would help beginners to get a deeper insight into the kernel of this fact. Our present purpose is to give such a proof which seems to have some interest in itself and to be new.

In what follows let the vectors of the plane be denoted by bold-face and real numbers by Greek letters. We shall prove the following

THEOREM. *Suppose we have defined addition and multiplication of any two vectors of the plane as well as multiplication of a vector by a scalar (i.e., a real number) with the following properties*:

(1) *addition is vector-addition*; it is therefore commutative, associative, and unique subtraction is possible;

(2) *multiplication by scalars is performed as usual*; therefore we have $\alpha(\mathbf{a}+\mathbf{b})=\alpha\mathbf{a}+\alpha\mathbf{b}$, $(\alpha+\beta)\mathbf{a}=\alpha\mathbf{a}+\beta\mathbf{a}$, $(\alpha\beta)\mathbf{a}=\alpha(\beta\mathbf{a})$ for arbitrary real numbers α, β and vectors $\mathbf{a}$, $\mathbf{b}$;

(3) *multiplication of vectors is associative*: $\mathbf{a}(\mathbf{bc})=(\mathbf{ab})\mathbf{c}$;

(4) $(\alpha\mathbf{a})\mathbf{b}=\alpha(\mathbf{ab})=\mathbf{a}(\alpha\mathbf{b})$;

(5) *divisors of zero do not exist*: $\mathbf{ab}=0$ implies either $\mathbf{a}=0$ or $\mathbf{b}=0$;

(6) *distributivity holds*: $\mathbf{a}(\mathbf{b}+\mathbf{c})=\mathbf{ab}+\mathbf{ac}$ and $(\mathbf{b}+\mathbf{c})\mathbf{a}=\mathbf{ba}+\mathbf{ca}$.

Then, apart from an affine transformation, our vector system is just the set of all vectors of the complex plane with the well-known operations. In other words, if (1)–(6) *hold, then the vectors of the plane form a field isomorphic to the complex number field.*

Proof. Clearly, only the statement concerning multiplication needs a verification.

First we show that multiplication of vectors is commutative. Let us consider a vector $\mathbf{x}$ not parallel to its own square $\mathbf{x}^2$. Such an $\mathbf{x}$ certainly exists, for if $\mathbf{y}\neq 0$ is any vector parallel to $\mathbf{y}^2$, *i.e.* $\mathbf{y}^2=\lambda\mathbf{y}$ (plainly $\lambda\neq 0$), then each vector $\mathbf{x}\neq 0$ not parallel to $\mathbf{y}$ has the required property. In fact, if we had $\mathbf{x}^2=\mu\mathbf{x}$, then from the equations

$$(\mu\mathbf{y}-\mathbf{yx})\mathbf{x}=0 \quad \text{and} \quad \mathbf{y}(\lambda\mathbf{x}-\mathbf{yx})=0$$

we would get

$$\mu\mathbf{y}=\mathbf{yx}=\lambda\mathbf{x}$$

on account of (5). But this would mean that $\mathbf{x}$ and $\mathbf{y}$ are parallel, contrary to our hypothesis. Choosing a vector $\mathbf{x}$ with the prescribed property, it is evident that the vectors $\mathbf{x}$ and $\mathbf{x}^2$ span the plane. Hence each vector $\mathbf{a}$ may be represented (uniquely) in the form

$$\mathbf{a}=\alpha\mathbf{x}+\beta\mathbf{x}^2$$

with some real numbers α, β. For vectors of this type it is obvious, in view of the above axioms, that they commute. Therefore we obtain

$$\mathbf{ab}=\mathbf{ba}$$

for all vectors $\mathbf{a}$, $\mathbf{b}$.

For the square of the vector $\mathbf{a}=\alpha\mathbf{x}+\beta\mathbf{x}^2$ we have;

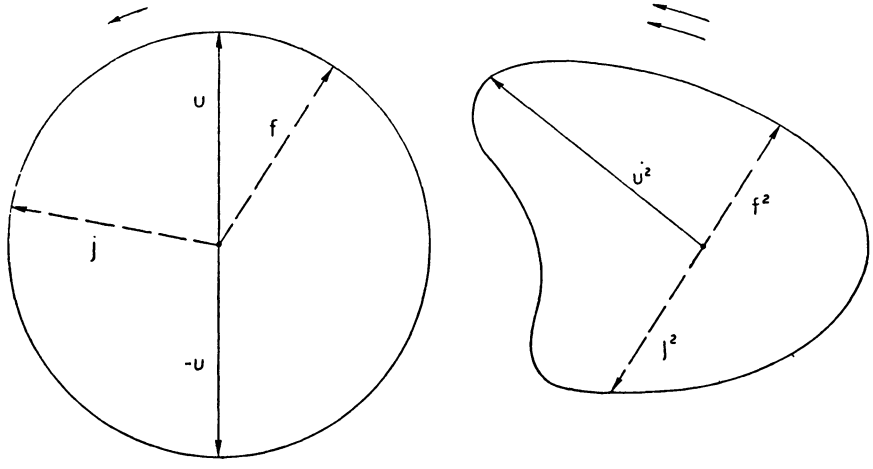
$$\mathbf{a}^2=\alpha^2\mathbf{x}^2+2\alpha\beta\mathbf{x}^3+\beta^2\mathbf{x}^4.$$

Hence it is seen that if we let $\mathbf{a}$ (*i.e.* α and β) vary a little, then $\mathbf{a}^2$ can vary but a little. Thus we get that if $\mathbf{a}$ varies continuously, then so does $\mathbf{a}^2$.

Now let $\mathbf{u}$ be an arbitrary unit vector and turn $\mathbf{u}$ through the angle π until it arrives at the position $-\mathbf{u}$. Meanwhile, according to what has been said above, the square vector has to rotate in one and the same direction, for, on the contrary, there would exist two non-parallel unit vectors $\mathbf{k}$ and $\mathbf{l}$ such that $\mathbf{k}^2=\lambda\mathbf{l}^2$ with $\lambda>0$. Hence, by commutativity, we have $(\mathbf{k}+\sqrt{\lambda}\mathbf{l})(\mathbf{k}-\sqrt{\lambda}\mathbf{l})=0$ and thus from (5) we are led to either of the equations

$$\mathbf{k}+\sqrt{\lambda}\,\mathbf{l}=0 \quad \text{and} \quad \mathbf{k}-\sqrt{\lambda}\,\mathbf{l}=0$$

contradicting the assumption on $\mathbf{k}$ and $\mathbf{l}$. Since $(-\mathbf{u})^2=\mathbf{u}^2$, we see that the square vector turns through the angle 2π, or -2π.



Let us now consider simultaneously the rotating of the unit vector and of its square (Figure). It is evident that during the rotation of the unit vector $\mathbf{u}$ through an angle 2π ($\mathbf{u}^2$ will then turn through an angle $\pm 4\pi$) $\mathbf{u}$ coincides with a vector $\mathbf{f}$ which is parallel to its own square in the same direction: $\mathbf{f}^2=\rho\mathbf{f}$ with $\rho>0$, and with a vector $\mathbf{j}$ whose square lies in the same direction as the vector $-\mathbf{f}$, that is, $\mathbf{j}^2=-\sigma\mathbf{f}$ with $\sigma>0$. Put $\mathbf{e}=(1/\rho)\mathbf{f}$ and $\mathbf{i}=(1/\sqrt{\rho\sigma})\mathbf{j}$, then we obtain

$$\mathbf{e}^2=\frac{1}{\rho^2}\mathbf{f}^2=\frac{1}{\rho^2}\rho\mathbf{f}=\mathbf{e}$$

and

$$\mathbf{i}^2=\frac{1}{\rho\sigma}\mathbf{j}^2=-\frac{1}{\rho\sigma}\sigma\mathbf{f}=-\mathbf{e}.$$

The vectors $\mathbf{e}$ and $\mathbf{i}$ are surely not parallel, since otherwise their squares would have the same direction. Hence it follows that $\mathbf{e}$ and $\mathbf{i}$ span the plane and therefore we may write each vector $\mathbf{a}$ in the plane in the form

$$\mathbf{a}=\xi\mathbf{e}+\eta\mathbf{i}$$

with suitable real numbers ξ and η. We have still to know which vector of the plane is the vector $\mathbf{ei}$.

We proceed to prove that $\mathbf{ei}=\mathbf{i}$, or, more generally, $\mathbf{ea}=\mathbf{a}$ for all vectors $\mathbf{a}$. The equation $\mathbf{e}^2=\mathbf{e}$ implies $\mathbf{e}^2\mathbf{a}=\mathbf{ea}$ or $\mathbf{e}(\mathbf{ea}-\mathbf{a})=0$ for all $\mathbf{a}$. Since there are no divisors of zero, it follows that $\mathbf{ea}-\mathbf{a}=0$, *i.e.* $\mathbf{ea}=\mathbf{a}$, as stated.

From the results obtained so far we may conclude that multiplication of two vectors is to be performed as though $\mathbf{e}$ and $\mathbf{i}$ were equal to the real number 1 and the imaginary unit $\sqrt{-1}$, respectively, *i.e.* we multiply according to the rules of multiplication of complex numbers.

Now use an affine transformation of the plane which carries the vectors $\mathbf{e}$ and $\mathbf{i}$ into two unit vectors perpendicular to each other; such a transformation clearly always exists. Then define the operations in the new vector system as induced by this affine transformation. It is obvious that we arrive just at the vectors of the complex plane with the well-known definitions of operations.

This completes the proof of the theorem.

Remark 1. As the proof above shows, the requirement of associativity can be replaced by commutativity. Also, it is worth while noticing that if we suppose neither associativity nor commutativity, the theorem is no longer valid. It is readily checked that if for two non-parallel vectors $\mathbf{a}$ and $\mathbf{b}$ we put

$$\mathbf{a}^2 = \mathbf{b}^2 = \mathbf{a}, \qquad \mathbf{ab} = -\mathbf{ba} = \mathbf{b}$$

(associativity fails to hold: $\mathbf{b}^2\mathbf{b}\neq\mathbf{bb}^2$), then we obtain an algebra without divisors of zero, but not isomorphic to the complex number system.

Remark 2. In the above proof we have made use of continuity; this can, however, be avoided, as is shown by the following argument.

For an arbitrary vector $\mathbf{u}\neq 0$ the vectors $\mathbf{au}$ and $\mathbf{bu}$ span the plane provided that the same holds for $\mathbf{a}$ and $\mathbf{b}$ (zero divisors do not exist!), therefore for any given vector $\mathbf{v}$ we have

$$\mathbf{v} = \alpha\mathbf{au} + \beta\mathbf{bu} = (\alpha\mathbf{a} + \beta\mathbf{b})\mathbf{u} = \mathbf{xu}$$

with $\mathbf{x}=\alpha\mathbf{a}+\beta\mathbf{b}$ and α, β suitable real numbers. Hence right-division and, similarly, left-division by any non-zero vector exists. We conclude that the vectors not equal to 0 form a group under multiplication with, say, $\mathbf{e}$ as identity. Then each vector has the form $\gamma\mathbf{e}+\delta\mathbf{d}$ with $\mathbf{d}$ an arbitrary vector not parallel to $\mathbf{e}$. Hence the multiplication is commutative.

Now from $\mathbf{d}^2=\lambda\mathbf{e}+\mu\mathbf{d}$ we have $(\mathbf{d}-\frac{1}{2}\mu\mathbf{e})^2=(\lambda+\frac{1}{4}\mu^2)\mathbf{e}$ where the real number $\lambda+\frac{1}{4}\mu^2=\kappa$ must be negative, since $\kappa\geqq 0$ would imply that the vectors $\mathbf{d}-\frac{1}{2}\mu\mathbf{e}\pm\sqrt{\kappa}\mathbf{e}$ are divisors of zero. Consequently, for the vector

$$\mathbf{i} = \frac{\mathbf{d} - \frac{1}{2}\mu\mathbf{e}}{\sqrt{-\kappa}}$$

we get $\mathbf{i}^2=-\mathbf{e}$, *q.e.d.*

THE BERNOULLI DIFFERENTIAL EQUATION

M. R. SPIEGEL, Rensselaer Polytechnic Institute

In the differential equation of Bernoulli

$$\frac{dy}{dx} + Py = Qy^n \tag{1}$$

where P and Q are functions of x, the usual method of solution is to place

$$v = y^{1-n} \tag{2}$$

and solve the resulting linear equation. This linear equation is

$$\frac{dv}{dx} + (1 - n)Pv = (1 - n)Q. \tag{3}$$

Since an integrating factor of (3) is $e^{\int_0^x (1-n)Pdx}$, we may write (3) as

$$\frac{d}{dx}\left[ve^{\int_0^x (1-n)Pdx}\right] = (1 - n)Qe^{\int_0^x (1-n)Pdx} \tag{4}$$

which yields the solution

$$v = \left\{e^{\int_0^x (n-1)Pdx}\right\}\left\{\int_0^x (1 - n)Qe^{\int_0^x (1-n)Pdx}dx + v_0\right\}. \tag{5}$$

Here v_0 is the value assumed by v when $x=0$. If we take y_0 to be the value assumed by y when $x=0$, the solution of (1) is

$$y = \left\{e^{-\int_0^x Pdx}\right\}\left\{y_0^{1-n} + (1 - n)\int_0^x Qe^{\int_0^x (1-n)Pdx}dx\right\}^{1/(1-n)}. \tag{6}$$

When the differential equation (1) is solved in the various textbooks, it is usually pointed out that one assumes n not equal to 0 or 1. The purpose of this article is to show that the solutions obtained for (1), in the supposed exceptional cases where $n=0$ or 1, may actually be obtained from the solution (6).

For the case $n=0$, there is no difficulty. Letting $n=0$ in (6) we obtain

$$y = \left\{e^{-\int_0^x Pdx}\right\}\left\{y_0 + \int_0^x Qe^{\int_0^x Pdx}dx\right\} \tag{7}$$

which is the solution of (1) in the case where $x=0$, $y=y_0$,

For the case $n=1$, there is a difficulty, for (6) then contains an indeterminate factor of the form 1^∞. The limit of (6) as $n\to 1$ may be shown, by the usual methods of dealing with indeterminate forms, to equal

(8) $$y_0 e^{\int_0^x (Q-P)dx}$$

which is the solution of (1) with $n=1$ such that $y=y_0$ when $x=0$ as may be easily verified.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1036. *Proposed by W. R. Ransom, Tufts College*

How large an asteroid could a man jump clear off of?

E 1037. *Proposed by A. G. Anderson, Oberlin College*

Show that a necessary and sufficient condition that a real cubic equation $ax^3+bx^2+cx+d=0$ have one real and two pure imaginary roots is that $bc=ad$ and $ac>0$.

E 1038. *Proposed by S. B. Townes, University of Hawaii*

Show that if a number of $3n$ digits is divisible by 27, then any number with the same digits cyclically permuted will also be divisible by 27.

E 1039. *Proposed by I. W. Burr, Purdue University*

Minimize the product of two perpendicular central chords of a given ellipse.

E 1040. *Proposed by H. E. Stelson, Michigan State College*

On page 20 of Rider and Fischer, *Mathematics of Investment*, is the statement, "It does not appear to have been generally recognized that the assumptions underlying the Residuary method are equivalent to those of the Merchant's Rule and hence the two methods yield identical results."

Prove that the Merchant's Rule and the Residuary (Interest at End) method give the same results for the case where the payments are all equal except the last and where the interest payment amounts to more than a single payment.

SOLUTIONS

The Problem of the Three Cowboys

E 1001 [1952, 104]. *Proposed by C. S. Ogilvy, Syracuse University*

Three cowboys who have to watch over cattle in a square pasture divide the square into three equal rectangles by parallel lines in the obvious way. Each then stations himself in the center of one of the rectangles and is responsible only for the cattle in his rectangle.

Show that the square can be divided into three equal areas in another way and a cowboy can be stationed at a point of each, so that (1) the three cowboys have the same "maximum ride," which is the distance from a cowboy's station to the most distant point of his area; (2) this maximum ride is less than it was before; and (3) every point in the square is entrusted to the man whose station is nearest it.

This problem is posed but not solved in Steinhaus, *Mathematical Snapshots* (1950), pp. 34–35.

Solution by Leon Bankoff, Los Angeles, Calif. On one of the parallel lines which trisect the square in the obvious way, two points are located at distances $a/4$ from the sides, where a is a side of the square. Lines through these points directed toward the interior of the square and inclined at 120° to each other meet at a point through which we draw a third line bisecting the larger of the two areas. The area of each of the two trapezoids thus formed is clearly equal to $(a/2)(2a/3)$, or $a^2/3$, so that the three areas are equal.

Cowboy A is stationed in the pentagonal area at the intersection of the two longer diagonals. Cowboys B and C are stationed in the trapezoidal areas at points which are reflections of station A with respect to the boundaries A–B and A–C. Hence stations A, B, C are vertices of an equilateral triangle, the sides of which are bisected perpendicularly by the three boundaries.

All conditions are now fulfilled.

(1) Each cowboy has the same maximum ride (from his station to the outer end of an oblique boundary). This is easily verified by placing the configuration on a rectangular coordinate system.

(2) Since A is now stationed at the center of a diagonal shorter than the original one, his maximum ride, as well as that of the others, is now less than it was before.

(3) Every point in the square is entrusted to the man nearest it, since all points on the common boundaries are equidistant from the stations in the areas separated by them.

Also solved by Julian Braun, W. E. Briggs, F. F. Dorsey, E. I. Gale, Michael Goldberg, J. D. Haggard, J. M. Kingston, L. A. Ringenberg, C. M. Sandwick, Sr., Henry Zatzkis, and the proposer.

The problem has, of course, infinitely many solutions. Kingston and the proposer gave solutions which are symmetrical in a diagonal of the square. It

would be an interesting problem to find the situation which minimizes the maximum ride of the three cowboys.

m-fold Factor of a Polynomial

E 1002 [1952, 104]. *Proposed by T. M. Apostol, California Institute of Technology*

Let A, B, C represent polynomials in a given number-field F, and let A' denote the derivative of A. Suppose B is an m-fold factor of A, *i.e.*, $A=B^mC$ and $B\nmid C$. It is well known that B is at least an $(m-1)$-fold factor of A'. The example $B=(x-1)^3$, $A=(x-1)^3(x^3-1)^2$ shows that it is possible for B to be an m-fold factor of A'. Is it possible for B to be an $(m+1)$-fold factor of A'?

Solution by G. A. Baker, Jr., California Institute of Technology. We know that if a polynomial P is a prime m-fold factor of a polynomial Q, then it is exactly an $(m-1)$-fold factor of Q'. By the prime factorization theorem we have

$$B = p_1^{j_1}p_2^{j_2}\cdots p_r^{j_r}, \qquad A = p_1^{m_1}p_2^{m_2}\cdots p_n^{m_n},$$

where the p_i are distinct prime factors, $r\leqq n$, $mj_i\leqq m_i$ $(1\leqq i\leqq r)$, and for some k, $m_k<(m+1)j_k$ (otherwise B^{m+1} would be a factor of A). Since p_i is exactly an (m_i-1)-fold factor of A', if B^{m+1} is a factor of A' then for all i, $(1\leqq i\leqq r)$, $m_i-1\geqq(m+1)j_i$, a contradiction. It follows that B cannot be an $(m+1)$-fold factor of A'.

Also solved by R. Z. Vause, Jr., and the proposer.

The above solution assumes that the underlying field F is of characteristic zero. Vause showed that a contrary state of affairs may exist if F is of positive characteristic p. Suppose, for example, F is the modular field having $p=5$, and take

$$B = x - 1, \qquad A = (x-1)^5(2x^3 - x^2 + x).$$

Then $A'=(x-1)^7$.

A Criterion for $f(x, y)=g(x)+h(y)$

E 1003 [1952, 105]. *Proposed by J. G. Wendel, Yale University*

Let $f(x, y)$ be continuous on the unit square, $0\leqq x, y\leqq 1$. Show that $f(x, y)$ can be written in the form $f(x, y)=g(x)+h(y)$ if and only if

$$\int_0^1\int_0^1\int_0^1\int_0^1 \{f(x, y)+f(u, v)\}^2\,dx\,dy\,du\,dv = \int_0^1\int_0^1\int_0^1\int_0^1 \{f(x, y)+f(u, v)\}\{f(x, v)+f(u, y)\}\,dx\,dy\,du\,dv.$$

Solution by J. E. Wilkins, Jr., Nuclear Development Associates, Inc. It is obvious that this equation holds when $f(x, y)=g(x)+h(y)$. Conversely, the

equation asserts that Schwarz' inequality is an equality for the functions $f(x, y)+f(u, v)$ and $f(x, v)+f(u, y)$. These functions must then be proportional, and in fact equal, for it is easy to see that either $f(x, y)$ vanishes identically or the factor of proportionality is unity. If we set $u=v=0$ we find that

$$f(x, y) = f(x, 0) + \{f(0, y) - f(0, 0)\} = g(x) + h(y),$$

as desired.

Also solved by Hyman Kaufman and Jerome Shipman (jointly), and the proposer.

Editorial Note. The assumption of the continuity of $f(x, y)$ is used to ensure that $f(x, y)+f(u, v)$ and $f(x, v)+f(u, y)$ are proportional everywhere, and not merely almost everywhere. More stringent restrictions on $f(x, y)$ lead to other criteria for the desired decomposition. For example, for properly restricted $f(x, y)$ a criterion is that $\partial^2 f/\partial x\partial y \equiv 0$. The interest of the criterion of the problem lies in the weakness of the restriction put on $f(x, y)$.

An Approximation for $\sqrt[n]{a}$

E 1004 [1952, 105]. *Proposed by L. R. White, Washington, D. C.*

(1) The approximation $\sqrt[n]{a}=1+(a-1)/n$ for n large and a close to 1 is well known and frequently used. Show that for large n and all a

$$\sqrt[n]{a} = 1 + (\ln a)/n$$

is a good approximation.

(2) Find

$$\lim_{n\to\infty} \{(k - 1 + \sqrt[n]{a})/k\}^n.$$

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. (1) For all $a>0$

$$\sqrt[n]{a} = e^{(\ln a)/n} = 1 + (\ln a)/n + O(1/n^2).$$

(2) Using the preceding result we find

$$\lim_{n\to\infty} \{(k - 1 + \sqrt[n]{a})/k\}^n = \lim_{n\to\infty} \{1 + (\ln a)/kn + O(1/n^2)\}^n$$

$$= e^{(\ln a)/k} = a^{1/k}.$$

Also solved by Alan Berndt, W. E. Briggs, Kenneth Bush, P. L. Chessin, W. W. Gerlinger, Emil Grosswald, Milton Handel, Vern Hoggatt, Hyman Kaufman and Jerome Shipman (jointly), L. A. Ringenberg, W. R. Scott, M. R. Spiegel, and S. C. Venkataraman.

An Application of the Newton-Raphson Method

E 1005 [1952, 105]. *Proposed by D. W. Dubois, University of Oklahoma*

Let $x_0 \neq 0$ and $a > 0$ be two real numbers. Define

$$x_{n+1} = x_n/2 + a/2x_n, \qquad n = 0, 1, 2, \cdots .$$

Find all values of x_0 and a for which the sequence $\{x_n\}$ converges, and find the limits.

I. *Solution by Hyman Kaufman and Jerome Shipman, Laboratory for Electronics, Boston, Mass.* The expression is recognized as the $(n+1)$st iterate in the well known Newton-Raphson method for finding the roots of $y = a - x^2$, *i.e.*, for solving $x^2 = a$. From the theory of the Newton-Raphson method, or from a glance at the graph of $y = a - x^2$, it can be easily shown that for $x_0 > 0$, $\{x_n\}$ converges to $\sqrt{a}$, while for $x_0 < 0$, $\{x_n\}$ converges to $-\sqrt{a}$.

II. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn.* If we plot $y = x/2 + a/2x$ and $y = x$ we get a geometric insight into the convergence process. The curve is a hyperbola with $x = 0$ and $y = x/2$ as the asymptotes. The intersections are $(\sqrt{a}, \sqrt{a})$ and $(-\sqrt{a}, -\sqrt{a})$. Also, the points of intersection are the maximum and minimum points of the hyperbola. From the plot it follows that the sequence converges for all values of $a > 0$ and $x_0 \neq 0$. If $x_0 > 0$, then $x_n \to \sqrt{a}$; if $x_0 < 0$, then $x_n \to -\sqrt{a}$.

Also solved by Kenneth Bush, A. L. Epstein, E. P. Graney, R. E. Greenwood, Emil Grosswald, C. S. Ogilvy, L. A. Ringenberg, and the proposer.

Greenwood called attention to Whittaker and Robinson, *The Calculus of Observations*, pp. 79–81. A late solution by A. E. Livingston was received.

Packaged Radios

E 1006 [1952, 180]. *Proposed by Leo Moser, University of Alberta*

A wholesale jobber received radios from the manufacturer packed in two types of cartons. Each type held a different number of radios. The shipping clerk found that by judiciously selecting the right number of one or both kinds, he could fill almost any size order without opening a carton. Indeed, there were exactly six orders possible which would require a carton to be opened and repackaged.

One day the manufacturer discontinued the smaller carton and sent a new size, containing a different number of radios. The shipping clerk calculated that with the new set-up there were ten different orders which would be impossible to fill without repacking. How many radios were contained in this new carton?

Solution by C. M. Sandwick, Sr., Easton High School, Easton, Pa. If $(m, n)=1$, the number of positive integers which are not of the form $Am+Bn$, where A and B are non-negative integers, is $(m-1)(n-1)/2$. Moreover, the value of each integer which is not of this form is less than $(m-1)(n-1)$.

Let a be the number of radios packed in each small carton and let b be the number in each large carton of the two original types. Let c be the number of radios packed in each carton of the type which is substituted for the smaller cartons.

We have $(a-1)(b-1)/2=6$ and $(c-1)(b-1)/2=10$. Hence $5(a-1)=3(c-1)$, or $5a-3c=2$. Only one solution of this equation yields a corresponding value of b which is a positive integer greater than a. The unique solution is $a=4$, $c=6$, and $b=5$. Thus, each of the new cartons contained six radios.

When using the original cartons, containing 4 and 5 radios, the clerk could not ship 1, 2, 3, 6, 7, or 11 radios without opening a carton. Using cartons containing 5 and 6 radios, he could not ship 1, 2, 3, 4, 7, 8, 9, 13, 14, or 19 radios without opening a carton.

Also solved by Leon Bankoff, C. R. Berndtson and J. E. Wilkins, Jr. (jointly), Julian Braun, F. F. Dorsey, William Douglas, Vern Hoggatt, Ray Jurgensen, J. M. Kingston, M. S. Klamkin, Sidney Kravitz, Azriel Rosenfeld, Charles Salkind, R. Z. Vause, Jr., Lila Peck Walker, LeRoy Warren, and the proposer.

(a, b)=(a+b, [a, b])

E 1007 [1952, 180]. *Proposed by Harvey Berry, University of Kentucky*

Prove that the G.C.D. of two numbers is equal to the G.C.D. of their sum and their L.C.M.

Solution by C. F. Pinzka, Princeton, N. J. If g is the G.C.D. of a and b, we may write $a=a'g$ and $b=b'g$, where a' and b' are mutually prime. Then the sum and L.C.M. of a and b are $(a'+b')g$ and $a'b'g$, respectively. Both have the common factor g, which is also their G.C.D., since $a'+b'$ and $a'b'$ are mutually prime.

Also solved by A. N. Aheart, F. Bagemihl, Alan Berndt, Julian Braun, Herbert Emich, C. V. Fronabarger, J. D. Haggard, Melvin Henriksen, Vern Hoggatt, Ray Jurgensen, J. D. E. Konhauser, George Millman and Bernard Chovitz (jointly), R. G. Paxman, Azriel Rosenfeld, C. M. Sandwick, Sr., J. S. Shipman, R. H. Sprague, W. L. Strother, D. R. Sudborough, R. P. Tapscott, C. W. Trigg, R. Z. Vause, Jr., Lila Peck Walker, Maud Willey, and R. R. Williams, Jr.

This problem is the same as Problem No. 116, *Mathematics Magazine*, Nov. 1951. Simple counter-examples show that the problem does not extend directly to more than two numbers.

A Recurrent Construction Problem

E 1008 [1952, 180]. *Proposed by R. H. Moorman, Tennessee Polytechnic Institute*

Through a given point draw a line cutting the sides of a given angle so as to form a triangle of given perimeter.

(*Editorial Note*: This is the same as problem 3320 [1929, 291], whose solution is herewith reproduced.)

Solution by D. L. Barrick, University of Colorado. Construction: Let D be the given point, EAF the given angle, and $2p$ the given perimeter. On AE take AZ equal to p and on AF take AY equal to p. Construct a circle tangent to AE and AF at Z and Y respectively. Through D construct a line tangent to the minor arc ZY meeting AE and AF at B and C respectively. This tangent is the required line.

Proof: The perimeter of ABC equals $2AZ$ equals $2p$.

Discussion: There is one and only one solution if D is outside of the given angle or its vertical angle, or if D falls on AZ, AY, or minor arc ZY not including points A, Z, or Y. There are two solutions if D falls within the area bounded by AZ, AY, and minor arc ZY. Otherwise there is no solution. It is assumed here that the sides of the triangle are not to lie upon the prolongations of EA and FA.

Also solved by I. D. Abella, Alan Berndt, C. W. Bruce, F. F. Dorsey, C. V. Fronabarger, J. D. Haggard, John Jones, Jr., M. S. Klamkin, J. D. E. Konhauser, Sam Kravitz, B. R. Leeds, B. Martin, C. F. Pinzka, Charles Salkind, C. M. Sandwick, Sr., A. Sisk, R. H. Sprague, and C. W. Thompson.

N. A. Court pointed out that this problem was discussed at great length by S. A. J. Lhuilier (1750–1840), *Éléments d'analyse géométrique et d'analyse algébrique, appliquées a la recherche des lieux géométriques*, p. 219, ex. 5, Paris and Geneva, 1809. It was also taken up by Eugene Catalan (1814–1894) in his *Théorèmes et problèmes de géométrie élémentaire*, p. 68, prob. 5, 6th ed., Paris, 1879. It has since appeared repeatedly in the problem departments of various periodicals, and is stated in several books on college geometry, such as Court, *College Geometry*, ex. 3, p. 80; Davis, *Modern College Geometry*, ex. 4, p. 38; Taylor and Bartoo, *Introduction to College Geometry*, ex. 2, p. 43.

The following two allied problems, solutions of which are easily suggested by that of the given problem, were proposed by Konhauser and Abella, respectively.

(1) Through a point between the sides of an angle draw a line cutting the sides of the angle so as to form a triangle of minimum perimeter.

(2) Through a point draw a line cutting off from a given triangle a quadrilateral of given perimeter.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4508. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

Find a polynomial $F(x)$ of lowest degree such that $F(x)+\omega^s$ is divisible by $(x-\omega^s)^r$ for $s=0, 1, 2, \cdots, p-1$, where ω is a primitive pth root of unity and r is a given positive integer. (This generalizes a problem in Goursat-Hedrick, *A Course in Mathematical Analysis*, v. 1, p. 32.)

4509. *Proposed by D. J. Newman, Harvard University*

Prove that the most general multiplicative function $f(n)$ such that $f(n+1) \sim f(n)$ is $f(n)=n^a$ for some constant a.

4510. *Proposed by H. P. Thielman, Iowa State College*

If $a_1, a_2, a_3, \cdots$ is a sequence of real numbers such that $a_{n+1}=1/n^p a_n$, where $p>0$, and $n=1, 2, 3, \cdots$, prove that the sequence a_n is monotone non-increasing if and only if $a_1=(\pi/2)^{p/2}$.

4511. *Proposed by W. V. Parker, Alabama Polytechnic Institute*

Solve the Diophantine equation

$$x^2 + y^2 + (xy)^2 = z^2.$$

4512. *Proposed by E. P. Starke, Rutgers University*

Prove that every positive rational number with odd denominator is a sum of a finite number of distinct terms from the sequence $1/3, 1/5, 1/7, \cdots$.

SOLUTIONS

Integers with Large Prime Divisors

4436 [1951, 266]. *Proposed by D. J. Newman, Harvard University*

What is the probability that an arbitrary integer have a prime divisor which is larger than its square root?

I. *Solution by J. D. Swift, University of California, Los Angeles.* The probability is log 2. In the proof of this result we shall make use of Tchebychef's theorem: $\Pi(x)=O(x/\log x)$, and Merten's theorem: $\sum_{p\leqq x}1/p=\log\log x+C+o(1)$. The letters p and q will be reserved for primes.

Let $A(n)$ be the number of positive integers a such that $a\leqq n$, and $p\,|\,a$ for some $p>\sqrt{a}$. Now the integers a are those multiples kp of primes less than or equal to n such that $k\leqq p-1$ and $kp\leqq n$. Therefore

$$A(n)=\sum_{p\leqq n}\min\,(p-1,\,[n/p]).$$

It is clear that no duplications occur in this method of counting; *e.g.* a product pq is counted as a multiple of q if $q>p$ and not as a multiple of p. Since $p-1$ will be the minimum up to $\sqrt{n}$, we have

$$\begin{aligned}A(n)&=\sum_{p\leqq\sqrt{n}}(p-1)+\sum_{\sqrt{n}<p\leqq n}[n/p]\\&=O(\sqrt{n}\,\pi(\sqrt{n}))+n\sum_{\sqrt{n}<p\leqq n}1/p+O(\pi(n)-\pi(\sqrt{n}))\\&=n(\log\log n-\log\log\sqrt{n}+o(1))+O(n/\log n)\\&=n\log 2+o(n).\end{aligned}$$

Therefore $\lim A(n)/n=\log 2$.

II. *Solution by John Todd, National Bureau of Standards.* S. D. Chowla and John Todd have shown that the density of integers n, of which the largest prime factor is greater than any fixed multiple of $n^{1/2}$ is $\log 2=.6931$. See *Canadian Journal of Mathematics*, v. 1, pp. 297–299.

Also solved by P. T. Bateman, J. B. Kelly, E. H. Umberger, and the Proposer.

Evaluation of a Hypergeometric Function

4438 [1951, 343]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Show that the polynomial of degree $2n$ defined by the hypergeometric function $F(-2n,\,\frac{1}{2};\,n+1;\,x)$ takes on the value 1 when $x=4$.

Solution by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y. We make use of a well-known integral representation for the hypergeometric function (Whittaker and Watson, *Modern Analysis*, American Edition, p. 293) to obtain

$$F(-2n,\,\tfrac{1}{2};\,n+1;\,4)=\frac{\Gamma(n+1)}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_0^1 u^{-1/2}(1-u)^{n-1/2}(1-4u)^{2n}du.$$

It is clear that this reduces to 1 in view of the following:

$$\int_0^1 u^{-1/2}(1-u)^{n-1/2}(1-4u)^{2n}du = 2\int_0^{\pi/2} \cos^{2n}\theta(1-4\sin^2\theta)^{2n}d\theta$$

$$= 2\int_0^{\pi/2} (\cos 3\theta)^{2n}d\theta$$

$$= 2\int_0^{\pi/2} \cos^{2n}\phi d\phi = \frac{\Gamma(n+\frac{1}{2})\sqrt{\pi}}{\Gamma(n+1)}.$$

Also solved by F. G. Fender, Anne C. Moldauer, O. E. Stanaitis, Chih-yi Wang, and the Proposer.

Triangle *ABC* and a Locus Problem

4439 [1951, 343]. *Proposed by Josef Langr, Prague, Czecho-Slovakia*

A line m cuts the sides BC, CA, AB of a triangle ABC at L, M, N. Show that the midpoints of the segments AL, BM, CN are on a line n, and find the locus of the point of intersection of m and n as m moves parallel to itself.

Solution by C. S. Ogilvy, Syracuse University. Locate the triangle on oblique coördinate axes with A at the origin, B at the point $(b, 0)$ and C at $(0, c)$. Let L, M, N have the coördinates (p, q), $(0, r)$ and $(-s, 0)$ respectively. Solving for the intersection of m and BC gives

$$p = \frac{bs(c-r)}{cs+br}, \qquad q = \frac{rc(s+b)}{cs+br}.$$

Using these values it is easy to verify that the three midpoints $(p/2, q/2)$, $(b/2, r/2)$ and $(-s/2, c/2)$ are collinear, the slope of n being $(r-c)/(b+s)$.

If μ is the slope of m, then $y=\mu x+r$ is the equation of m. The equation of n is $(d-2y)/(a-2x)=(r-c)/(b+s)$. Using the fact that $\mu=r/s$, we eliminate r and s between the equations of m and n, finding the locus of their intersections to be the parabola

$$\mu x^2 - 2xy + (1/\mu)y^2 + 2cx + 2by - bc = 0,$$

whose axis is parallel to m.

Also solved by Emil Grosswald, R. F. Hays, M. S. Klamkin, George Millman (first part), W. O. Pennell, O. J. Ramler, L. A. Ringenberg, Azriel Rosenfeld (first part), A. Sisk, F. Underwood, and the Proposer.

Editorial Note. Ramler notes that n is the Newton line of the quadrilateral composed of the sides of triangle ABC together with m. He also shows that the parabola is inscribed in the medial triangle of ABC, a fact that is easily verified analytically in the above solution.

In generalizing the problem, Grosswald allows m to vary so as always to pass through a fixed point P. The locus of intersections of m and n is then in general a curve of the third degree having three real asymptotes parallel to the three

sides of ABC. If, however, P is on a side of ABC, then all lines n also pass through a fixed point Q, and the locus of intersections is a hyperbola passing through P and Q and having asymptotes parallel to the other two sides of ABC.

A Summation

4440 [1951, 343]. *Proposed by R. G. Stoneham, University of California, Berkeley.*

Evaluate

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}B_{2k}\pi^{2k}}{(2k+1)!}$$

where B_{2k} are the even Bernoulli numbers.

Solution by O. E. Stanaitis, St. Olaf College, Northfield, Minnesota. From the well-known formula

$$x \cot x = 1 + \sum_{k=1}^{\infty} (-1)^k \frac{2^{2k}B_{2k}}{(2k)!} x^{2k},$$

by integrating between the limits 0 and $\pi/2$ we have

$$\int_0^{\pi/2} x \cot x dx = \pi/2 + \frac{\pi}{2} \sum_{k=1}^{\infty} \frac{(-1)^k B_{2k}\pi^{2k}}{(2k+1)!}.$$

Integration by parts gives

$$\int_0^{\pi/2} x \cot x dx = -\int_0^{\pi/2} \log \sin x dx = \frac{\pi}{2} \log 2,$$

a familiar definite integral. Hence the value of the proposed sum is $1-\log 2$.

Also solved by Hyman Kaufman, M. S. Klamkin, W. Seidel, Chih-yi Wang, and the Proposer.

Polynomials with Positive Coefficients

4441 [1951, 343]. *Proposed by R. M. Cohn, Rutgers University*

If a polynomial equation $f(x)=0$, with integral coefficients has no positive roots, there exists a polynomial $g(x)$ with integral coefficients such that all coefficients of $f(x)\cdot g(x)$ are positive.

Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn. It will suffice to prove the theorem for $q(x)=x^2-ax+b$, $(a, b>0, a^2<4b)$. Let $p(x)=(1+x)^k$. Then all the coefficients of $q(x)\cdot p(x)$ will be positive if

$$\binom{k}{r} - a\binom{k}{r-1} + b\binom{k}{r-2} > 0 \tag{1}$$

for $r=1, 2, \cdots, k+1$. This is always possible for k sufficiently large.

$f(x)$ can be factored into linear factors like $x+c$, $c \geqq 0$, and quadratic factors like $q(x)$. We may then take $g(x)$ to be $(1+x)^K$ where K is the sum of the k's determined as above for the individual quadratic factors of $f(x)$. Since the coefficients of $f(x)$ are integers, the coefficients of $f(x) \cdot g(x)$ will be positive integers.

Also solved by Emil Grosswald, L. J. Paige, George Piranian, O. E. Stanaitis, M. A. Kropnitsky and the Proposer.

Editorial Note. The existence of suitable k may be seen as follows. (1) reduces to

$$(k - r + 2)(k - r + 1) - ar(k - r + 2) + br(r - 1) > 0$$

which is true if

$$2k > (2 + a)r - 3 + \sqrt{1 + (2a + 4b)r - (4b - a^2)r^2} \tag{2}$$

and the radical is real; if the radical is imaginary (1) is true for all k. Evidently the right member of (2) has a maximum value and we need only choose k greater than this maximum.

Paige and Piranian refer to a proof by E. Meissner (*Math. Annalen*, v. 70 (1911), p. 223), and to the following generalization by D. R. Curtiss (*Math. Annalen*, v. 73 (1913) p. 424): *If $f(x)$ is any polynomial with all its coefficients real, there exist polynomials $f_1(x)$ such that when the product $f_2(x) = f_1(x) \cdot f(x)$ is arranged according to ascending or descending powers of x, the number of variations of sign presented by its coefficients is exactly equal to the number of positive roots of $f(x)$.* See also Pólya and Szegö, *Aufgaben and Lehrsatze*, v. II, problem 190, section 5.

Zeros of a Complex Polynomial

4442 [1951, 343]. *Proposed by Michael Golomb, Purdue University*

If a polynomial (not a constant) with complex coefficients is of the same modulus at two points z_1, z_2 of the z-plane, show that it has at least one zero in the open half-plane

$$R\{\bar{z}(z_1 - z_2)(|z_1| - |z_2|)\} > 0$$

if $|z_1| \neq |z_2|$, and at least one zero in each of the closed half-planes

$$R\{\bar{z}(z_1 - z_2)\} \geqq 0, \qquad R\{\bar{z}(z_1 - z_2)\} \leqq 0$$

if $|z_1| = |z_2|$.

Solution by S. H. Gould, Purdue University. Let the coefficient of the highest power of z be unity. If w_i are the zeros of $f(z)$, we may write

$$\prod |z_1 - w_j| = |f(z_1)| = |f(z_2)| = \prod |z_2 - w_j|,$$

so that at least one w_i lies in the half-plane $|z_2 - z| \geqq |z_1 - z|$, since otherwise we would have $\prod |z_1 - w_j| > \prod |z_2 - w_j|$.

If $z_j = x_j + iy_j$, the inequality $|z_2 - z| \geqq |z_1 - z|$ becomes

$$(x_2 - x)^2 + (y_2 - y)^2 \geqq (x_1 - x)^2 + (y_1 - y)^2,$$
$$2x(x_1 - x_2) + 2y(y_1 - y_2) \geqq x_1^2 + y_1^2 - x_2^2 - y_2^2,$$
$$2R\{\bar{z}(z_1 - z_2)\} \geqq |z_1|^2 - |z_2|^2,$$

so that

$$R\{\bar{z}(z_1 - z_2)(|z_1| - |z_2|)\} > 0 \qquad \text{if } |z_1| > |z_2|,$$
$$R\{\bar{z}(z_1 - z_2)\} \geqq 0 \qquad \text{if } |z_1| = |z_2|.$$

The rest of the problem results from interchanging z_1 and z_2.

Also solved by Paul Cohen and the Proposer.

Real Zeros of Polynomials

4443 [1951, 422]. *Proposed by R. H. Breusch, Amherst College, Massachusetts.*

If

$$P_n(z) \equiv \sum_{p=0}^{[n/2]} (-1)^p \frac{z^{n-2p}}{(n-2p)!} = \frac{z^n}{n!} - \frac{z^{n-2}}{(n-2)!} + \cdots,$$

prove that the number of real zeros of $P_n(z)$ is

$$N_n = \frac{2n}{e\pi} + \frac{\log n}{e\pi} + O(1).$$

Solution by the Proposer. **1.** Let

$$R_n(x) \equiv \sum_{p=0}^{\infty} (-1)^p \frac{x^{n+2p}}{(n+2p)!} = \frac{x^n}{n!} - \frac{x^{n+2}}{(n+2)!} + \cdots.$$

Since

(a) $R_2(x) = 1 - \cos x \geqq 0$

(b) $R_n'(x) = R_{n-1}(x)$ for $n \geqq 1$

(c) $R_n(0) = 0$ fos $n \geqq 1$,

it follows by complete induction that $R_n(x)$, $n \geqq 3$, is positive and increasing for positive values of x.

2. Call a_n the (one and only) positive value such that $R_n(a_n) = 1$. Now

$$R_{n+2}(x) = P_n(x) \pm \begin{cases} \sin x \\ \cos x \end{cases}.$$

Therefore for $x > a_{n+2}$, $P_n(x) \neq 0$, since

$$|P_n(x)| > 1 - \begin{cases} |\sin x| \\ |\cos x| \end{cases}.$$

For $0<x<x_{n+2}$, the graph of $P_n(x)$ will intersect the x-axis as many times (or at the utmost, once more or once less) as the $\begin{Bmatrix}\sin\\ \cos\end{Bmatrix}$ curve, that is $\frac{a_{n+2}}{\pi}+O(1)$ times. And since $P_n(x)$, as an even or an odd function, has its zeros symmetric with respect to 0,

$$N_n = 2a_{n+2}/\pi + O(1).$$

3. Let Q_n be given by

$$R_n(x) = \frac{x^n}{n!}\left(1 - \frac{x^2}{(n+1)(n+2)} + \cdots\right) = \frac{x^n}{n!}Q_n(x).$$

Since

$$2n^{n+1/2}e^{-n} < n! < 3n^{n+1/2}e^{-n},$$

$$R_n\left(\frac{n}{e}\right) < \frac{n^n}{e^n n!} < 1.$$

Now

$$R_n\left(\frac{n}{2}\right) > \frac{n^n}{2^n n!}\left(1 - \frac{1}{4}\right) > \frac{(e/2)^n}{4\sqrt{n}} > 1 \qquad \text{for } n \geqq 9.$$

Thus $n/e<a_n<n/2$, and $1-\frac{1}{4}<Q_n(a_n)<1$. The rest of the proof proceeds as follows:

$$\frac{3}{4}\frac{a_n^n}{n!} < R_n(a_n) = 1 < \frac{a_n^n}{n!},$$

$$2(n/e)^n\sqrt{n} < n! < a_n^n < 4n!/3 < 4(n/e)^n\sqrt{n},$$

$$\frac{n}{e}e^{\log n/2n+\log 2/n} < a_n < \frac{n}{e}e^{\log n/2n+\log 4/n},$$

$$a_n = \frac{n}{e}\left(1 + \frac{\log n}{2n} + O\left(\frac{1}{n}\right)\right) = \frac{n}{e} + \frac{\log n}{2e} + O(1),$$

$$a_{n+2} = \frac{n+2}{e} + \frac{\log (n+2)}{2e} + O(1) = \frac{n}{e} + \frac{\log n}{2e} + O(1),$$

and finally,

$$N_n = \frac{2n}{e\pi} + \frac{\log n}{e\pi} + O(1).$$

More detailed care with the last steps will yield the result

$$N_n = \frac{2n}{e\pi} + \frac{\log n}{e\pi} + k_n, \qquad |k_n| < 4.$$

Isosceles Tetrahedron, Hyperbolic Sets of Lines

4446 [1951, 702]. Corrected. *Proposed by Victor Thébault, Tennie Sarthe, France*

In an equifacial tetrahedron the lines which join the vertices to the circumcenters (or the orthocenters) of the opposite faces form a hyperbolic system. If the lines are concurrent, the tetrahedron is regular and conversely.

Solution by the Proposer and W. E. Byrne, Virginia Military Institute. Lines joining the midpoints of opposite edges of an equifacial tetrahedron pass through the centroid G, are mutually perpendicular and are axes of symmetry of the tetrahedron. Using them as axes of a rectangular coordinate system, the vertices may be taken as

$$A(a, -b, -c) \qquad B(-a, b, -c)$$
$$C(-a, -b, c) \qquad D(a, b, c)$$

with a, b, c positive. The tetrahedron $ABCD$ is regular if and only if $a=b=c$. The faces ABC, DCB, CAD, BAD are congruent and are symmetric two by two with respect to the axes.

If a line δ is drawn from vertex D to a point D' in face ABC, the lines α, β, γ, symmetric to δ with respect to the axes, pass through the other vertices and meet the opposite faces in points A', B', C' homologous to D' in face ABC. If δ has the equations

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} = 0 \qquad lmn \neq 0,$$

then α, β, γ are given by

$$\frac{x-a}{l} = \frac{y+b}{-m} = \frac{z+c}{-n}, \qquad \frac{x-a}{-l} = \frac{y-b}{m} = \frac{z+c}{-n},$$
$$\frac{x+a}{-l} = \frac{y+b}{-m} = \frac{z-c}{n},$$

respectively. If δ does not cut the axes Gx, Gy, Gz and is not parallel to face ABC, then the four lines δ, α, β, γ are hyperbolic, *i.e.*, it is easily established that there is a line through D which meets α, β, γ; likewise a line through A which meets β, γ, δ; etc. Evidently, if A', B', C', D' are circumcenters (or orthocenters) we have a special case.

If the tetrahedron is regular (all faces equilateral triangles) the circumcenters

of the faces are also centroids of the faces. Thus, if δ is taken as the line joining D to the centroid of face ABC; $\delta, \alpha, \beta, \gamma$ will meet at G.

Lower Bound for the Zeros of an Analytic Function

4447 [1951, 423]. *Proposed by D. J. Newman, Harvard University*

Let $f(z)$ be analytic in $|z| \leqq 1$. Suppose $|f(z)| \leqq 2$ there and $f(0) = 1$. Show that $f(z)$ has no zeros for $|z| \leqq \frac{1}{2}$.

Solution by Peter Ungar, University College, London, England. Let $f(a) = 0$.

$$g(z) = \frac{\bar{a}z - 1}{z - a} f(z)$$

is regular in $|z| < 1$, and $|g(z)| \leqq 2$ on $|z| = 1$. By the maximum modulus theorem

$$2 \geqq |g(0)| = \frac{1}{|a|} |f(0)| = \frac{1}{|a|},$$

whence $|a| \geqq \frac{1}{2}$. Equality can occur only in case $g(z) \equiv g(0) = 1/a$, so that

$$f(z) = \frac{z - a}{a(\bar{a}z - 1)}.$$

Also solved by P. T. Bateman, R. P. Boas, Jr., R. H. Brown, Paul Cohen, B. Kelly, M. S. Klamkin, A. J. Lohwater, C. S. Ogilvy, M. S. Robertson, W. Seidel, O. E. Stanaitis, J. L. Ullman, H. S. Wall, and the Proposer.

Editorial Note. Most of the solvers note that the equality sign in $|z| \leqq \frac{1}{2}$ of the original statement should be omitted. Cohen and Robertson remark that the original statement remains true if 2 and $\frac{1}{2}$ are replaced by M and $1/M$, $M > 1$. Jensen's theorem (Titchmarsh, *Theory of Functions*, Oxford, 1932, 3.61) and Schwartz's lemma (Titchmarsh, *loc. cit.*, 5.2) are employed in several solutions. Bateman and Stanaitis refer to previous proofs by Landau (*Bull. Soc. Math. France*, v. 33, 1905, pp. 251–261; *Tôhoku Math. Journ.*, v. 5, 1914, pp. 97–116).

A Diophantine Equation

4448 [1951, 495]. *Proposed by Jekuthiel Ginsburg, Yeshiva College, New York City*

Solve in positive integers

$$z^4 = \frac{ax^2 + by^2}{a + b},$$

where a and b are given integers.

I. *Solution by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York.* The given equation will be satisfied if

$$an(z^2 + x) = m(y - z^2), \qquad m(z^2 - x) = bn(y + z^2),$$

m and n being arbitrary integers. Therefore

$$x:y:z^2 = |\, m^2 - 2bmn - abn^2 \,| : |\, m^2 + 2amn - abn^2 \,| : (m^2 + abn^2),$$

is a solution provided the constant of proportionality, k, is so chosen that $k(m^2+abn^2)$ is a perfect square.

Other solutions may be obtained by solving $z^2=m^2+abn^2$. See L. E. Dickson, *History of the Theory of Numbers*, v. 2, p. 425. One form of the solution is

$$m:n:z = |\, p^2 - abq^2 \,| : 2pq : (p^2 + abq^2).$$

II. *Solution by P. Somanatham and K. Subbarao, M. R. College, Vizianagram, South India.* Set $x=p-bq$, $y=p+aq$. Then

$$\frac{ax^2 + by^2}{a + b} = p^2 + abq^2.$$

Now numbers of the form p^2+abq^2 are multiplicative, that is, the product of two is another of the same form. The formula is

$$(1) \qquad (u^2 + abv^2)(m^2 + abn^2) = (mu \pm abnv)^2 + ab(mv \pm nu)^2.$$

Hence, if $z=p_1^2+abq_1^2$, (1) implies

$$z^2 = p_2^2 + abq_2^2, \qquad p_2 = |\, p_1^2 - abq_1^2 \,|, \qquad q_2 = 2p_1q_1.$$

Then, by (1) again,

$$z^3 = p_3^2 + abq_3^2, \qquad p_3 = |\, p_1p_2 \pm abq_1q_2 \,|, \qquad q_3 = |\, p_1q_2 \mp p_2q_1 \,|.$$

By repeated use of (1) as a recurrence relation we have

$$z^s = p_s^2 \pm abq_s^2, \qquad p_s = |\, p_1p_{s-1} \pm abq_1q_{s-1} \,|, \qquad q_s = |\, p_1q_{s-1} \mp p_{s-1}q_1 \,|,$$

so that infinitely many solutions of $z^s=p^2+abq^2$ and hence of

$$z^s = \frac{ax^2 + by^2}{a + b}$$

may be obtained for every positive integer s.

In particular for $s=4$, if the above substitutions are carried out, a solution of the proposed problem is obtained:

$$x = |\, p_1^4 - 4bp_1^3q_1 - 6abp_1^2q_1^2 + 4ab^2p_1^3q_1 + a^2b^2q_1^4 \,|,$$
$$y = |\, p_1^4 + 4ap_1^3q_1 - 6abp_1^2q_1^2 - 4a^2bp_1^3q_1 + a^2b^2q_1^4 \,|,$$
$$z = p_1^2 + abq_1^2.$$

Aslo solved by E. Morgantini, R. Venkatachalam Iyer, and the Proposer.

Series Conditionally Convergent Everywhere on the Unit Circle

4449 [1951, 495]. *Proposed by D. J. Newman, Harvard University*

The series $\sum z^n/n$ converges everywhere on the unit circle except at $z=+1$. Can $\pm$ signs be introduced so that $\sum \pm z^n/n$ will converge everywhere on the unit circle with no exceptions?

Solution by Frederick Bagemihl, University of Rochester, New York. The answer is affirmative. The series of Pringsheim,

$$\sum_{n=1}^{\infty} \frac{(-1)^{[\sqrt{n}]}}{n} z^n,$$

as is well known, converges everywhere on the unit circle with no exceptions. (*Münch. Berichte* XXX, 1900; quoted also in Knopp, *Problem Book in the Theory of Functions*, vol. I, New York, 1948.)

The series of problem 4384 [1951, 573] is very similar and possesses the same property.

Also solved by Joshua Barlaz, M. S. Klamkin, V. Linis, David Mandelbaum, O. E. Stanaitis, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Linear Transformations in n-Dimensional Vector Space. By H. L. Hamburger and M. E. Grimshaw. Cambridge University Press, 1952. 10+195 pages. $4.50.

This book was written to "introduce the ideas and methods of the theory of linear transformations in Hilbert space by using them to present the elements of the theory in a finite dimensional vector space."

In the first chapter, the vectors in a finite dimensional space are introduced in their concrete realization as n-tuples of complex numbers. The elements of the unitary structure and geometry of finite dimensional spaces are studied. Theorems about linear manifolds are proved, and applications are made to the solution of linear equations. The elementary algebra of linear transformations concludes the chapter.

The preliminary spectral properties of such special linear transformations as the hermitian, normal, unitary transformations, and projectors are established in the second chapter.

Chapter III contains a proof of the spectral theorem for hermitian transformations H by Hilbert's method (characteristic values are obtained by maximizing the form (Hx, x) on the unit sphere). Eigenvalue inequalities are established, and the functional calculus for hermitian transformations is developed. Commutativity properties of hermitian transformations are studied, and the chapter concludes with a development of the foregoing for normal transformations.

Chapter IV deals with the general linear transformation on n-dimensional space. Various spectral properties are established. A functional calculus for regular functions of linear transformations is developed. The Jordan canonical form is derived first for nilpotent and then for arbitrary linear transformations. Commutativity properties of general linear transformations are studied.

In the fifth and final chapter, finite-dimensional unitary spaces are now introduced abstractly for the purpose of analyzing the pencil $H-\lambda G$, with H hermitian and G positive. Conditions are stated for linear transformations to be hermitian or normal in some scalar product. Applications are made to the dynamical theory of small oscillations.

This book has a certain amount to recommend it. There is a wealth of material to be found in it which is usually not contained in an elementary text. The eigenvalue inequalities, the physical applications, and the analytical techniques are but a few examples of this. Some of the more powerful methods of modern functional analysis are illustrated in the simplified finite dimensional situation in such a manner as to make them readily accessible to the beginning student.

On the other hand, the reviewer found many objectionable features to this book. The notation is made clumsy by using superscripts on letters to denote vectors in a set of vectors and subscripts on letters for complex numbers in a set. The non-invariant concept of "real vector" in unitary space barely merits the attention the authors pay it. The authors leave the reader with the impression that separability is an inherent rather than an assumed (and somewhat inessential) property of Hilbert space (cf. p. 5). At several points, the authors actually go counter to their purpose of developing the finite dimensional theory in a manner most closely adaptable to Hilbert space, and nothing is saved by their procedure: on page 31, they define a regular transformation as one for which the inverse exists and a singular transformation as one of rank less than n; on page 36, they establish the associative law for multiplication of linear transformations by writing down the general element of the matrix obtained by each method of bracketing; on page 42, the spectrum of a linear transformation is defined as the set of all eigenvalues, where an eigenvalue is a number in the point spectrum of the transformation—this is not the spectrum for a transformation on Hilbert space and is unnecessarily misleading in view of the simple possibility of defining the spectral values as those numbers λ for which $A-\lambda I$ is singular. The concept of rank is used freely, but determinants are studiously avoided "because determinants can be used in Hilbert space only in very special cases." A proof of the spectral theorem is adopted which the authors point out is

adaptable only to completely continuous hermitian transformations in Hilbert space. Modern developments have made available simpler, more illuminating, and more elegant proofs of the spectral theorem which yield at the same time the simultaneous spectral decomposition of any family of commuting normal transformations. It should be remarked, however, that the proof given leads naturally to the discussion of the eigenvalue inequalities. The important polar decomposition of a linear transformation into the product of a partially isometric (unitary in the finite case) transformation and a positive one is not discussed. Our final objection is to the lack of motivation presented for the development of the Jordan canonical form.

R. V. KADISON
Institute for Advanced Study

The Design and Analysis of Experiments. By Oscar Kempthorne. New York, John Wiley & Sons, Inc. London, Chapman & Hall, Limited, 1952. 19+631 pages. $8.50.

The aim of this book, according to the Preface, is "to give a description of the design of experiments from as broad a view as possible and to relate the subject matter of this field of statistics to the general field of statistics and to the general problem of experimental inference." This aim the author has carried out well. He gives a systematic treatment of the mathematical theory upon which designs of experiments are based, enabling the research worker to make more intelligent use of these designs, both in planning them and analyzing them.

The formulation and testing of hypotheses are the important features of scientific method in general and of statistical method in particular. A hypothesis can be verified, although not proved, by experiments. That is, the results of the experiment may be in accord with the hypothesis, but they might equally well be in accord with some other hypothesis. On the other hand, a hypothesis can be disproved by the results of the experiment. Now the testing of a hypothesis requires the collection of observations and it is the function of experimental design to provide a pattern of the observations to be collected so that they will be relevant to the hypothesis to be tested and so that they will be accumulated as economically as possible, from the standpoint of time, labor, money, *etc.*

The earlier chapters of the book discuss the principles of statistical designs, elementary statistical notions, the theory of least squares, *etc.* Then come chapters on such topics as multiple regression and the analysis of variance, multiple classifications, randomization and randomized experiments (including randomized blocks and Latin squares), plot technique, factorial experiments, confounding, split-plot experiments, fractional replication, and lattice designs.

Although the treatment is very mathematical in character, a considerable amount of descriptive matter, including illustrative examples, is given.

Before concluding the review of this book it may not be inappropriate to

state its relation to two other recent books on the same subject, Cochran and Cox, *Experimental Designs*, and Mann, *Analysis and Design of Experiments*. It complements the former, which has to do with how an experiment is actually designed, in that it explains the underlying reason for designing it in a particular way. It is much more comprehensive and contains much more explanatory matter than Mann's brief treatise.

The author has in mind three groups of users: "the experimenter, who needs to understand the theory and practice of the statistical designs; the consulting statistician, who wants a description of basic theory and methodology of designs; and the mathematical statistician, who is concerned with the validity of the application of statistical techniques to the analysis of data." A member of any one of these groups would be rewarded by a diligent study of the book or of those parts in which he has a special interest.

P. R. RIDER
Washington University

Statistical Methodology Reviews, 1941–1950. Edited by Oscar Krisen Buros. John Wiley and Sons, Inc., New York: Chapman and Hall, Ltd., London, 1951. Pp. x+457. Price $7.00.

In attempting to write a review of Professor Buros' book, one cannot help having a curious feeling. Various authors wrote books on statistical methodology. Then each of these books was reviewed in several journals. Then Professor Buros collected the reviews, edited them and published his book. Now, I am writing a review of Professor Buros' review of the various reviews. If all goes well, somebody may publish a review of my review of . . . , *etc.* All this sounds like a malicious joke on my part. Actually, of course, the situation calls for a joke but the present one is not intended to be malicious. On the contrary in my opinion, both Dr. Buros' aims and his performance deserve deepest respect.

As stated in the Preface, "This volume is the third in a series of publications prepared to assist research workers and statisticians to locate and evaluate the statistical methodology books in all fields in which statistical methods are used." This primary objective is certainly achieved by Professor Buros and whoever wishes to learn several opinions on a given statistical book published in the decade, 1941–50, will find them collected or, at least, abstracted in the present volume. However, the collection of this kind of information is not the only objective of Professor Buros. In addition to this he has two more objectives which, to the present writer's mind, are more important than the first. One of them is to improve the quality of the reviews (i) "by stimulating editors to take greater pains to choose competent reviewers who have the industry and the courage to contribute frankly critical reviews after a careful study of the book" and (ii) "by stimulating reviewers 'to take their responsibility more seriously' by refusing to review books that they cannot, or will not, appraise

competently and honestly." Finally, the last objective of Dr. Buros is "to discourage the writing and publication of stereotyped textbooks written by persons ignorant of modern developments in statistical theory."

It is difficult to predict with certainty whether or not the last two mentioned aims of Dr. Buros will be attained. Much depends on how widely his book will be seen by persons concerned. All one can say is that a number of reviewers will have good reason to blush when reading their own writings printed side by side with those by other authors all concerned with the same book. One could also risk a similar assertion regarding some of the authors of books. The only question is whether or not these persons will look at the collection of Professor Buros or will try to keep away from it.

There is still another useful purpose that Professor Buros' book can serve although he does not mention it directly. The present period in the life of universities in this country and also in several countries abroad is characterized by a drive towards statistics. Departments of statistics or, at least statistical cells within other departments such as mathematics, are yearly established in more and more institutions of higher learning. In each such instance somebody takes the initiative and usually this somebody is not a professional statistician. His first problem is, then, to find and to bring in a person who could start and develop a suitable program of instruction and research in statistics. Naturally, the choice of this person is most important because, given the opportunity, he will establish a tradition which would be very difficult to eradicate. In situations of this kind, the chairman of the mathematics or of the economics or of the agriculture department, whoever is in charge of finding a suitable statistician, will find it profitable to consult Professor Buros' book. For his prospective candidate may happen to be the author of a book which he should not have written. Alternatively this candidate may be one of the reviewers "who cannot or will not" appraise other authors' works competently and honestly. Judgments in such matters are frequently difficult and a confrontation of several opinions on the same subject is likely to be very useful.

J. NEYMAN
University of California

Introduction to Mathematical Thinking. By Friedrich Waissman. With a foreword by Karl Menger. Translated by T. J. Benac. Frederick Ungar Publishing Co., New York.

The German text of Dr. Waissman's book, subtitled "The Formation of Concepts in Modern Mathematics," appeared first in 1936. Like several other books before and since its time, its aim was to give the non-mathematician a taste of the special flavor of mathematics, and to provide him with an insight into the nature of mathematical activity. Unlike some other books, the aim of this book —as Professor Menger says in his foreword to it—was realized, even though there may be as many disagreements as agreements with Dr. Waissman's discussion of, say, the principle of mathematical induction.

The aim of any translation is obvious. It is also obvious that the prime requirements of a good translation are: consistently good grammar and syntax, reasonably familiar words, and a precise transmission of the original meaning. There are too many sentences in the present translation which do not meet these requirements. When one also observes a somewhat careless editing and printing, one is driven to say that the present translation is a disservice to a fine book.

BEN DUSHNIK
University of Michigan

NEW BOOKS RECEIVED

La Geometrie Descriptive et ses Applications. By Andre Delachet and Jean Moreau. Paris, Presses Universitaire de France, 1952. 120 pages. 150 francs.

The Rational and the Superrational, Vol. II. By C. J. Keyser. New York, Scripta Mathematica. 8+259 pages. $4.25.

A Course of Pure Mathematics. Tenth Edition (with index). By G. H. Hardy. New York, Cambridge University Press. 12+509 pages. $4.75.

Sur Les Cercles Bitangents a la Parabole. By A. Durand. Paris, France, Librairie Vuibert, 1952. 32 pages.

Tables of Coulomb Wave Functions, Vol. 1, Applied Mathematics Series 17. National Bureau of Standards. 141 pages. $2.00.

Tables for the Analysis of Beta Spectra, Applied Mathematics Series 13. National Bureau of Standards, 1952. 61 pages. 35 cents.

Advanced Statistical Methods in Biometric Research. By C. R. Rao. New York, John Wiley and Sons Co., 1952. xvii+390 pages. $7.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB NEWS

The Florida State University Mathematics Club was organized on November 6, 1951 with Dr. S. L. Jamison and Dr. H. E. Taylor acting as faculty sponsors.

The Indiana Beta chapter of Kappa Mu Epsilon was installed at Butler University on May 16, 1952.

The Louisiana State University chapter of Pi Mu Epsilon presented the fol-

lowing awards on May 1, 1952: senior award, Jasper A. Welch, Jr.; freshman award, Robert E. Cavanaugh.

The University of Missouri chapter of Pi Mu Epsilon conducted a competitive examination in calculus during 1951–52, awarding the following prizes: James Fithian, $15; Delmar Van Meter, $10; Donald Meyer, $5.

The University of North Carolina chapter of Pi Mu Epsilon has established awards for the two highest scores achieved in a statewide high school mathematics contest. The chapter also has established a library consisting of undergraduate texts and periodicals.

During 1951–52, the *Ricci Mathematical Journal*, official publication of the Ricci Mathematics Academy, Boston College, appeared in November, February, March, and May. Members of the Academy presented the mathematics department with a dissectible cone and a large scale pyramind of plexiglass.

The St. Louis University chapter of Pi Mu Epsilon reports that the sixth Annual Essay Contest for undergraduate students was conducted by Professor Lorenz. The senior division was won by Mr. Ying-nien Yu, a student at Parks College. His paper was entitled, *Daniel Bernoulli's hydrodynamical equation.* Mr. Roland Nokes also of Parks College won the junior award for his essay, *Daniel Bernoulli.* The Garneau Mathematics Award for the highest ranking senior majoring in mathematics was awarded to Miss Louise Renard.

The mathematics club of the College of Saint Rose, Albany, New York, in April sponsored a High School Geometric Design Contest to engender a keener interest in mathematics among high school students. Over one hundred twenty entries were received from some eighteen high schools in the Albany area.

DO YOU WANT TO BE A MATHEMATICIAN?

Mathematics Department, University of Georgia

1. **Question:** Who should be called a mathematician?
   Answer: The desire among mathematicians themselves is to classify, as mathematicians, only persons who through their own efforts are able to add to existing mathematical theory or aid in the analysis of scientific problems of an advanced nature and contribute materially to their solution.
2. **Question:** What is mathematical research?
   Answer: By research mathematicians mean the creation of new mathematics. More than 6000 pages of mathematical research were published in mathematical journals in the United States during 1949.
3. **Question:** What training should a mathematician have?
   Answer: The mathematician should have advanced post graduate training equivalent to that expected for the Doctor of Philosophy Degree and some additional training in independent research.
4. **Question:** What are the professional advantages of post graduate mathematical training?
   Answer: Professional opportunities for mathematicians may be classified under

(a) university teaching
(b) college teaching
(c) industrial and business employment
(d) governmental employment including employment by the Navy
(e) independent consultation
(f) writing of books

5. **Question:** What does a mathematician earn?
Answer: (a) Salaries paid to mathematicians range from $3000 to $15,000. There are but few mathematicians in the United States whose salary is so large as the latter figure. However, many professions closely allied to mathematics pay even larger salaries. For example, a number of actuaries are paid salaries in excess of $20,000.
(b) Many mathematicians earn considerable sums from royalties on textbooks and other writings. There are several persons in the United States who received as much as $25,000 per year royalties on college textbooks on mathematics.
(c) There is a growing tendency for the United States Navy and many corporations to pay mathematicians for work on special problems. Thus many university professors and others earn considerable sums in addition to their salaries for research work under government or industrial contract.
6. **Question:** Is there a shortage of mathematicians?
Answer: At the present time there is a shortage of mathematicians and it is very difficult to get highly skilled persons to fill positions. At the conclusion of the War a survey made by the Mathematical Association of America showed that demand exceeded supply in the ratio of thirty to one. This situation has considerably improved since that time. However, the demand for mathematicians is still great.
7. **Question:** What are the qualifications necessary for success as a mathematician?
Answer: To succeed as a mathematician one must have a first class analytical mind and willingness to work coupled with a love for the subject.
8. **Question:** How long does it take to become a mathematician?
Answer: The training period for a mathematician is essentially the same as that for any other high class profession, such as medicine. Three or four years of post graduate work in addition to an undergraduate major may be regarded as essential. However, the best men in mathematics, as in any other profession, spend their entire lives on their subject.
9. **Question:** What subjects other than mathematics should a prospective mathematician take in college?
Answer: A person who wishes to become a mathematician should strive first for a good "arts" education. This should include physics, some chemistry, and philosophy. It is necessary for the mathematician to be able to read modern foreign languages and the prospective mathematician should study sufficient French and German to read mathematical literature writ-

ten in these languages. Italian is also desirable and it is likely that in the future Russian will be a great asset.

10. **Question:** What does a mathematician do?
Answer: (a) The mathematician at the University teaches college students and advanced graduate students. He also is continually engaged in his own research or in aiding others with their research.
(b) The mathematician in government employ or in the employ of the Navy or in industry is usually engaged on a research project in conjunction with others, notably engineers and physicists. His contribution to these joint projects is valuable and is as important as that of any other persons working on them.
11. **Question:** How does a mathematician seek employment?
Answer: If a mathematician is an active publisher in his field, employment is apt to seek him. This is particularly true of employment in the university world. Government employment is sought through civil service and industrial employment through contacting the firms in which one is interested.
12. **Question:** What are the opportunities for the mathematician to contribute to the advancement of civilization?
Answer: There is probably no more permanent or useful addition to civilization than mathematical research. The man who substantially adds to mathematical knowledge has the satisfaction that he has added to the advancement of mankind in one of the most useful and enduring ways. Moreover contributions by mathematicians to physics, engineering, astronomy, philosophy, biology, chemistry, and economics advance civilization through advancing these subjects.
13. **Question:** What are some of the satisfactions from mathematics other than salary?
Answer: Satisfactions that come to a mathematician are from his research, from his writing, from his teaching, and from his associations with other people of outstanding intellectual ability.

INTRODUCING CONGRUENCES

R. LARIVIERE, University of Illinois in Chicago

Too many students, taking traditional mathematics courses, complete their mathematical studies before any congruence relations are taught, though the idea of congruence is one of the most rewarding of the simpler mathematical concepts. The introduction of the topic in mathematical clubs and in some of the better classes can remedy this unfortunate situation to some extent.

Some properties of congruences are being used already to explain the familiar criteria of divisibility by the digits and by eleven. Little more is required to provide an explanation not only for these rules but also for lack of a worthwhile rule for divisibility by seven. The basis of the widely used checks of casting out the nines and casting out the elevens in multiplication and division can be included in the discussion of divisibility criteria.

Helping to determine whether the roots of a quadratic equation are rational or not is an especially easy application which is often overlooked. If the discriminant D_1 of an equation is the integer 5322, or even the reduced fraction 5326/933, the last digits show without further computation that the roots of the equation are irrational, for $n^2 \equiv \pm 1,0$ (5). Another equation, which has a discriminant $D_2 = 5339$, also has irrational roots since $(2n+1)^2 \equiv +1$ (8), and the last three digits show that the residue modulo eight is three. Such restrictions are not sufficient, and discriminants like 5329 or 2009 must be subjected to further trial though they satisfy both of the above congruences.

The partial check of equations like the following is another application which can be explained to an eager elementary group. This quick check is obtained by solving the equations modulo some small primes which do not divide the denominator of the values of either of the unknowns.

$$102x + 512y = 317, \quad 231x - 517y = 897; \qquad x = \frac{-623153}{-171006}, \quad y = \frac{6089}{-57002}$$

$$4x + y \equiv 2(7), \quad 0x - (-y) \equiv 1(7); \qquad x \equiv 2(7), \quad y \equiv 1(7).$$

$$2x + 2y \equiv 2(5), \qquad x - 2y \equiv 2(5); \qquad x \equiv 3(5), \quad y \equiv 3(5).$$

$$\frac{623153}{171006} \equiv \frac{6}{3}\,(7) \equiv 2(7) \equiv \frac{3}{1}\,(5) \equiv 3(5).$$

$$\frac{6089}{-57002} \equiv \frac{6}{-1}\,(7) \equiv 1(7) \equiv \frac{-1}{-2}\,(5) \equiv 3(5).$$

Hence the above solutions are satisfactory with respect to divisibility by seven and five.

Editorial Note: This article suggests an important source of club programs. Many exciting and interesting topics, though quite elementary, are postponed to advanced courses which many students do not have an opportunity to elect. Readers are invited to contribute notes on such topics.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

FELLOWSHIPS OF THE EDUCATIONAL TESTING SERVICE

The Educational Testing Service is offering for 1953–54 its sixth series of research fellowships in psychometrics leading to the Ph.D. degree at Princeton

University. Open to men who are acceptable to the Graduate School of the University, the two fellowships each carry a stipend of $2500 a year and are normally renewable.

Fellows will be engaged in part-time research in the general area of psychological measurement at the offices of the Educational Testing Service and will, in addition, carry a normal program of studies in the Graduate School. Competence in mathematics and psychology is a prerequisite for obtaining these fellowships. The closing date for completing applications is January 16, 1953. Information and application blanks may be obtained from: Director of Psychometric Fellowship Program, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey.

PRELIMINARY ACTUARIAL EXAMINATIONS

PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1952 Preliminary Actuarial Examinations are as follows:

First Prize of $200

Freimer, Marshall	Harvard University
Lew, John S.	Yale University

Additional Prizes of $100

Bauman, Norman	Harvard University
Guinane, J. E.	Brown University
Horowitz, Michael	Cornell University
Kerr-Lawson, Angus	University of Toronto
Krause, Ralph M.	Harvard University
Muckenhoupt, Benjamin	Harvard University
Potthoff, Richard	Swarthmore College

Ordinarily only one $200 prize is offered, but this year two candidates tied for first place. Each candidate was awarded a $200 prize.

The Society of Actuaries has authorized a similar set of prizes for the 1953 examinations based on the Part 2 scores.

The Preliminary Actuarial Examinations consist of the following three examinations:

Part 1. *Language Aptitude Examination.*

(Reading comprehension, meaning of words and word relationships, antonyms, and verbal reasoning.)

Part 2. *General Mathematics Examination.*

(Algebra, trigonometry, coordinate geometry, differential and integral calculus.)

Part 3. *Special Mathematics Examination.*

(Finite differences, probability and statistics.)

The 1953 Preliminary Actuarial Examinations will be prepared by the Educational Testing Service and will be administered by the Society of Actuaries at centers throughout the United States and Canada in May 1953. The closing date for applications is March 15, 1953.

Detailed information concerning the Examinations can be obtained from: The Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

PERSONAL ITEMS

The Honorary Doctor of Science degree was conferred upon Mr. E. C. Molina by the Newark College of Engineering on June 6, 1952.

Albama Polytechnic Institute announces the following: Dr. H. C. Wang of the Institute for Advanced Study has been appointed Research Professor of Mathematics; Dr. W. C. Royster of the University of Kentucky and Dr. J. C. Morelock of the University of Florida have been appointed to assistant professorships; Associate Professor J. C. Eaves has been promoted to a professorship; Assistant Professor V. E. Dietrich has resigned to accept a position as Professor of Physics at North Central College.

Baldwin-Wallace College reports: Assistant Professor J. A. Wilson is now Acting Head of the Physics Department; Assistant Professor Del Willard has accepted a position with the Cadillac Motor Car Company, Cleveland Tank Division.

Georgia Institute of Technology announces: Associate Professors C. W. Hook and I. E. Perlin have been promoted to professorships; Dr. M. H. M. Esser and Dr. G. K. Overholtzer have been appointed to assistant professorships.

Lafayette College makes the following announcements: Assistant Professor C. W. Saalfrank of Rutgers University has been appointed Head of the Department of Mathematics; Mr. Ralph Playfoot and Mr. Samuel Stoddard have been promoted to assistant professorships.

At Marshall College: Assistant Professor Bernice F. Wright has been promoted to an associate professorship; Mr. F. D. White has been appointed to an instructorship.

Yeshiva University announces that The Yeshiva Institute of Mathematics is offering courses leading to the degree of Master of Science in Mathematics. The faculty consists of the following: Professor Herman Baravalle, Adelphi College; R. S. Beard; Professor C. R. Boyer, Brooklyn College; Professor Jesse Douglas, Columbia University; Professor Samuel Eilenberg, Columbia University; Professor A. A. Frankel, Hebrew University, Jerusalem, Israel; Professor H. F. Fehr, Columbia University; Professor Abe Gelbart, Syracuse University; Professor Jekuthiel Ginsburg, Yeshiva University; Professor C. N. Shuster, New

Jersey State Teachers College; and Associate Professor R. C. Yates, United States Military Academy.

Mr. F. S. Badger of Alliance College has been appointed Dean of Students.

Mr. W. H. Badgley, formerly a graduate student at Vanderbilt University, is now a mathematician with Redstone Arsenal.

Mr. D. O. Banks, who has been a graduate assistant at Oregon State College, is a mathematician with Hanford Works, Richland, Washington.

Mr. James Bercos, previously a graduate student at the University of Georgia, has a position as Chief Statistician with the United States Naval Ordnance Plant, Macon, Georgia.

Mr. H. F. Blasch, formerly a student at Hofstra College, is now Research Engineer and Instructor with the United States Army, Fort Monmouth, New Jersey.

Dr. Paul Brock of Reeves Instrument Company has accepted a position as a mathematician with Consolidated Engineering Corporation, Pasadena, California.

Mr. C. C. Buck of the University of Nebraska is now a graduate student at the University of Michigan.

Dean J. W. Clawson of Ursinus College has been named Dean Emeritus and Professor Emeritus of Mathematics.

Associate Professor C. R. DePrima has returned to California Institute of Technology after a year's leave of absence; during the past year he has been at the Office of Naval Research, Washington, D. C.

Dr. A. H. Diamond who has been Visiting Professor of Mathematics at the University of Kansas has been appointed Chief of the Mathematics Branch, Mathematical Sciences Division, Office of Ordnance Research.

Professor G. M. Ewing, who has been on leave of absence from the University of Missouri and has been serving as a mathematician with Sandia Corporation, Albuquerque, New Mexico, has returned to the University.

Dr. J. D. Haggard of Kansas State Teachers College, Pittsburg, has been appointed Visiting Assistant Professor of Mathematics in the College of the University of Chicago for the academic year 1952–53.

Dr. P. C. Hammer, a group leader of hydrodynamics with I. B. M. Corporation, has been appointed to an associate professorship at the University of Wisconsin.

Mr. B. F. Handy is now at the Institute for Numerical Analysis of the National Bureau of Standards, Los Angeles, California.

Dr. Melvin Henriksen of the University of Alabama has been appointed to an instructorship at Purdue University.

Dr. S. H. Herzfeld, who has been doing research for the Office of Rubber Research, Reconstruction Finance Corporation, is now a consultant for the Office of Scientific Research, Air Research and Development Command, Baltimore, Maryland.

Mr. F. A. Kros, formerly a graduate student at the University of Colorado,

has accepted a position as a mathematician with White Sands Proving Ground, Las Cruces, New Mexico.

Professor Walter Leighton is on leave of absence from his position as Chairman of the Department of Mathematics of Washington University and is serving as a consultant for the Office of Scientific Research, Air Research and Development Command, Baltimore, Maryland.

Assistant Professor W. S. Massey of Brown University has been promoted to an associate professorship.

Professor Benjamin Evans Mitchell of Alabama Polytechnic Institute has been appointed to an assistant professorship at Louisiana State University.

Mr. P. S. Null of West Virginia University has accepted a position as Stress Analyst with Douglas Aircraft Company, Long Beach, California.

Dr. Anne F. O'Neill of Smith College has been appointed to an assistant professorship at Wheaton College, Norton, Massachusetts.

Mr. S. T. Paine, formerly of the Department of Meteorology, University of California at Los Angeles, has a position as Research Mathematician with Convair Aircraft Company, San Diego, California.

Dr. D. H. Porter of Indiana University has been appointed to an instructorship at Marion College.

Associate Professor Adrienne Rayl of the Birmingham Center of the University of Alabama has been promoted to a professorship.

Mr. Gordon Ritchie is teaching at Thessalon High School, Ontario, Canada.

Mr. R. W. Royston has been appointed to an assistant professorship at Washington and Lee University.

Associate Professor Robert Schatten of the University of Kansas has been promoted to a professorship.

Dean T. M. Simpson, Jr., of Randolph-Macon College has retired after thirty-three years of service.

Mr. W. B. Stovall, who has been with the Underway Training Element, United States Fleet Sonar School, San Diego, California, is returning to the Bureau of Vital Statistics, State Board of Health, Jacksonville, Florida, in the position of Chief.

Mr. R. E. Thomas of State University of Iowa has accepted a position as a mathematician with Battelle Memorial Institute, Columbus, Ohio.

Mr. M. W. Tolman, previously a student at the University of South Dakota, has a position as Junior Geophysicist with Shell Oil Company, Tulsa, Oklahoma.

Assistant Professor J. W. Warwick of Wellesley College has accepted a position as Scientific Consultant with the Harvard College Observatory, High Rolls, New Mexico.

Dr. J. G. Wendel of the Rand Corporation has been appointed to an assistant professorship at Louisiana State University.

Reverend B. C. Zimmerman who has been a Jesuit Missionary of Cayo, British Honduras, Central America died on April 22, 1952; he had been a member of the Association for twenty-one years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE THIRTY-THIRD SUMMER MEETING OF THE ASSOCIATION

The thirty-third summer meeting of the Mathematical Association of America was held at Michigan State College, East Lansing, Michigan, on Monday and Tuesday, September 1–2, 1952, in conjunction with the summer meetings of the American Mathematical Society, the Institute of Mathematical Statistics, the Econometric Society, and the Pi Mu Epsilon Fraternity. A total of seven hundred and thirty-four adults were registered, including the following three hundred and twenty-one members of the Association.

C. R. Adams, M. W. Al-Dhahir, Bess E. Allen, E. B. Allen, W. R. Allen, C. B. Allendoerfer, A. G. Anderson, R. D. Anderson, R. V. Andree, J. J. Andrews, H. A. Antosiewicz, Beulah M. Armstrong, K. J. Arnold, Miriam C. Ayer, W. L. Ayres, H. M. Bacon, J. M. Barbour, I. A. Barnett, C. F. Barr, D. Y. Barrer, R. C. F. Bartels, P. T. Bateman, W. D. Baten, H. M. Beatty, H. W. Becker, E. M. Beesley, E. G. Begle, J. H. Bell, Arthur Bernhart, Dorothy L. Bernstein, T. A. Bickerstaff, R. H. Bing, D. W. Blackett, Shirley A. Blackett, Barbara B. Blair, H. D. Block, C. R. Blyth, R. P. Boas, H. F. Bohnenblust, Julia W. Bower, Angeline J. Brandt, A. T. Brauer, G. U. Brauer, J. L. Brenner, Leonard Bristow, J. C. Brixey, Myrtle C. Brown, R. H. Bruck, C. C. Buck, R. C. Buck, C. E. Buell, R. A. Burrows, J. H. Bushey, Jewell H. Bushey, A. T. Butson, H. E. Campbell, E. L. Canfield, K. H. Carlson, R. E. Carr, W. B. Carver, Jeremiah Certaine, Josephine H. Chanler, Abraham Charnes, Y. W. Chen, S. S. Chern, F. Marion Clarke, Helen E. Clarkson, D. E. Coffee, R. H. Cole, N. B. Conkwright, S. D. Conte, Geraldine A. Coon, T. F. Cope, A. H. Copeland, Max Coral, R. R. Coveyou, V. F. Cowling, H. S. M. Coxeter, J. W. Coy, C. C. Craig, A. B. Cunninghan, Grace M. Cutler, D. A. Darling, Robert Davies, Violet B. Davis, W. M. Davis, R. B. Deal, W. A. Dolid, D. W. Dubois, W. L. Duren, P. S. Dwyer, J. M. Earl, W. F. Eberlein, P. D. Edwards, J. G. Elliott, David Ellis, Benjamin Epstein, D. H. Erkiletian, H. P. Evans, W. H. Fagerstrom, D. T. Finkbeiner, C. D. Firestone, K. W. Folley, J. R. Foote, M. K. Fort, J. S. Frame, Evelyn Frank, C. H. Frick, C. V. Fronabarger, W. B. Fulks, H. K. Fulmer, A. E. Gault, H. M. Gehman, J. J. Gergen, K. G. Getman, J. W. Givens, A. J. Goldman, R. A. Good, A. W. Goodman, Lillian Gough, S. H. Gould, Cornelius Gouwens, Beulah Graham, A. A. Grau, L. M. Graves, R. E. Graves, F. L. Griffin, H. C. Griffith, Betty Grossman, Franklin Haimo, P. R. Halmos, P. C. Hammer, H. W. Handsfield, Frank Harary, M. C. Hartley, E. R. Heineman, R. G. Helsel, M. S. Hendrickson, Anna S. Henriques, I. N. Herstein, Fritz Herzog, G. W. Hess, T. H. Hildebrandt, J. E. Hoffman, R. V. Hogg, D. L. Holl, Aughtum S. Howard, W. J. Huebner, Ralph Hull, J. W. Hurst, W. R. Hutcherson, C. G. Jaeger, R. D. James, T. A. Jeeves, L. W. Johnson, R. E. Johnson, L. S. Johnston, B. W. Jones, P. S. Jones, Wilfred Kaplan, Leo Katz, Dora E. Kearney, M. W. Keller, L. M. Kelly, D. E. Kibbey, E. C. Kiefer, J. M. Kingston, Evelyn K. Kinney, S. C. Kleene, L. A. Knowler, F. W. Kokomoor, W. H. Kruskal, R. M. Lakness, R. E. Langer, Leo Lapidus, E. H. Larguier, H. D. Larsen, Margaret M. LaSalle, C. G. Latimer, J. R. Lee, Walter Leighton, Tadeusz Leser, F. A. Lewis, A. J. Lohwater, C. I. Lubin, R. W. MacDowell, C. C. MacDuffee, Saunders MacLane, H. M. MacNeille, Ella Marth, Beckham Martin, W. T. Martin, Kenneth May, June M. McArtney, Dorothy McCoy, N. H. McCoy, L. E. Mehlenbacher, B. E. Meserve, D. M. Mesner, E. J. Mickle, W. E. Milne, Benjamin Ernest Mitchell, Josephine M. Mitchell, Harriet F. Montague, Mabel D. Montgomery, D. C. Morrow, F. C. Mosteller, H. T. Muhly, C. W. Munshower, W. L. Murdock, S. B. Myers, Zeev Nehari, A. L. Nelson, Ivan Niven, M. L. Norden, E. A. Nordhaus, E. S. Northam, Ilse L. Novak, D. G. O'Connor, Ruth E. O'Donnell, E. G. Olds, R. R. Otter, J. C. Oxtoby, C. D. Parker, Mary H. Payne, L. L. Pennisi, F. W. Perkins, H. P. Pettit, George Piranian, A. E. Pitcher, J. C. Polley, J. E. Powell, G. B. Price, C. B.

Rader, Tibor Rado, Henry Rainbow, J. F. Randolph, G. E. Raynor, Mina S. Rees, P. V. Reichelderfer, W. T. Reid, C. E. Rhodes, H. B. Ribeiro, F. A. Rickey, P. R. Rider, D.D. Rippe, J. H. Roberts, R. A. Rosenbaum, P. C. Rosenbloom, Arthur Rosenthal, M. F. Rosskopf, E. H. Rothe, Charles Saltzer, Hans Samelson, A. C. Schaeffer, R. D. Schafer, J. A. Schatz, Henry Scheffe, E. D. Schell, Edith R. Schneckenburger, Augusta L. Schurrer, C. H. W. Sedgewick, Esther Seiden, D. H. Shaffer, M. E. Shanks, Sister Mary Henrietta, R. L. Shively, Edward Silverman, M. F. Smiley, G. W. Smith, Ernst Snapper, W. S. Snyder, Elizabeth S. Sokolnikoff, C. E. Springer, E. P. Starke, F. H. Steen, H. E. Stelson, C. F. Stephens, B. M. Stewart, Ruth W. Stokes, R. R. Stoll, E. H. Stone, M. H. Stone, H. C. Stotz, D. D. Strebe, P. C. Sweetland, Gabor Szego, A. D. Talkington, O. E. Taulbee, Mildred E. Taylor, William Clare Taylor, H. P. Thielman, W. J. Thron, H. E. Tinnappel, John Todd, Leonard Tornheim, A. W. Tucker, Bryant Tuckerman, J. W. Tukey, Eugene Usdin, E. P. Vance, G. W. Walker, R. M. Walter, S. S. Walters, D. L. Webb, G. C. Webber, C. P. Wells, Irene L. Wente, W. M. Whyburn, L. R. Wilcox, J. E. Wilkins, G. P. Williams, Y. K. Wong, A. W. Wortham, R. L. Young, B. K. Youse, J. H. Zant, W. M. Zaring, R. K. Zeigler, J. L. Zemmer, J. A. Zilber.

Sessions of the Association were held on Monday afternoon at 2:00 with President Saunders MacLane presiding, on Tuesday morning at 9:00 with Vice-President Jewell H. Bushey presiding, and on Tuesday evening at 7:30 with Vice-President F. L. Griffin presiding. All sessions were held in Room 118, Physics-Mathematics Building of Michigan State College. A feature of the sessions was the first series of Earle Raymond Hedrick Lectures delivered by Professor Tibor Rado. The Program Committee for the meeting consisted of E. P. Vance, Chairman, O. E. Lancaster, and Kenneth May.

FIRST SESSION OF THE ASSOCIATION

"Illustrations of Game Theory," by Professor Harold W. Kuhn, Princeton University.

"On Minimizing the Oscillation of a Function," by Dr. J. E. Wilkins, Nuclear Development Associates. (Presented by title.)

"What Machines Cannot Do," by Professor P. C. Rosenbloom, University of Minnesota.

The Earle Raymond Hedrick Lectures: "Derivatives and Jacobians," Lecture I, by Professor Tibor Rado, Ohio State University.

SECOND SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Derivatives and Jacobians," Lecture II, by Professor Tibor Rado, Ohio State University.

Symposium on the Mathematical Training of Engineers, Chairman, Professor Ralph Hull, Purdue University.

"Mathematics and the Engineering Curriculum," by Professor E. B. Allen, Rensselaer Polytechnic Institute.

"Mathematics Used for Engineering Designs," by Dr. C. H. Harry, Bureau of Aeronautics, Department of the Navy.

"Problems of Mathematicians Who Teach Engineers," by Professor Elizabeth S. Sokolnikoff, University of Wisconsin.

THIRD SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Derivatives and Jacobians," Lecture III, by Professor Tibor Rado, Ohio State University.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Mathematics Library in the Physics-Mathematics Building. This meeting was preceded by joint meetings of the Executive and Finance Committees on Sunday evening and Monday morning. Of the forty members of the Board, there were thirty present, including all the officers of the Association. Among the more important items of business transacted were the following:

Professor W. H. Fagerstrom was present to discuss the contests for high school students which have been conducted for several years by a committee of the Metropolitan New York Section. The Board voted to commend the section on its activity in this field and to recommend that it continue to conduct contests on approximately the present scale, taking care to avoid conflict with analogous activities of other sections and other organizations.

The Board voted to authorize the publication of a joint List of Members with the American Mathematical Society. It is expected that copies will be distributed to members of the Association in odd-numbered years.

On the recommendation of the Executive Committee, the Board voted to re-elect Professor Edith R. Schneckenburger of the University of Buffalo as Associate Secretary of the Association for a second term of five years, beginning on January 1, 1953.

The Board of Governors voted to instruct the Secretary to send copies of the following resolution to the Governor of the State of Oklahoma, to the presiding officers of the two houses of the Oklahoma Legislature, to the Attorney General of Oklahoma, and to the Presidents of Oklahoma A and M College and the University of Oklahoma, and to read the resolution at the next session of the Association and to publish the resolution in full in the account of this meeting:

"The Board of Governors of the Mathematical Association of America have considered the effect upon mathematics and mathematicians of the Oklahoma Loyalty Oath Law of April 9, 1951.

"The Board wishes to convey to the heads of government in Oklahoma their view that many mathematicians regard this type of legislation as dangerous to individual liberty and to the principles which protect freedom of thought in a university from political infringement, and as ineffective for the purpose of eliminating disloyal influences from the universities. Even those who sign such oaths often do so more in a spirit of tolerance for the patriotic motives which conceived the law than in respect for the law itself, or more in fear than in concurrence.

"Whenever such laws are enacted, it is to be expected that at least a few of the American idealists, the sincerely religious people, and the courageous lovers

of liberty will rebel, with tragic effects upon themselves, their families, and upon the universities where they work. It is not to be expected that such legislation will be effective in eliminating from the faculties men who are dangerous to the national welfare, so that the injury caused is a useless waste, as at Oklahoma A and M College, where a department of mathematics which had achieved much recognition for its mathematical work was seriously damaged."

MEETINGS OF OTHER ORGANIZATIONS

The sessions of the American Mathematical Society began on the afternoon of Tuesday, September 2, and continued through Friday. The Colloquium lectures were delivered by Professor Alfred Tarski of the University of California on the topic: "Arithmetical classes and types of algebraic systems." An invited address with the title "On the topology of 3-manifolds" was delivered by Professor E. E. Moise of the University of Michigan.

The Institute of Mathematical Statistics and the Econometric Society held sessions beginning on Tuesday morning and continuing through Friday afternoon. The Pi Mu Epsilon Fraternity held its sessions on Sunday evening and on Monday morning and evening.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements for the meeting consisted of J. S. Frame, Chairman; J. H. Bell, H. M. Gehman, Fritz Herzog, Leo Katz, E. A. Nordhaus, H. E. Stelson, B. M. Stewart, J. W. T. Youngs.

Registration headquarters were in the lobby of Phillips Hall, and accommodations for arrivals were available at all hours of the day or night. Rooms in Phillips Hall and Snyder Hall were open for occupancy from Sunday afternoon, August 31, until Saturday noon, September 6. Meals were served in the cafeteria of Phillips Hall.

An informal coffee hour was held on Monday evening in the Conference Room of the Physics-Mathematics Building. There was an informal tea on Tuesday afternoon in Phillips Lower Lounge. Both of these events were arranged by the members of the Michigan State College Mathematics Department and their wives.

The banquet for the mathematical organizations was held on Wednesday evening in the Union building. Professor C. B. Allendoerfer acted as toastmaster, and introduced representatives of each of the five mathematical organizations. President John A. Hannah of Michigan State College spoke on the scientific aspects of the Point Four Program. Professor J. J. Gergen of Duke University presented a resolution of thanks to our hosts for their warm hospitality and excellent arrangements, which was adopted by an enthusiastic vote.

On Thursday evening members of the Department of Music of Michigan State College presented a Sonata Recital including classical and modern music for the bassoon, oboe, viola, violin and piano.

The annual party of the Institute of Mathematical Statistics took place on Thursday evening at the Coral Gables Ballroom.

Conducted tours of the campus were held on Tuesday and Wednesday mornings, and there was a trip through the Oldsmobile plant on Wednesday afternoon. On Thursday was held an all day trip to the Henry Ford Museum and Greenfield Village in Dearborn.

An exhibit of mathematical textbooks was available during the week in the Physics-Mathematics Building.

All those who attended the meeting are grateful to the members of the staff of Michigan State College for their hospitality and for the efficiency with which the details of the meeting were carried out.

H. M. GEHMAN, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

Thirty-fourth Summer Meeting, Laval University, Quebec, Canada, August 31–September 1, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS

KENTUCKY, University of Louisville, Spring, 1953.

LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Howard University, Washington, D. C., December 6, 1952.

METROPOLITAN NEW YORK, Spring, 1953.

MICHIGAN, Western Michigan College of Education, Kalamazoo, April 11, 1953.

MINNESOTA

MISSOURI, William Jewell College, Liberty, Spring, 1953.

NEBRASKA

NORTHERN CALIFORNIA

OHIO

OKLAHOMA

PACIFIC NORTHWEST, Montana State University, Missoula, June, 1953.

PHILADELPHIA, University of Delaware, Newark, November 29, 1952.

ROCKY MOUNTAIN, University of Colorado, Boulder, April, 1953.

SOUTHEASTERN, Alabama Polytechnic Institute, Auburn, March 13–14, 1953.

SOUTHERN CALIFORNIA, Los Angeles City College, March 14, 1953.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanic Arts, State College, January 2–3, 1953.

TEXAS

UPPER NEW YORK STATE, United States Military Academy, West Point, Spring, 1953.

WISCONSIN, Mount Mary College, Milwaukee, May, 1953.



TRIGONOMETRY, PLANE AND SPHERICAL

By LLOYD L. SMAIL, Lehigh University, 405 pages, (with tables) $3.75

A basic text for a standard college course in trigonometry, with due attention to both the numerical and theoretical aspects of the subject. The arrangement has been planned to give utmost flexibility, so as to make the book adaptable to courses of varying lengths and needs.

ELEMENTS OF STATISTICAL METHOD. New 3rd Edition

By ALBERT E. WAUGH, University of Connecticut. 531 pages, $5.50

This text is designed to introduce the student to statistical concepts and nomenclature and to encourage him to think in statistical terms. Every effort is made to keep the discussion at the beginner's level and to present basic ideas in such a way that the student will find it easy to continue under his own power. *Statistical Tables and Problems,* Third Edition, is available to accompany the text and develops in the student a real appreciation of statistical data.

BUSINESS STATISTICS. New 3rd Edition

By JOHN R. RIGGLEMAN, Consulting Economist and Statistician; and IRA N. FRISBEE, University of California. 818 pages, $6.00

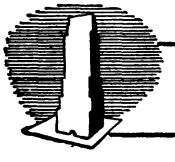
This text views business statistics from the point of view of the businessman, who judges the value of statistics according to their usefulness in securing practical results. The third edition has been expanded particularly with regard to machine tabulation, business cycles, business forecasting, and marketing analysis. *Graphic Methods for Presenting Business Statistics* by John R. Riggleman, Second Edition, 256 pages, $3.50 provides a direct, practical and detailed discussion of the principles and practice of modern statistical charting which shows the student how to make graphic charts and how to use them effectively.

ESSENTIAL BUSINESS MATHEMATICS. New 2nd Edition

By L. R. SNYDER, City College of San Francisco. Ready in November

A general revision of this very popular business mathematics text, the second edition has been revised and brought up to date on all topics. In some instances material has been condensed but without sacrifice of clarity or teachability; in others, topics have been expanded to provide fuller coverage. *A Student's Workbook for Essential Business Mathematics,* Second Edition, will also be available to accompany the text. Assignments in the workbook follow the same number arrangement as do the units and exercises in the text.

Send for copies on approval



McGRAW-HILL BOOK COMPANY, INC.

330 WEST 42ND STREET, NEW YORK 36, N. Y.

WILEY BOOKS

An effective stimulant to creative and original thinking . . .

FOUNDATIONS OF MATHEMATICS

By RAYMOND L. WILDER

Research Professor of Mathematics, University of Michigan

This new book presents mathematics not as a discipline to be accepted without question of source and validity; rather it inquires critically into the nature of this subject and the foundations upon which it is built. Professor Wilder provides information regarding fundamental principles and methods indispensable for an understanding of modern mathematics. The axiomatic method, theory of sets, number systems, and basic algebraic notions are scrutinized. The book also examines these important topics:

- the source and evolution of mathematical concepts and mathematical existence
- the nature and importance of the contradictions and paradoxes, and methods of avoiding them
- the origin and growth of various present day foundations theories—theories such as those based on the Zermelo axioms, the methods of symbolic logic, Intuitionism, etc.
- and the position of mathematics as a branch of human knowledge

Ready now. ***305 pages.*** ***$5.75.***

The theory of multivariate analysis and its applications . . .

ADVANCED STATISTICAL METHODS IN BIOMETRIC RESEARCH*

By C. RADHAKRISHNA RAO, *Professor of Statistics, Indian Statistical Institute, Calcutta.* Here is a unified treatment of mathematical distributions, statistical inference, and diverse computational techniques. Most of the examples are drawn from the field of biometric research. Highlighting Rao's presentation is the introduction of discriminatory topology, useful in ordering statistical groups to discover their affinities and differences.

1952. ***390 pages.*** ***$7.50.***

A basic book to ground the reader for advanced work . . .

ELEMENTARY ANALYSIS

By KENNETH O. MAY, *Carleton College.* "Professor May has **not only** explained fundamental concepts with patience and clarity, but he has also made the student **really think** about them in his well-written problems."—JULIA WELLS BOWER, Chairman, Department of Mathematics, Connecticut College.

1952. ***635 pages.*** ***$5.00.***

Experimental design in detail . . .

METHODS OF STATISTICAL ANALYSIS*, *2nd Edition*

By CYRIL H. GOULDEN, *Canadian Department of Agriculture.* Detailed methods of analysis, geared to give the reader a knowledge of the principles necessary to understand experimental design. This book includes valuable information on basic designs, factorial experiments, and incomplete block designs. Introduced for the first time in a general book are: modern methods of computation for partial and multiple correlation; the discriminant function; factorial experiments; treatment of non-orthogonal data; probit analysis; and quality control.

1952. ***467 pages.*** ***$7.50.***

** A Wiley Publication in Statistics, Walter A. Shewhart, Editor.*

Send for on-approval copies today.

JOHN WILEY & SONS, Inc. 440-4th Ave., New York 16, N.Y.

FOUNDATIONS OF ALGEBRAIC TOPOLOGY

By Samuel Eilenberg and Norman Steenrod

The need for an axiomatization of homology and cohomology theory has long been felt by topologists, and it has led them to the axiomatizing of several stages in the construction of homology groups. Here, for the first time, is an axiomatization of the complete transition from topology to algebra.

Samuel Eilenberg and Norman Steenrod are professors of mathematics at Columbia and Princeton universities respectively.

No. 15 in the Princeton Mathematical Series.

356 pages. *$7.50*

SYMMETRY

By Hermann Weyl

Displaying the great variety of applications of the principle of symmetry in the arts and in nature, this book demonstrates its philosophical and mathematical significance. Numerous illustrations, making this a handsome book, are taken from many fields of art and science.

Starting from the vague notion of symmetry-harmony of proportions, the author gradually develops first the geometric concept of symmetry in its several forms, and finally rises to the general abstract mathematical idea underlying all these varied forms.

176 pages. Profusely illustrated. *$3.75*

Order from your bookstore

PRINCETON UNIVERSITY PRESS

Princeton, New Jersey

THE SLAUGHT PAPERS

The Herbert Ellsworth Slaught Memorial Papers are a series of brief expository pamphlets published as supplements to the American Mathematical Monthly. Two numbers have appeared:

1. *Fourier's Series; The Genesis and Evolution of a Theory* by R. E. Langer. v + 86 pages. Paper.

2. *Outline of the History of Mathematics (6th edition)* by R. C. Archibald. iv + 114 pages. Paper.

Copies at one dollar each postpaid may be ordered from:

The Mathematical Association of America
University of Buffalo
Buffalo 14, New York

Algebra Texts *by Raymond W. Brink*

Second Editions

College Algebra

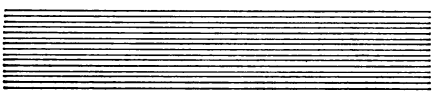
Notable for its accuracy, logic and flexibility, this very complete, first-year text includes a review of high-school algebra. Contains exercises, study aids, diagrams and tables. *490 pp., $3.75*

Algebra: College Course

Intended for students not requiring a review of high-school algebra, this book concentrates on the development of technical proficiency and basic understanding. *378 pp., $3.25*

Intermediate Algebra

Designed for college students having only one year of high-school algebra, this text presents in full the first fifteen chapters of *College Algebra, 2nd Edition,* with problems and tables. *295 pp., $3.00*



Appleton - Century - Crofts, Inc.
35 West 32nd Street, New York 1, N.Y.

HEATH COLLEGE TEXTS

DIFFERENTIAL EQUATIONS

ALFRED L. NELSON, KARL W. FOLLEY, AND MAX CORAL, Wayne University

Comprehensiveness and emphasis on applications make this text suitable for engineering students as well as for students of pure mathematics. Also, this book includes a good chapter on numerical methods. 309 pages. $3.25

ELEMENTS OF ANALYTIC GEOMETRY

WILLIAM L. HART

Provides the content, in plane and solid analytic geometry, which is essential as preparation for calculus and for the applications of analytic geometry itself in engineering, the physical sciences, and statistics. 271 pages (239 pages text). $3.00

INTRODUCTION TO THE MATHEMATICS OF BUSINESS

WILLIAM L. HART

An elementary but reasonably complete treatment, with a simplified approach to annuities and life insurance. Includes a substantial review of pertinent parts of intermediate algebra and selected elementary topics in statistics. 428 pages (326 pages text). $3.25

D. C. HEATH

AND COMPANY



SALES OFFICES: NEW YORK CHICAGO SAN FRANCISCO ATLANTA DALLAS

HOME OFFICE: BOSTON

3 The Experts Rate Tops

ALGEBRA FOR COMMERCE AND LIBERAL ARTS

By A. K. BETTINGER and W. A. DWYER. A clear and simple exposition of the algebraic foundations and skills needed for the mathematics of finance and statistics. Well-graded illustrative examples, drill exercises, world problems, and review exercises. 225 pages. Answers available. $3.00

Currently being used at Creighton, Drexel Institute, St. Louis Preparatory Seminary, Detroit, Hofstra, North Dakota, Northwestern, and many other schools.

MATHEMATICS FOR FINANCE AND ACCOUNTING

By J. B. COLEMAN and W. O. ROGERS. The first text of its kind to correlate mathematics with accounting and business administration courses. Up-to-date mortality tables, numerous review exercises and problems highlight the text. 310 pages. Answers available. $4.00

Adopted by Pennsylvania State College, South Carolina, Wyoming, Marquette, Duke, Seattle, Idaho, Kentucky, Creighton, among others.

COLLEGE ALGEBRA

By HARRY A. BENDER. Explains fundamental concepts and emphasizes manipulation by encouraging the student to develop his analytical skills through the numerous general problems and exercises. Clear explanations for formulas and expressions eliminate the need for close supervision. 452 pages. $3.75

Adopted by Rhode Island, Tulane, Pennsylvania State College, New York Polytechnic Institute, John Carroll University, Franciscan Preparatory University, Creighton, San Jose State College among others.

You are cordially invited to send for examination copies.

PITMAN **2 West 45th Street** **New York 36**



OFF THE PRESS . . .

And Ready for Your Classes

Fundamental Procedures Of Financial Mathematics

RASSWEILER & RASSWEILER

A student without experience in mathematics beyond arithmetic can digest this practical book dealing with business and investment mathematics. The book is designed to give the student a thorough grounding in percentage and its application to business through commissions, taxes, pricing, interest and discount, and negotiable instruments. The authors include sets of problems divided into practice problems and situation problems. $3.25

Theory of Numbers

B. M. STEWART

"In my opinion it will overshadow comparable existing texts," says one college professor who read the manuscript. The author calls attention to basic chapters and includes optional related topics. In addition to the excellent assortment of illustrative examples and exercises which test comprehension and growth, there is more advanced material which serves to prepare students for further study in equivalence relations, abstract mathematical systems, groups of transformations, matrices domains and fields. $5.50

THE MACMILLAN COMPANY

60 FIFTH AVE., NEW YORK 11, N.Y.

Prentice-Hall

BASIC MATHEMATICS FOR ENGINEERING AND SCIENCE

By Walter R. Van Voorhis and Elmer E. Haskins, Fenn College

The subjects in this text are unified under the function concept, so basic to scientific mathematics. Within this framework most of the traditional material of algebra, trigonometry and analytic geometry, plus a few additional topics selected with a view to strengthening the students' preparation for calculus, is presented in an integrated pattern designed to stimulate a high degree of interest throughout.

619 Pages 6" x 8½" Published 1952

MODERN ELEMENTARY STATISTICS

By John E. Freund—Alfred University

Designed for students in the social and natural sciences who have very little background in mathematics. It emphasizes the *meaning* of statistics rather than the acquisition of mathematical skills. Theoretical distributions are introduced as early as Chapter 3 on a more or less intuitive level. Chapter 7 has a discussion and repeated emphasis on the meaning of probability statements. For the first time, the modern theory of the testing of hypotheses is presented on the non-technical level.

Approx. 418 Pages 6" x 8½" Published 1952

PLANE TRIGONOMETRY—Third Edition

By Fred W. Sparks—Texas Technological College
and Paul K. Rees—Louisiana State University

PLANE TRIGONOMETRY, Third Edition, like earlier editions, covers all essentials—including logarithms, graphs of trigonometric functions, and trigonometric equations.

The method used by the U. S. Air Force for designating directions is explained and problems in elementary air navigation have been added to most exercises dealing with the solution of triangles. The United States Naval Academy used the Revised Edition in their classes, and has also adopted the new Third Edition.

275 Pages with tables 6¼" x 9¼" Published 1952
199 Pages without tables



GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

INDEX TO VOLUME 59, 1952

THE AMERICAN MATHEMATICAL MONTHLY

GENERAL MATHEMATICAL PAPERS

ALGEBRA

FRANK, EVELYN. On Certain Determinantal Equations, 300–309.

GOODMAN, A. W. and W. M. ZARING. Euclid's Algorithm and the Least-Remainder Algorithm, 156–159.

LEAVITT, W. G. Mappings of Vector Spaces and the Theory of Matrices, 219–222.

MOLLER, RAYMOND. Sums of Powers of Numbers having a Given Exponent Modulo A Prime, 226–230.

ORE, OYSTEIN. The General Chinese Remainder Theorem, 365–370.

RIORDAN, JOHN. A Recurrence Relation for Three-Line Latin Rectangles, 159–162.

SALZER, H. E. An Elementary Note on Powers of Quaternions, 298–300.

SAMET, P. A. An Equation in Gaussian Integers, 448–452.

SZEKERES, G. A Canonical Basis for the Ideals of a Polynomial Domain, 379–386.

ZARING, W. M. See Goodman, A. W.

ANALYSIS

BAGCHI, HARI DAS, and CHATTERJEE, PHATIK CHAND. Note on a Second Functional Equation, Connected with the Function $\wp(z)$, 91–92.

BUSH, K. A. Continuous Functions without Derivatives, 222–225.

CARLITZ, L. Note on a Paper of Bagchi and Chatterjee, 683–684.

CHATTERJEE, PHATIK CHAND. See Bagchi, Hari Das.

CHURCHILL, R. V. Integral Transforms and Boundary Value Problems, 149–155.

ERDÉLYI, A. See Weber, Maria.

FORT, M. K., JR. Some Properties of Continuous Functions, 372–375.

MCSHANE, E. J. Partial Orderings and Moore-Smith Limits, 1–11.

PHIPPS, C. G. Maxima and Minima Under Restraint, 230–235.

TAYLOR, A. E. L'Hospital's Rule, 20–24.

WEBER, MARIA and ERDÉLYI, A. On the Finite Difference Analogue of Rodriques' Formula, 163–168.

WIGNER, E. P. Simplified Derivation of the Properties of Elementary Transcendentals 669–683.

APPLIED MATHEMATICS, STATISTICS

GALE, DAVID. An Indeterminate Problem in Classical Mechanics, 291–295.

ROBBINS, HERBERT. A Note on Gambling Systems and Birth Statistics, 685–686.

EDUCATION, HISTORY

BLISS, G. A. Autobiographical Notes, 595–606.

BUSH, L. E. The William Lowell Putnam Mathematical Competition, 538–542.

COOLIDGE, J. L. The Origin of Polar Coordinates, 78–85.

———. The Unsatisfactory Story of Curvature, 375–379.

LANGER, R. E. The Things I Should Have Done, I Did Not Do, 443–448.

RICHARDSON, MOSES. Mathematics and Intellectual Honesty, 73–78.

GEOMETRY

GADDUM, J. W. The Sums of the Dihedral and Trihedral Angles in a Tetrahedron, 370–371.

MOSER, LEO. On the Different Distances Determined by n Points, 85–91.

MUSSELMAN, J. R. On the Rectangular Hyperbola, 11–19.

ROBINSON, ROBIN. Robinson's Constant, 296–297.

THÉBAULT, VICTOR. Perspective and Orthologic Triangles and Tetrahedrons, 24–28.

UNDERWOOD, R. S. Functions of N Variables in Extended Analytic Geometry, 453–460.

WALKER, A. W. The Differential Equation of a Conic and Its Relation to the Aberrancy, 531–538.

WHYTE, L. L. Unique Arrangements of Points on a Sphere, 606–611.

LOGIC

QUINE, W. V. The Problem of Simplifying Truth Functions, 521–531.

SWIFT, J. D. Algebraic Properties of N-Valued Propositional Calculi, 612–621.

MATHEMATICAL NOTES

Edited by F. A. FICKEN, University of Tennessee

ADNEY, J. E. See Herstein, I. N.
BELLMAN, RICHARD. A Note on Scalar Functions of Matrices, 391–392.
CAMPBELL, J. G. A Criterion for the Polynomial Solutions of a Certain Riccati Equation, 388–389.
CARLITZ, LEONARD. Note on a Paper of Shanks, 239–241.
———. Note on an Arithmetic Function, 386–387.
CHURCH, ELSIE. A Certain Cubic Transformation, 314–315.
DANSKIN, J. M. Dresher's Inequality, 687–688.
ELKIN, J. M. A General Rule for Divisibility Based on the Decimal Expansion of the Reciprocal of the Divisor, 316–318.
FULTON, C. M. Weierstrass Tensors, 544–547.
HERSTEIN, I. N. and ADNEY, J. E. A Note on the Automorphism Group of a Finite Group, 309–310.
HODGE, P. G. JR. On the Sign of a Function in a Closed Interval, 461–462.
KOLODNER, I. I. On the Applicability of a Certain Formula in the Theory of Linear Differential Equations, 168–170.
LOTKIN, MARK. Polynomials Having a Root Approaching π, 96–98.
MCKEAN, H. P. JR. A New Proof of the Completeness of the Hermite Functions, 621–622.
MCLEAN, DAVID. Cubic Equations in Groups, 624–626.
MAXFIELD, J. E. Sums and Products of Normal Numbers, 98.
MILLER, D. G. Postulates for Boolean Algebra, 93–96.
MOPPERT, C. F. Deduction of Cardano's Formula by Conformal Mapping, 310–314.
NIBLETT, J. D. A Theorem of Nesbitt, 171–174.
OPATOWSKI, I. Laplace Transform of $(\text{erf } \sqrt{t})^2$, 392.
PORGES, ARTHUR. A Continued Fraction Cipher, 236.
ROOM, T. G. Composition of Rotations in Euclidean Three-space, 688–692.
RUDERMAN, H. D. Two New Inequalities, 29–32.
SANDHAM, H. F. The Approximation of Radicals by Rational Means, 622–624.
THÉBAULT, VICTOR. A Note on Orthopolar Triangles, 542–543.
THURSTON, H. S. A Correction, 32.
TORNHEIM, LEONARD. The Sylvester-Franke Theorem, 389–391.
WALTERS, S. S. On Ascoli's Theorem, 237–238.
WRIGHT, E. M., Correction, 99.

CLASSROOM NOTES

Edited by G. B. THOMAS, Massachusetts Institute of Technology.

ADLER, CLAIRE. A Modern Trick, 328.
ALBERTI, FURIO. A Theorem on Evolutes, 177; 406.
APOSTOL, T. M. Term-wise Differentiation of Power Series, 323–326.
BALLANTINE, J. P. Inverse Matrices in Interpolation, 178–180.
BALLANTINE, J. P. and JERBERT, A. R. Distance From a Line, or Plane, to a Point, 242–243.
BARROW, D. F. See Huff, G. B.
BOURNE, SAMUEL. Coordinate Geometry from the Vector Point of View, 245–248.
BRAUER, ALFRED. The Proof of the Law of Sines, 319.
CORLISS, J. J. Volumes of Revolution, 37–40.
EDITORIAL. Differentials, 403–406.
FORT, M. K. JR. Differentials, 392–395.
FUCHS, L. and SZELE, T. Introduction of Complex Numbers as Vectors of the Plane, 628–631.
FULTON, C. M. A Simple Proof of the Binomial Theorem, 243–244.
GREEN, J. W. On the Envelope of Curves Given in Parametric Form, 626–628.
HAMILTON, H. J. Cauchy's Form of R_n from the Iterated Integral Form, 320.
———. Toward Understanding Differentials, 398–403.
HART, J. J. A Correction for the Trapezoidal Rule, 33–37; 406.
HUFF, G. B. and BARROW, D. F. A Minute Theory of Radical Equations, 320–323.
JERBERT, A. R. See Ballantine, J. P.
LEWIS, D. C. A Connection between Taylor's Theorem and Linear Differential Equations, 692–693.
LYON, RICHARD and WARD, MORGAN. The Limit for e, 102–103.
MISRA, B. B. On Lines of Invariability of Continuous Functions, 550.
NAGLE, T. D. A Method for the Solution of the General Cubic, 326–328.
NORRIS, M. J. Integrability of Continuous Functions, 244–245.
OGILVY, C. S. Rulings, 547–549.
OSBORN, ROGER. A Note on Indeterminate Forms, 549–550.
PHIPPS, C. G. The Relation of Differential and Delta Increments, 395–398.
RICHMOND, D. E. Inverse Square Orbits. A Simple Treatment, 694–696.
SPIEGEL, M. R. Student's Choice in Mathematics—Fundamental Reasoning or Blind Subservience to Rules, 99–100.
———. Reciprocal Quadratic Equations, 175–177.
———. The Bernoulli Differential Equation, 632–633.

SZELE, T. See Fuchs, L.
WALKER, A. W. An Elementary Integral, 103–104.
WARD, J. A. An Extension of Plane Analytic Geometry, 463–464.
WARD, MORGAN. See Lyon, Richard.
WITTY, W. H. Generalization of Clairaut's Differential Equation, 100–102.

PROBLEMS AND SOLUTIONS

Edited by HOWARD EVES, State University of New York, and E. P. STARKE, Rutgers University

AUTHORS

Numbers refer to pages, boldface type indicating a problem solved and solution published; italics, a problem solved but the solution not published; ordinary type a problem proposed.

Abella, I. D., *639*.
Ablow, C. M., *417*.
Acton, F. S., **337**.
Aheart, A. N., *107*, *250*, *470*, *638*.
Aitken, A. C., *335*.
Alder, H. L., *191*.
Alonso, Pompeys, *185*.
Amos, D. E., *108*.
Anderson, A. G., 633.
Andersson, J., 41, *466*, **553**.
Andree, R. V., *106*.
Ankeny, N. C., *338*.
Anning, Norman, *191*, 328, *558*.
Anselone, P. M., *107*, *250*, *331*, *411*.
Apostol, T. M., 104, 110, *331*, *472*, *557*, *635*, *706*.
Arens, H. C., *338*.
Arens, R., *417*.
Atkins, Ferrel, **250**, *252*, *259*.
Aude, H. T. R., *330*.
Bagemihl, F., *44*, **330**, *411*, *556*, *638*, **650**.
Baker, G. A., Jr., **635**.
Baldwin, J. W., *408*, *411*, *412*.
Ballantine, J. P., *558*.
Bankoff, Leon, *106*, *107*, *184*, *250*, *250*, **251**, *252*, *330*, *331*, *408*, *411*, **634**, *638*.
Barlaz, J., *650*, *706*.
Barrick, D. L., **639**.
Bateman, P. T., **338**, *556*, *641*, *648*.
Bender, P., *42*.
Bergmann, H. G., *106*, *107*.
Berkofsky, Louis, *107*.
Berndt, A., *636*, *638*, *639*.
Berndtson, C. R., *638*.
Berry, H. H., *106*, 180, *330*.
Block, H. D., *417*, **555**.
Blumenthal, L. M., *47*, 332.
Boas, R. P., *648*.
Boggs, A. B., **191**, *191*.
Botsford, J. L., **252**, *411*, *412*.
Bowersox, E. R., *330*.
Brauer, A., *338*.
Braun, J. H., *42*, **43**, *106*, *107*, *184*, *185*, *186*, *250*, *252*, *253*, *330*, *408*, *411*, *468*, *634*, *638*, **697**, *701*.
Breusch, R. H., **645**.
Briggs, W. E., *330*, *634*, *636*.
Brown, B. H., *181*.
Brown, R. H., *648*.
Browne, D. H., *106*, *108*, *191*, *418*, *470*, *472*.
Bruce, C. W., *331*, *639*.
Buchman, Aaron, *411*.
Buck, R. C., *417*.
Buker, W. E., *106*, *330*.
Burr, I. W., 633.
Bush, Kenneth, *411*, *412*, *636*, *637*.
Bushey, J. H., *338*.
Byrne, W. E., *473*, **647**.
Camier, E. D., **48**.
Carlitz, L., *191*, *412*, *470*, 701, **706**, *706*.
Carpenter, F. M., *410*.
Carter, W. O., *330*.
Chessin, P. L., *330*, *636*, 697.
Chovitz, B., *638*.
Christopher, J., *331*.
Chung, K. L., 407.
Clark, A. G., *189*.
Clark, B., *330*.
Clark, R., 465, *466*.
Clutterham, D. R., *330*.
Cohen, Carl, *113*.
Cohen, P., *645*, *648*, **704**, *705*.
Cohn, R. M., *644*.
Coleman, A. J., 45.
Cormier, R. J., *330*.
Court, N. A., **188**, *639*.
Coxeter, H. S. M., **558**.
Currier, A. E., 465.
Danzl, Arthur, 181, *698*.
Darraugh, J. E. *330*.
Dernham, Monte, *106*.
Discepoli, Fred, *250*, *252*.
Disch, John, 329.
Dodes, I. A., *106*, *331*, *331*, *410*.
Dorsey, F. F., *184*, *186*, *250*, *330*, *331*, *411*, *634*, *638*, *639*.
Douglas, William, *330*, *638*.
Droussent, L., *48*, **261**.
Dubois, D. W., 105, *637*.
Dutka, Jacques, *259*, **339**, 413.
Eaves, J. C., *330*.
Eisinger, R. P., *408*.
Eisman, S. H., *191*.
Eldridge, R., *330*.
Ellis, David, 41, *467*.
Emich, Herbert, *330*, *331*, *408*, *411*, *467*, *638*.
Epstein, A. L., *107*, *183*, *637*.
Erdös, Paul, 254, **255**, **259**, 412, 554.
Fan, Ky, 46, *48*, 109, *116*, 186.
Federbush, P. G., 469.

Fender, F. G., *642*.
Fine, N. J. *259*, *335*, *338*, **416**, *421*.
Finkel, Daniel, *113*.
Foote, S. A., *411*.
Frame, J. S., **334**, *706*.
Franck, Abraham, *411*.
Franz, E. A., *107*, *416*.
Freund, J. E., *191*.
Friedland, A. J., 551.
Friedman, D. H., *250*.
Fronabarger, C. V., *106*, *107*, *108*, *185*, *250*, *252*, *330*, *331*, *411*, **466**, *638*, *639*.
Fulks, W., *44*.
Furstenberg, Harry, 329.
Gaddum, J. W., *47*, *556*.
Gale, D., **46**, 407.
Gale, E. I., *468*, *634*,
Gambill, R. A., *330*, *411*.
Gerlinger, W. W., *636*.
Germain, C. B., *411*.
Gerstenhaber, Murray, 254.
Ginsburg, J., *649*.
Gold, B. K., *106*, *107*, *250*.
Goldberg, K., 469.
Goldberg, M., **107**, 181, **409**, *634*, **699**.
Golomb, M., 465, *645*.
Golomb, S. W., *44*, 697.
Gough, E. S. J., *106*.
Gould, S. H., *331*, **644**.
Graney, E. P., *637*.
Greenspan, Bernard, *411*.
Greenwood, R. E., *637*.
Grinstein, Louisa, *106*.
Grossman, George, 332.
Grossman, H. D., 413.
Grosswald, Emil, *107*, **116**, *331*, *411*, *412*, *636*, *637*, *642*, *644*, *705*, *707*.
Gunderson, N. G., **331**.
Gunning, Robert, *108*.
Haggard, J. D., *634*, *638*, *639*
Hahn, S. W., *411*.
Handel, M., *636*.
Hartung, D., *330*.
Hausmann, B. A., *411*.
Hays, R. F., *642*.
Hellerstein, Simon, *107*.
Henderson, G. P., *259*, *335*, **419**.
Hendler, A. S., *704*.
Henriksen, M., *638*.
Henry, Boyd, *250*.
Herlihy, Frank, *106*, *107*, **186**.
Herschfeld, A., *417*.
Herzog, Fritz, *189*.
Hildebrand, J. L., *701*.
Hoggatt, Vern, *250*, *330*, *331*, *411*, *412*, 465, *467*, **467**, *636*, *638*, *638*, *701*.
Hood, R. T., *184*.
Horton, R. E., *107*, *108*.
Hoyle, H. B., Jr., *330*.
Hsu, L. C., *257*, *412*, *421*.
Hu, N. C. C., *330*.
Hummel, P. M., *191*.
Hyde, A. R., *701*.
Indritz, J., **556**.
Isaacs, Rufus, 187, *417*.
Iyer, R. V., *649*.
Jasper, S. J., *107*.
Johnson, L. G., 41.
Johnston, G. R., *44*.
Jones, J., Jr., *639*.
Jones, L. G., *337*.
Jordan, T. L., Jr., *186*.
Jurgensen, Ray, *408*, *411*, *638*, *638*.
Kao, Simon T., 254.
Katz, S., *706*.
Kaufman, H., *411*, *636*, *636*, **637**, *643*, *707*.
Keeping, E. S., *186*, *410*.
Kelly, B., *648*.
Kelly, J. B., 109, **191**, *338*, *417*, *641*.
Kingsbury, R. S., *107*.
Kingston, J. M., *185*, *250*, *634*, *638*.
Kirmser, P. G., *44*, *411*, *412*.
Kishi, T., *466*, *468*.
Kissling, R., *412*, **466**.
Klamkin, M. S., *42*, *44*, *107*, *108*, *188*, *189*, *191*, *410*, *411*, *412*, *466*, *467*, *472*, **636**, **637**, *638*, *639*, 640, *642*, **643**, **648**, *648*, *650*, 702, *704*, *707*.
Klee, V. L., Jr., *47*, *556*.
Klimczak, W. J., *252*.
Kloeppel, P. K., *330*.
Konhauser, J. D. E., *408*, *638*, *639*.
Kravitz, Sam, *250*, *252*, *330*, *410*, *639*.
Kravitz, Sidney, *106*, *107*, *184*, *185*, *250*, *330*, *331*, *638*.
Kropnitsky, M. A., *644*.
Kruse, A. H., *338*, *417*.
La Borde, H. T., *705*.
Lange, L. H., *251*.
Langenhop, C. E., *417*.
Langr, J., *45*, *184*, *250*, *473*, *642*.
Lariviere, R., *330*.
Lassiter, J., *330*.
Latshaw, Elmer, *106*.
Layman, J. W., *106*.
Lee, H. L., *331*.
Leeds, B. R., *639*.
Lees, Milton, *411*.
Leifer, H. R., *106*, *251*.
LeLeiko, M., *338*.
Lemke, C. E., *338*.
Leonard, P. J., Jr., *107*.
Lessard, Roger, *191*.
Linis, V., *650*.
Livingston, A. E., *44*, **108**, *114*, 329, **410**, *637*.
Lohwater, A. J., *648*.
Lombardi, John, *106*.
Lorentz, G. G., **190**, 702.
Lowden, I., *330*.
Lumer, G., *114*, **115**, *191*, 333, *417*.
Malie, P. J., *106*.
Mandelbaum, David, *107*, *251*, *331*, *331*, *410*, *650*, *701*.
Marchand, E. W., *107*, **410**, 551.
Marer, Fred, *191*.
Marsaglia, G., *417*.
Marsh, D. C. B., *106*, *107*, *108*, *408*, *411*.
Martin, B., *410*, *639*.
Maxfield, M. W., *470*.
McConnon, Myles, *330*.
McCracken, Charles, Jr., *259*.
Mead, W. H., Jr., *701*.

Millar, J. G., *189*.
Miller, R. A., *707*.
Millman, G., *43*, *107*, *108*, *108*, *638*, *642*, *707*
Milosevich, Kovina, *251*, *411*, *467*.
Minker, Jack, *330*.
Misra, B. B., 329.
Moldauer, A. C., *642*.
Moliver, Martin, *107*, *108*.
Moorman, R. H., 180.
Morales, Rodolfo, *411*.
Morgan, F. M., *252*.
Morgantini, E., *649*.
Morrison, D., *330*, *338*, *417*.
Moser, Leo, 41, *107*, 180, *253*, *331*, *338*, **339**, *470*, 555, *637*, *705*.
Muguercia, R., 251.
Musselman, J. R., 469.
Nagara, Prasert Na, *106*, *183*, *184*, *185*, *251*, *252*, *330*, *331*, *408*, *411*, **416**.
Napp, E., *330*.
Narasimhamurthy, M., 551.
Neff, J. D., *330*. *331*.
Neustadt, L. W., *331*.
Newman, D. J., 110, 187, 254, *261*, 332, 407, 412, *420*, 469, 640, *641*, *648*, *650*.
Newman, Morris, *338*.
Newton, D. V., *701*.
Norris, M. J., *417*, *556*.
Norton, D. A., *191*.
Oakley, C. O., *331*.
Odland, Leola, *184*, *330*.
Oeder, Robert, *411*, *465*, *466*, *468*.
Ogilvy, C. S., 104, *184*, *189*, *251*, *331*, *331*, *408*, *417*, *468*, *556*, **557**, *634*, *637*, **642**, *648*, *701*, *707*.
Olds, C. D., *190*, 469.
Olmsted, M., *43*, *106*, *107*, *108*, *251*, *252*.
Oppenheim, J., *331*.
Ore, Oystein, 249.
Paige, L. J., *644*.
Parker, F. D., *106*, *107*, *330*.
Parker, S. T., *191*, *408*.
Parker, W. V., **190**, 640.
Paxman, R. G., *638*.
Payne, A. H., **181**.
Pearl, Martin, *106*, *107*, *108*.
Peiser, A. M., *706*.
Pennell, W. O., *183*, *642*.
Petracca, A., *472*.
Pinzka, C. F., *185*, *186*, **189**, *191*, *251*, **259**, *330*, *411*, **638**, *639*.
Piranian, G., *644*.
Piza, P. A., 249, *418*.
Rabson, G., *331*.
Rado, Richard, **255**.
Raine, P. W. A., *106*.
Raisbeck, G., *44*.
Ramler, O. J., **45**, *108*, **472**, *642*, 697.
Ransom, W. R., *106*, *107*, *330*, 407, **633**.
Redheffer, R. M., *44*, *707*.
Reich, E., *44*.
Rhodes, A. P., *330*.
Richman, J. A., *106*.
Ridley, J. W., Jr., *106*.
Ringenberg, L. A., **42**, *43*, *44*, **106**, *107*, *108*, *113*, *408*, *411*, *412*, *418*, *466*, *556*, *634*, *636*, *637*, *642*.
Robertson, M. S., *420*, *648*, *707*.
Robison, G. B., 551.
Rose, G. F., *191*.
Rose, I. H., *337*.
Rosenbaum, Joseph, **252**.
Rosenberg, A., *337*.
Rosenfeld, A., **44**, *106*, *251*, *330*, *331*, *331*, *331*, *408*, *411*, *466*, *467*, *638*, *638*, *642*, *707*.
Rosser, J. B., **337**.
Rouleau, W. G., *556*.
Salkind, C., *330*, 465, *638*, *639*.
Samet, P. A., *338*.
Sanders, J. E., *106*, *418*.
Sandham, H. F., *642*, *706*.
Sandwick, C. M., Sr., **418**, *634*, **638**, *638*, *639*.
Schaefer, P. T., *330*.
Schafer, R. D., *191*.
Scheuer, E. M., *330*.
Schoenfeld, L., **338**.
Scott, E. J., **707**.
Scott, W. R., **117**, **338**, *417*, *636*.
Seely, N. T., Jr., *113*.
Seidel, W., *44*, *108*, **330**, 331, *556*, *557*, *643*, *648*.
Serbyn, W. D., *42*, *44*, *108*.
Shanks, E. B., *417*.
Shapiro, H. S., 45, 187, 554, 696.
Shields, W. H., 696.
Shipman, J., 411, *636*, *636*, **637**, *638*.
Shook, C. A., 249.
Simester, J. H., *107*, *108*.
Sisk, A., *639*, *642*.
Skolnik, Samuel, **262**, *417*.
Smith, O. Dale, *259*.
Somanatham, P., **649**, **703**.
Spiegel, M. R., **44**, **107**, *108*, **113**, *189*, 254, **411**, 411, 554, *636*, **641**.
Sprague, R. H., *638*, *639*.
Springer, C. E., 110.
Stallard, F. W., *330*.
Stalley, R. D., *183*.
Stanaitis, O. E., *44*, *107*, *108*, *114*, *189*, *191*, *258*, *335*, *338*, *410*, *411*, *412*, **413**, *472*, *557*, *642*, **643**, *644*, *648*, *650*, **698**, *706*, *707*.
Stanoyewich, Chaslaw, *108*.
Starke, E. P., **417**, 640.
Steinberg, Robert, **420**, **471**.
Stelson, H. E., 633.
Stephanie, M., *553*.
Stetter, Hans, *191*.
Stimac, Michael, *116*.
Stoddard, B. B., *106*.
Stoneham, R. G., *643*.
Strother, W. L., *467*, *638*.
Sturdley, Arthur, **411**.
Subbarao, K., **649**.
Sudborough, D. R., *638*.
Swanson, C. A., *411*, **700**.
Swift, Elijah, *184*.
Swift, J. D., **641**.
Sylwester, Alfred, *411*.
Szekeres, G., **110**.
Szegö, G., *335*, *420*.
Tapscott, R. P., *252*, *638*.
Taussky, O., *338*, 554.

Taylor, A. E., *47*.
Terwey, P., Jr., *330*, *331*.
Thébault, V., 41, 46, 187, 249, **333**, 407, 412, *416*, 551, **647**, 702.
Thielman, H. P., 640.
Thompson, C. W., *639*.
Thompson, S. L., *259*.
Thompson, S. T., **185**, **250**.
Thomsen, D. L., *108*.
Tierney, J. A., *43*, *331*, *411*.
Todd, J., **641**.
Townes, S. B., 633.
Treuenfels, P., *42*, *411*, *556*.
Trigg, C. W., *106*, *107*, *108*, *251*, *251*, **252**, **329**, *638*.
Trost, E., **470**, *706*.
Ullman, J. L., *338*, *420*, *648*.
Umberger, E. H., *641*.
Underwood, F., **337**, *418*, *642*.
Underwood, R. S., *558*.
Ungar, P., **648**.
Vales, J., *472*.
Van Voorhis, W. R., *186*, *330*.
Varga, R. S., 254.
Varineau, V. J., *338*.
Vause, R. Z., Jr., **183**, *185*, *251*, *330*, *331*, *635*, *636*, *638*, *701*.
Venkataraman, C. S., **112**, 333, *411*, *636*.
Wagner, R. W., *417*.
Wahab, J. H., *470*.
Walker, G. W., *106*, *184*, *251*, **253**, *330*, **331**, **417**, *421*.
Walker, L. P., *638*, *638*.
Wall, D. D., 249.
Wall, H. S., *648*.
Walters, S. S., *417*.
Wang, C., *412*, *557*, *642*, *643*.
Ward, L. E., *106*, *108*.
Ward, Morgan, *191*, **257**, *338*
Warren, L., *638*.
Wayne, Alan, *108*.
Wegner, L. H., 329.
Weidlich, J. E., *106*.
Weiner, L., **336**.
Weinstock, R., *417*.
Wendel, J. G., 105, *417*, *636*.
Wenner, A. M., *106*.
Wertheim, M. S., 412.
White, L. R., 105.
Wilansky, A., *47*, 186, 249, **555**.
Wild, R. E., *42*, *44*, *108*, *189*, **252**, *470*, *706*.
Willcox, A. W., *106*.
Wilkins, J. E., Jr., **635**, *638*, 697.
Willey, M., *638*.
Williams, G. T., 46.
Williams, R. R., Jr., *638*.
Wolf, W. T., *330*.
Wood, W. D., *185*.
Woods, Roscoe, **108**, **184**, *466*, **552**.
Wright, E. M., **705**.
Young, F. H., *411*, *556*.
Zatzkis, H., *634*.
Zimmerberg, H. J., **188**.
Zirkel, E. J., *251*, *466*.
Zuckerman, H. S., **48**.

SOLUTIONS

Numbers in boldface type refer to problems, those in lightface to pages.

E-962,, 181. **E-963**, 42. **E-967**, 43, 329. **E-968**, 44. **E-969**, 44. **E-970**, 44. **E-971**, 105. **E-972**, 106. **E-973**, 107. **E-974**, 108. **E-975**, 108. **E-976**, 183. **E-977**, 184. **E-978**, 185. **E-979**, 186. **E-980**, 250. **E-981**, 250. **E-982**, 251. **E-983**, 252. **E-984**, 252, 408. **E-985**, 253. **E-986**, 329. **E-987**, 330. **E-988**, 331. **E-989**, 331. **E-990**, 331. **E-991**, 408. **E-992**, 408. **E-993**, 410. **E-994**, 410. **E-995**, 411. **E-996**, 466. **E-997**, 466. **E-998**, 467. **E-999**, 551. **E-1000**, 553. **E-1001**, 634. **E-1002**, 635. **E-1003**, 635. **E-1004**, 636. **E-1005**, 637. **E-1006**, 637. **E-1007**, 638. **E-1008**, 639. **E-1009**, 697. **E-1010**, 699. **E-1011**, 700.

4358, 255. **4359**, 46. **4398**, 47. **4399**, 48. **4400**, 110. **4401**, 112. **4402**, 113. **4403**, 115. **4404**, 116. **4405**, 116. **4406**, 187. **4407**, 188. **4408**, 189. **4409**, 189. **4410**, 190. **4411**, 257. **4412**, 258. **4413**, 259. **4414**, 261. **4415**, 413. **4416**, 261. **4417**, 333. **4418**, 334. **4419**, 336. **4420**, 337. **4421**, 338, 470. **4422**, 338. **4423**, 415. **4424**, 416. **4425**, 417. **4426**, 419. **4427**, 420. **4428**, 470. **4431**, 471. **4432**, 472. **4433**, 555. **4434**, 556. **4435**, 557. **4436**, 640. **4438**, 641. **4439**, 642. **4440**, 643. **4441**, 643. **4442**, 644. **4443**, 645. **4446**, 647. **4447**, 648. **4448**, 648. **4449**, 650. **4450**, 702. **4451**, 704. **4452**, 705. **4453**, 706. **4455**, 707.

RECENT PUBLICATIONS

Edited by E. P. VANCE, OBERLIN College

NEW BOOKS RECEIVED

52–53, 121, 197, 266–267, 342–343, 423, 474–475, 560–561, 655.

REVIEWS

Names of authors are in ordinary type, those of reviewers in italics.

Bettinger, A. K. and Dwyer, W. A. *Algebra for Commerce and Liberal Arts*, W. R. HUTCHERSON, 192–194.
Bibb, S. F. See Palmer, C. I.
Buros, O. K. *Statistical Methodology Reviews*, J. NEYMAN, 653–654.

Crathorne, A. R. See Rietz, H. L.
Davis, D. R. *The Teaching of Mathematics*, P. S. JONES, 194–196.
Dwyer, W. A. See Bettinger, A. K.
Fehr, H. F. *Secondary Mathematics, A Functional Approach for Teachers*, J. R. MAYOR, 263–264.
Feller, William. *An Introduction to Probability Theory and Its Applications*, F. A. C. SEVIER, 265–266.
Ferrar, W. L. *Finite Matrices*, C. C. MACDUFFEE, 422–423.
Fulmer, H. K. and Walter Reynolds, *College Algebra*, W. R. HUTCHERSON, 192–194.
Golomb, M. and Shanks, M. *Ordinary Differential Equations*, W. M. WHYBURN, 50–52.
Grimshaw, M. E. See Hamburger, H. L.
Hamburger, H. L. and Grimshaw, M. E. *Linear Transformations in n-Dimensional Vector Space*, KADISON, R. V., 650–652.
Herland, Leo. *Dictionary of Mathematical Sciences, Volume 1* (German-English), E. P. VANCE, 120.
Jaeger, J. C. *An Introduction to Applied Mathematics*, W. S. LOUD, 559–560.
Kaltenborn, H. S. *Meaningful Mathematics*, R. L. SWAIN, 473–474.
Kempthorne, Oscar. *The Design and Analysis of Experients*. P. R. RIDER, 652–653.
Kinney, L. B. and Purdy, C. R. *Teaching Mathematics in the Secondary School*, D. A. JOHNSON, 709–710.
May, A. E. *Mathematics of Finance*, D. E. SOUTH, 118–119.
May, K. O. *Elementary Analysis*, W. N. HUFF, 708–709.
Munroe, M. E. *Theory of Probability*, KENNETH MAY, 50.
Nagell, Trygve, *Introduction to Number Theory*, A. L. WHITEMAN, 339–341.
Nehari, Zeev. *Conformal Mapping*, E. J. MICKLE, 421–422.
Palmer, C. I. and Bibb, S. F. *Practical Mathematics, Part II, Algebra with Applications*, W. R. HUTCHERSON, 192–194.
Pease, E. M. J. *Intermediate College Algebra*, W. R. HUTCHERSON, 192–194.
Peck, L. C. *Elements of Algebra*, W. R. HUTCHERSON, 192–194.
Peters, J. W. See Rietz, H. L.
Phillips, H. B. *Differential Equations*, W. M. WHYBURN, 50–52.
Powell, J. E. and Wells, C. P. *Differential Equations*. W. M. WHYBURN, 50–52.
Purdy, C. See Kinney, L. B.
Rees, P. K. and Sparks, F. W. *Intermediate Algebra*, W. R. HUTCHERSON, 192–194.
Reynolds, Walter, See Fulmer, H. K.
Rietz, H. L., Crathorne, A. R., and Peters, J. W. *College Algebra*, W. R. HUTCHERSON, 192–194.
Rosenbach, J. B. and Whitman, E. A. *Essentials of College Algebra*, W. R. HUTCHERSON, 192–194.
Shanks, M. See Golomb, M.
Smith, F. C. *The Mathematics of Finance*, D. E. SOUTH, 117–118.
Sparks, F. W. See Rees, P. K.
Struik, D. J. *Lectures on Classical Differential Geometry*, C. E. SPRINGER, 119–120.
Thébault, Victor. *Les Récréations Mathématiques*, E. P. STARKE, 341–342.
Wade, T. L. *The Algebra of Vectors and Matrices* I. H. ROSE, 264–265.
Waissman, Friedrich. *Introduction to Mathematical Thinking*, BEN DUSHNIK, 654–655.
Wells, C. P. See Powell, J. E.
Whitman, E. A. See Rosenbach, J. B.
Wylie, C. R. Jr. *Advanced Engineering Mathematics*, W. S. LOUD, 599–560.

CLUBS AND ALLIED ACTIVITIES

Edited by H. D. LARSEN, Albion College

CLUB REPORTS

University of Alabama, 344.
Baldwin-Wallace College, 55.
Beloit College, 54.
Boston College, 199.
Brown University, 200.
University of Buffalo, 343.
University of California, 347.
Case Institute of Technology, 54
Central Michigan College, 344
Central Missouri State College, 346.
University of Chicago, 268.
Columbia College, 343.
Cornell University, 125.
University of Detroit, 124
D'Youville College, 270.
Harvard University, 126.
Hunter College, 199.
Immaculate Heart College, 55.
Iowa State College, 197–198.
Kansas State Teachers College, Emporia, 269.
University of Kansas, 54.
Lehigh University, 56
Los Angeles City College, 198.
Marquette University, 270.
University of Miami, 125.
Michigan State College, 344.
University of Missouri, 124.
Montana State University, 122.
Mount Mary College, 122.
New York University, 122.
Northwestern University, 56.
Oberlin College, 346.
Oregon State College, 55.
University of Oregon, 268.
Pasadena City College, 268.
University of Richmond, 198.

University of Rochester, 345.
Rutgers University, 123.
Sacramento State College, 267.
Southwestern Louisiana Institute, 345.
Stanford University, 345.
University of Toledo, 125.
Upsala College, 123–124.
University of Washington, 126.

ARTICLES

UNIVERSITY OF GEORGIA MATHEMATICS DEPARTMENT. Do You Want to Major in Mathematics?, 561–563.
———. Do You Want to be a Mathematician?, 656–658.
LARIVIERE, R. Introducing Congruences, 658–659.
LARSEN, H. D. Club Topics, 475–478; 710, 711.
—— Club News, 655–656; 711. Pi Mu Epsilon National Meeting, 711–712.
READ, C. B. Books Containing Material for Undergraduate Mathematics Club Programs, 424–426.

NEWS AND NOTICES

Edited by EDITH R. SCHNECKENBURGER, University of Buffalo

GENERAL INFORMATION

Annual Conference for Teachers of Mathematics, 275–276.
Bureau of Ordnance, Employment Opportunities Bulletin, 127–128.
Essay Contest of the Institute for the Unity of Science, 271–272.
Fellowships of the American Association of University Women, 563–564.
Fellowships of the Educational Testing Service, 659–660.
Journal of the Graduate Institute for Applied Mathematics, 57.
Journal of Rational Mechanics and Analysis, 349.
Mathematics Institute at the University of Houston, 348.
Mathematics Institute at Louisiana State University, 348–349.
Meeting of Mathematics Division of A.S.E.E., 347–348.
Preliminary Actuarial Examinations and Prize Awards, 271.
Preliminary Actuarial Examinations. Prize Awards, 660–661.
Purchase of the A.M.S. Library by the University of Georgia, 270–271.
Report of the General Assembly of the International Mathematical Union, 478–479.
Request for Books, 272.
Stanford University Competitive Examination in Mathematics, 127.
Statistical Summer Session, 348.
Summer Courses, 272–275; 349–350.
Syllabuses for Courses in Modern Numerical Analysis, 349.

PERSONAL INFORMATION

Newly elected members of the Association, 64–66, 133–134, 279–281, 431–433, 483–486 568–571.

The following persons presented papers at meetings of the Association and its Sections.

Agan, R. J., 141.
Agmon, Samuel, 518.
Ainsworth, O. R., 497.
Aker, H. M., 72, 591.
Alexander, Mary K., 495.
Allen, E. B., 665.
Allen, E. F., 363.
Allendoerfer, C. B., 593.
Anning, N. H., 511.
Antonietta, Sister, 520.
Apostol, T. M., 498.
Arena, F. J., 212, 439.
Arnold, P. H., 491
Arsove, M. G., 593.
Artin, Emil, 437.
Ayre, H. G., 286.
Ayres, W. L., 583.
Bagemihl, Frederick, 289.
Ball, W. R., 361
Ballantine, J. P., 362
Baker, Frances E., 589.
Bardell, R. H., 71.
Barr, C. F., 144.
Barrow, D. F., 497.
Baten, W. D., 510.
Beach, J. W., 516.
Beaumont, R. A., 361.
Beberman, Max, 575.
Becker, H. W., 140, 584.
Beckmann, M. W., 141.
Bell, J. H., 511.
Bell, R. F., 593.
Bennett, A. A., 218.
Benson, D. C., 442
Bernstein, Dorothy L., 589.
Bettinger, A. K., 585.
Betz, Herman, 582.
Betz, S. A. E., 583.
Bird, M. T., 442.
Birnbaum, Z. W., 361.
Black, A. H., 135.
Blair, B., 147.
Block, H. D., 145, 147.
Bloom, G. M., 513.
Blum, W. H., 573.
Boas, R. P., Jr., 285.
Bonnell, C. R., 213, 581.
Bradford, J. C., 518.
Brand, Louis, 515, 516, 517.
Brandner, F. A., 502.
Brenner, J. L., 361.
Briggs, W. E., 143.
Bright, H. F., 520.
Bryan, N. R., 139.
Bush, K. A., 589.
Bussey, W. H., 213.
Butchart, J. H., 516.
Byrne, W. E., 509.
Cameron, R. H., 581.
Camp, E. J., 439.
Campbell, Frances L., 499.
Canfield, E. L., 501.
Carpenter, Dale, 498.
Charnes, Abraham, 572.
Chellevold, J. O., 501.
Chevalley, Claude, 216.
Chin, Louise H., 515.
Chittenden, E. W., 147.
Church, Elsie T., 215, 488.
Churchill, R. V., 511.
Clark, A. G., 143.
Clark, B. G., 493.
Clark, Stirling, 210.
Clarke, F. Marion, 140, 585.
Coble, A. B., 495.
Coburn, Nathaniel, 510.
Coffman, M. L., 519.
Cohen, A. C., Jr., 496.

Conkling, R. M., 494.
Copeland, Sr., A. H., 510.
Couch, W. A., 583.
Courtright, Henrietta, 491.
Cowan, R. W., 136, 496.
Cowling, V. F., 507.
Cox, H. M., 141, 142.
Cramlet, C. M., 361.
Craw, A. R., 209.
Currie, J. C., 135, 495.
Curtiss, J. H., 283, 436.
Daniells, Marian E., 146.
Davis, L. I., 514.
Derry, Douglas, 594.
Di Prima, R. C., 572.
Dodes, I. A., 579.
Donsker, M. D., 211.
Douglas, Jim, Jr., 518.
Duncan, D. C., 498.
Dunnington, G. Waldo, 488.
Duren, W. L., Jr., 68.
Durfee, William H., 588.
Dustheimer, O. L., 514.
Du Val, Patrick, 494.
Dye, L. A., 138, 497.
Eberlein, W. F., 590.
Edrei, Albert, 587.
Edwards, P. D., 577.
Eiseman, R. L., 210.
Ellis, David, 495.
Epperson, E. R., 513.
Epstein, Bernard, 437.
Ericksen, J. L., 357.
Ernsdorff, L. E., 504.
Evans, T., 72.
Eves, Howard, 360, 588.
Farnell, A. B., 143.
Federer, Herbert, 206.
Felice, Mary, Sister, 592.
Feller, William, 218.
Feyerherm, A. M., 503.
Fides, M., Sister, 507.
Ford, L. R., 509.
Foster, R. M., 579.
Freund, J. E., 289.
Frick, C. H., 209.
Frink, Orrin, Jr., 206.
Fronabarger, C. V., 583.
Fuchs, W. H., 289.
Fulton, C. M., 441.
Fulks, W. B., 213.
Gallie, T. M., Jr., 519.
Gergen, J. J., 493.
Godderz, H. W., 212.
Godfrey, E. L., 358, 577.
Goldberg, Michael, 434, 511.
Golomb, Michael, 359.
Goodman, A. W., 215, 506.
Goodner, D. B., 138.
Gosslee, David, 440.
Gould, S. H., 358, 577.
Grable, E. S., 508.
Graesser, R. F., 517.
Grau, A. A., 363.
Graves, G. H., 358.
Guy, Jr., W. T., 520.
Hadlock, E. H., 495.
Hadnot, B. F., 136.
Haggard, J. D., 491.
Halfar, Edwin, 141.
Hamilton, H. J., 498.
Hamilton, O. H., 364.
Handelman, G. F., 572.
Hanson, H. A., 511
Harrell, E. G., 72.
Harry, C. H., 665.
Hartley, M. C., 286.
Healy, P. W., 283.
Heineman, E. R., 519.
Henry, Boyd, 504.
Hendrix, Gertrude, 575.
Herriot, J. G., 442.
Hildebrandt, E. H. C., 141, 575.
Hill, P. R., Sr., 139.
Hind, A. T., Jr., 136.
Hinrichsen, J. J., 503.
Hlavaty, J. H., 216.
Hlavaty, Vaclav, 577.
Hohn, F. E., 574.
Holl, D. L., 147.
Hostetter, I. M., 362.
Householder, A. S., 67.
Hoyle, H. B., Jr., 497.
Howard, Aughtum S., 505.
Huff, G. B., 497.
Hull, R., 665.
Hurt, J. M., 520.
Hutcherson, W. R., 488.
Irwin, J. B., 359.
Jackson, L. K., 585.
Jackson, S. B., 435, 509.
Jamison, S. L., 139, 493.
Jones, B. W., 143, 586.
Jones, P. S., 510.
Jurney, H. M., 143.
Kaltenborn, H. S., 139.
Kempner, A. J., 143.
Kanter, L. H., 68, 489.
Karnes, H. T., 69.
Keppers, G. L., 147.
Kiernan, P. J., 438.
Kirkham, Don, 501.
Knox, S. R., 488.
Korevaar, Jacob, 359.
Kramer, Max, 517.
Kros, F. A., 586.
Kuhn, Harold W., 665.
Kyle, G. D., 363.
Lambert, R. J., 145.
Langenhop, C. E., 147.
Lanzano, Paolo, 582.
Lariviere, Rose, 574.
Larney, Violet H., 284.
LaSalle, Margaret M., 488.
Laush, G., 571.
Lazar, Nathan, 514.
Leach, Edgar, 575.
Leech, J. S., 586.
Leavitt, W. G., 141.
Lehmer, D. H., 499.
Leipnik, R. B., 360, 594.
Leser, Tadeusz, 215, 504.
Leutert, W. W., 210.
Levit, R. J., 135.
Lewis, F. A., 496.
Lindahl, C. H., 502.
Loewner, Charles, 290.
Lorch, E. R., 579.
Loud, W. S., 211.
MacDuffee, C. C., 286.
MacLane, G. R., 69.
MacLane, Saunders, 137, 138, 577, 590.
Mancill, J. D., 493.
Marie, Sister Gertrude, 358.
Marth, Ella C., 509.
Matilda, Sister M., 506.
May, Kenneth, 212.
Mayor, J. R., 490.
McCall, J., 68.
McEwen, W. H., 438.
McFarlan, L. H., 361.
McGlasson, A. G., 507.
McInnis, S. W., 135, 493.
Mears, Florence M., 509.
Mendelsohn, N. S., 440.
Menger, Karl, 206.
Meyer, B. C., 515.
Michalik, E. R., 572.
Mielke, P. T., 359.
Milkman, Joseph, 435.
Miller, R. A., 67.
Mills, Harlan, 502.
Mitchell, Benjamin Evans, 495.
Mitchell, W. W., Jr., 515.
Moore, G. E., 573.
Moore, Keith, 284.
Moser, L., 69.
Muhly, H. T., 140, 141.
Murphy, C. H., Jr., 436.
Neisius, W. V., 137, 497.
Nelson, W. K., 144.
Newton, T. A., 137.
Nibarger, Agnes, 284.
Norris, M. J., 581.
Nowlan, F. S., 574.
Ohbayashi, H., 360.
Ollmann, L. F., 579.
Ore, Oystein, 206.
Orleans, Joseph, 217.
Pall, Gordon, 574.
Parker, F. D., 588.
Parker, W. V., 497.
Parrish, H. C., 518.
Peach, M. O., 359.
Perlin, I. E., 136.
Peterson, T. S., 594.
Phillips, R. S., 500.
Phipps, C. G., 135, 138.
Pickard, W. L., 575.
Pipes, C. J., 70.
Pitt, W. B., 589.
Ploenges, E. W., 286.
Polya, George, 441.
Prenowitz, Walter, 206.
Prince, M. D., 136.
Pugsley, D. W., 505.
Purcell, E. J., 516, 517.
Rado, Tibor, 513, 665, 666.
Raiffa, Howard, 510.
Randolph, J. F., 206.
Rankin, W. W., 492, 494.
Ratner, L. T., 137.
Reid, A. T., 147.
Reves, G. E., 492.
Reynolds, T. L., 488.
Rhodes, Ida, 436.
Ribeiro, H. B., 141, 585.
Rice, J. D., 70.
Rickey, F. A., 489.
Richert, A. S., 490.
Riggs, C. L., 518.
Rinehart, R. F., 512.
Ringenberg, L. A., 285.
Robertson, Fred, 146, 504.
Robinson, H. A., 136.
Robinson, L. V., 435, 508.
Robinson, Raphael M., 441.
Robinson, W. J., 505.
Rodabaugh, L. D., 286.
Rosenbaum, R. A., 594.
Rosenbloom, P. C., 581, 665.
Ross, F. E., 506.
Rosskopf, M. F., 289.
Salzer, H. E., 436.
Sanger, R. G., 491.
Sard, A., 579.
Schaeffer, A. C., 71.
Schiffer, M., 441.
Schoenberg, I. J., 499.
Schult, Veryl G., 509.
Schwartz, B. L., 572.
Scott, L. L., 489.
Scott, W. R., 283.
Seebeck, C. L., Jr., 495.
Seiden, Esther, 289.
Seifert, George, 584.
Seybold, M. Anice, 286, 575.
Shanks, E. B., 139, 496.
Shanks, M. E., 71.
Shapiro, E. I., 580.
Shenton, W. F., 435.
Sherman, Seymour, 500.
Simmons, H. A., 286.
Sisk, Augustus, 138.
Smiley, M. F., 146.
Smith, C. B., 136, 494.
Smith, C. D., 68.
Smith, F. C., 439.
Smith, I. W., 439.
Smith, P. K., 68.
Smith, R. E., 508.
Smith, S. R., 144.
Smith, W. N., 587.
Snyder, W. S., 214.
Sobczyk, Andrew, 516.
Sokolnikoff, Elizabeth S., 591, 665.
Spiegel, M. R., 290.
Spragens, W. H., 68.
Springer, C. E., 519.
Staley, R. C., 582.
Stanaitis, O. E., 211, 439, 581.
Stancliff, Fenton, 514.
Stark, L. W., 139.

Steenrod, N. E., 206.
Steinbach, E. M., 510.
Stephen, Sister Marie, 285.
Stetter, Hans, 144.
Stewart, B. M., 511.
Stokes, R. A., 67.
Stoner, W. J., 148.
Stopher, E. C., 491.
Striegl, M., 71.
Strother, W. L., 494.
Sussman, Irving, 442.
Taam, Choy-tak, 583.
Talacko, J. V., 590.
Taylor, H. E., 137.
Taylor, Strawn, 214.
Temple, V. B., 67, 489.
Thielman, H. P., 503, 580.
Thomas, J. M., 496.
Thompson, S. L., 497.
Thurston, H. S., 137, 495.
Tierney, J. O., 210.
Todd, John, 436.
Toralballa, L. V., 71.
Trifan, Deonisie, 515.
Tucker, A. W., 206, 579.
Tucker, Donald, 144.
Ulrich, F. E., 70.
Underwood, R. S., 70, 519.
Vanaman, Sherman, Jr., 214.
Vanderburgh, E. L., 586.
Van Horn, C. E., 145.
Van Voorhis, W. R., 513.
Varner, W. W., 143.
Vinograde, Bernard, 146.
Wahlstrom, L. F., 591.
Walden, Earl, 517.
Walker, D. T., 138.
Walker, Lila P., 495.
Wall, J. G., 136.
Wallace, A. D., 67.
Waltz, A. K., 288.
Ward, J. A., 214.
Webster, M. S., 358.
Wedel, A. M., 146.
Weinberger, E. B., 572.
Welker, E. L., 575.
Westhafer, R. L., 516.
Wexler, Charles, 517.
Whaples, George, 576.
Wheeler, R. E., 506.
Whyburn, W. M., 487.
Wilburn, L., 70.
Wilkins, J. E., 665.
Williams, W. L., 494.
Wilson, R. H., 505.
Wilson, R. L., 494.
Wollan, G. N., 136, 493.
Wollan, R. S., 494.
Woods, Dale, 516.
Wren, F. L., 70, 136, 492.
Wrestler, Ferna E., 491.
Wright, H. A., 505.
Wylie, C. C., 503.
Yates, R. C., 437, 506.
Yeardley, Nelson, 501.
Young, F. H., 593.
Zant, J. H., 363.

Personal Mention. This section contains the names of officers of the Association and its Sections, and persons mentioned in the Department of News and Notices. The list does not include names of new members or of attendants at meetings.

Abbeduto, L. J., 352.
Abramowitz, Walter, 59.
Ader, O. B., 480.
Adkins, L. K., 129.
Ainsworth, O. R., 201.
Albero, L. V., 129.
Albert, A. A., 564.
Alkire, Don, 129.
Allen, R. W., 129.
Allison, G. F., 352.
Amick, T. C., 480.
Ammann, Andre, 351.
Anderson, A. G., 564.
Andersson, A. G., 565.
Andree, R. V., 363.
Andrews, J. J., 128.
Antosiewicz, Henry, 565.
Applebaum, F. H., 129.
Armstrong, P. N., 565.
Arnoff, E. L., 352.
Arnold, B. H., 277.
Arnold, K. J., 564.
Arnold, L. G., 203.
Arnold, P. H., 565.
Aronszajn, Nachman, 202.
Arzt, Sholom, 352.
Austin, D. G., 128.
Austin, George, 351.
Ayer, Miriam C., 352.
Ayre, H. G., 352, 573.
Bacon, R. P., 59.
Bade, W. G., 59.
Badger, F. S., 662.
Badgley, W. H., 662.
Bagemihl, Frederick, 352.
Baker, Frances E., 60.
Baker, Iola M., 129.
Baker, J. L., 352.
Ball, R. W., 565.
Banhagel, E. W., 60.
Banks, D. O., 662.
Baravalle, H., 661.
Barnett, H. H., 59.
Barnett, I. A., 483.
Barr, C. F., 586.
Barrett, J. H., 277.
Barry, J. Y., 278.
Bartholomay, A. F., 129.
Baum, W. R., 60.
Beard, R. S., 661.
Beasley, S. Louise, 60.
Becker, H. W., 584.
Bell, Clifford, 201.
Bellman, R. E., 278.
Bennett, O. B., 60.
Beraru, Jonas, 278.
Berberian, S. K., 129.
Bercos, J., 662.
Berg, L. E., 352.
Berg, W. D., 60.
Bernstein, B. A., 351.
Bernstein, Dorothy L., 202.
Bing, R. H., 70, 565, 590.
Bird, R. W., 278.
Birkhoff, Garrett, 58.
Birnbaum, Z. W., 565.
Blackman, Jerome, 480.
Blake, Archie, 480.
Blasch, H. F., 662.
Bleick, W. E., 351.
Blomquist, N. G., 200.
Blumenthal, L. M., 129.
Boas, R. P., 277.
Boldyreff, A. W., 278.
Boothby, W. M., 277.
Bothwell, F. E., 427.
Bott, Raoul, 128.
Bowker, J. G., 565.
Boyd, E., 565.
Boyer, C. R., 661.
Bradford, J. C., 566.
Bradshaw, R. G., 480.
Bradt, R. N., 352.
Brauer, R., 566.
Brewer, B. W., 277.
Bright, H. F., 480.
Britton, J. R., 142, 586.
Brock, P., 662.
Brooks, Foster, 512.
Brothers, W. H., 566.
Browder, Felix, 200.
Brown, B. H., 483.
Brown, Helen E., 427.
Brown, H. H., 480.
Brown, Kenneth E., 480.
Brown, R. W., 566.
Bruck, R. H., 565.
Brunk, H. D., 129.
Bryant, Robert, 352.
Buchanan, H. E., 60.
Buck, C. C., 662.
Burcham, P. B., 200.
Burgess, Cecil, 203.
Burnette, H. W., 352.
Burns, G. C., 566.
Bushey, Jewell H. (Mrs.), 57, 128.
Bushey, J. Hobart, 479.
Bussel, Bernard, 278.
Butchart, J. H., 515.
Cairns, S. S., 285.
Calabi, Eugenio, 201.
Callahan, C. M., 352.
Cameron, E. A., 128.
Cannon, Kathleen, 59.
Cansado, Enrique, 201.
Carlson, C. S., 211.
Carlson, K. H., 277.
Carr, R. E., 427.
Carroll, R. E., 353.
Carson, A. B., 130.
Carver, W. B., 480.
Cashwell, E. D., 130.
Cawley, John, 427.
Chand, Ottam, 200.
Chasen, S. H., 480.
Chatelain, Virginia L., 353.
Chatland, H., 592.
Chellevold, J. O., 427, 500.
Chernoff, Herman, 201, 202.
Chowla, Sarvadaman, 565.
Chrestenson, H. E., 130.
Church, Elsie T., 203.
Clark, B. G., 135.
Clark, R. A., 276.
Clarke, A. B., 128.
Clawson, J. W., 662.
Coble, A. B., 353.
Cobb, Reginald, 60.
Cochran, C. N., 203.
Coffman, M. L., 481.
Cohen, Eckford, 130.
Coke, Robert, 203.
Cole, Margaret B., 203.
Colquitt, L. A., 201, 508.
Colson, H. D., 211.
Colvin, B. H., 130.
Comba, Paolo, 278.
Comfort, E. G. H., 566.
Connelly, Damian, 57.
Conrad, Karl, 277.
Conrad, P. F., 566.
Conte, Samuel, 203.
Conwell, G. N., 203.
Conwell, H. H., 58.
Cook, C. H., 142.
Cook, J. M., 59.
Cosby, B., Jr., 59.
Cotlar, Mischa, 351.
Coulter, D. F., Jr., 427.
Court, N. A., 130.
Cowling, V. F., 479.
Cox, J. G., 427.
Coxeter, H. S. M., 277.
Cramer, G. F., 481.
Craw, A. R., 128.
Crell, C. C., 277.
Crisler, E. H., 564.
Crispin, J. W., 427.
Curran, Ann M., 201.
Curtis, C. W., 352.
Daly, D. L., 353.
Dancer, Wayne, 479.
Daniel, L. R., 200.
Dansky, Morris, 140.

Darsow, W. F., 276.
Daus, P. H., 498.
Davidson, D. O., 427.
Davis, H. S., 352.
Davis, J. E., 350.
Davis, L. E., 427.
Davis, M. S., 352.
Deal, D. E., 566.
Demers, M. R., 130.
Dean, R. A., 276.
De Prima, C. R., 662.
Derr, John, 201.
Diamond, A. H., 202, 662.
DiDonato, A. R., 427.
Diesel, J. W., 129.
Dietrich, V. E., 661.
Diliberto, S. P., 351.
Dix, L. E., 566.
Douglas, J., 661.
Dowds, R. E., 130.
Doyle, W. C., 427, 582.
Dragonette, Leila A., 351.
Drandell, Milton, 59.
Dresden, Arnold, 480.
Dubé, G. R., 278.
Dubisch, Roy, 440.
Duff, G. F. D., 201.
Dugundji, J., 59.
Durfee, W. H., 130.
Dutton, Arthur, 202.
Dye, H. A., 276.
Eagleson, Rosalind M., 276.
Earl, J. M., 350.
Eaves, J. C., 661.
Eckmann, Beno, 201.
Eddington, G. M., 60.
Edrei, Albert, 128, 481.
Ehlers, F. E., 277.
Eilenberg, S., 661.
Ekstrom, R. E., 130.
Ellis, David, 565.
Ellman, Norbert, 130.
Elyash, E. S., 200.
Englund, J. A., 203.
Epperson, E. R., 277.
Ernsdorff, L. E., 145.
Esser, M. H. M., 661.
Evans, Jacqueline P., 60.
Evans, Trevor, 427.
Evans, W. B., 200.
Eves, Howard, 203, 277.
Ewing, G. M., 662.
Ewy, D. J., 353.
Exner, R. M., 128.
Fagerstrom, W. H., 578.
Farnell, A. B., 128.
Faulkner, F. D., 351.
Fehr, H. F., 661.
Feige, Rudolf, 481.
Felder, Virginia I., 67.
Feller, W., 566.
Finkbeiner, D. T., 350.
Fishback, W. T., 353.
Fix, Evelyn, 351.
Flanders, Harley, 351.
Ford, Gloria C., 276.
Ford, L. R., 566.
Foreman, W. C., 490.
Forsyth, C. H., 480.
Forsythe, F. E., 203.
Foster, A. E., 427, 566.
Foster, A. L., 351.
Fowler, K. A., 59.
Frame, J. S., 58.
Frankel, A. A., 661.
Frick, C. H., 209, 508.
Friedenheit, A. R., 481.
Fuller, F. B., 276.
Fuller, L. E., 564.
Fulmer, H. K., 135, 492.
Gandy, W. W., 427.
Garfin, L., 566.
Gass, S. I., 427.
Gehman, H. M., 276, 564.
Gelbart, Abe, 428, 661.
Gentry, F. C., 276.
Gerstenhaber, Murray, 276.
Giarratana, J., 351.
Gibson, J. C., 130.
Giever, J. B., 200.
Ginsburg, Barry, 278.
Ginsburg, J., 661.
Glander, Harold, 60.
Golomb, Michael, 351.
Good, V. L., 566.
Gordon, R. D., 276.
Gormsen, S. T., 565.
Graesser, R. F., 483.
Grandy, C. C., 566.
Graves, L. M., 479.
Gray, D. O., 203.
Green, L. W., 278.
Greene, Laura Z., 283, 490.
Gregory, Christopher, 278.
Gregory, L. D., 428.
Griffin, F. L., 207, 479.
Griffith, H. C., 566.
Guhsé, M. P., 353.
Gunderson, N. G., 288, 588.
Gutierrez, Lino, 351.
Gutzman, W. W., 129.
Guy, D. L., 129.
Haefeli, H. G., 60.
Haggard, J. D., 662.
Halfar, Edwin, 140, 584.
Halmos, P. R., 351.
Hamilton, N. T., 564.
Hammer, P. C., 662.
Handy, B. F., 662.
Hanna, J. R., 130.
Hannon, H. H., 480.
Harden, J. C., Jr., 130.
Hardy, F. J., 428.
Hart, J. J., 130.
Hartley, M. C., 573.
Hartmann, C. F., 60.
Hatfield, F. C., 130.
Haynes, Nola L. A., 60, 582.
Heaps, H. S., 428.
Heath, Mary C., 59.
Heerema, Nicholas, 59.
Helms, L. L., 428.
Helson, Henry, 276, 278.
Hendrickson, M. S., 515.
Henriksen, M., 662.
Herbach, Leon, 350.
Herreshoff, J. B., IV, 58.
Hershner, I. R., 128.
Herstein, Israel, 351.
Herzfeld, S. H., 662.
Herzog, Fritz, 481.
Hewitt, E., 565.
Hill, A. G., 211.
Hille, Einar, 276.
Hiz, Henry, 203.
Hlavaty, Vaclav, 353.
Hochschild, G. P., 202, 278.
Hocking, J. G., 277.
Hodges, J. L., 351.
Hohn, F. E., 202.
Hokanson, J. S., 353.
Honig, Julius, 130.
Hook, C. W., 661.
Hoover, Norman, 129.
Hornback, J. H., 353.
Horsfall, I. O., 203.
Horwitz, David, 428.
Hostinsky, L. A., 566.
Houghton, D. B., 353.
Houser, D. M., 130.
Howell, S. W., 353.
Huffer, R. C., 58.
Hull, S. L., 60.
Humphreys, M. Gweneth, 508.
Huneke, H. V., 428.
Hyde, A. R., 60.
Hyers, D. H., 59.
Immel, E. R., 352.
Ishihara, Jiro, 428.
Ito, W. H., 428.
Iturralde, Verba W., 428.
Jacobs, J. A., 129.
Jacobson, Arvid, 203.
Jacobson, Nathan, 278.
James, R. C., 350.
James, R. D., 360, 483.
Jaramillo, T. J., 481.
Jenkins, Emerson, 480.
Jenner, W. E., 277.
Jennings, S. A., 58.
Jennings, Walter, 351.
Jerison, Meyer, 351.
Johnson, Elizabeth, 480.
Johnson, George, 203.
Johnson, L. W., 131, 363.
Johnson, Marvin, 480.
Jones, B. E., 276.
Jones, B. W., 142, 586.
Jones, L. O., 582.
Jones, P. S., 509.
Jordan, D. B., 353.
Jordan, T. L., 60.
Kalicki, Jan, 351.
Kallianpur, Gopinath, 351.
Kamel, Hyman, 564.
Kaufman, H., 567.
Keefer, Helen S., 428.
Keller, Joseph, 131.
Keller, M. W., 357.
Kelley, C. E., 428.
Kelly, J. B., 277.
Kemperman, J. H. B., 351.
Kennedy, E. S., 428.
Kennedy, Ruth E., 131.
Kenner, M. R., 428.
Kheiralla, A. A., 278.
Kibbey, D. E., 128.
Kiefer, E. C., 285, 573.
Kiefer, Jack, 564.
Kieval, H. S., 216, 578.
Killen, C. G., 67.
Kindle, Joseph, 276.
Kingston, J. M., 360, 592.
Kinney, A. E., 131.
Kinney, Evelyn K., 60.
Kinsman, Blair, 61.
Kinsolving, H. L., 128.
Kiokemeister, Fred, 428.
Kissin, Sidney, 278.
Klee, V. L., 209.
Kleinfeld, Erwin, 351.
Kline, J. R., 567.
Klinger, E. L., 351.
Klopfenstein, R. W., 277.
Klopsteg, P. E., 131.
Kneale, S. G., 428.
Knowler, L. A., 145, 500.
Knox, S. R., 567.
Kokomoor, F. W., 59.
Kokoris, L. A., 565.
Kraft, Charles, 203.
Kraft, D. H., 61, 428.
Kramer, Max, 480.
Kramer, S. B., 61.
Krans, Robert, 59.
Kravitz, Sidney, 429.
Kros, F. A., 662.
Kundert, E. G., 201.
Lambert, R. J., 353.
Lancaster, E. R., 429.
Landers, A. W., 350.
Landry, A. E., 58.
Lane, E. P., 481.
Lane, R. A. C., 567.
Langenhop, C. E., 131.
Larney, Violet H., 565.
Larsen, H. D., 509.
LaSalle, Margaret M., 487.
Latimer, C. G., 58.
Lawrence, J. N. P., 429.
Layton, L. L., 61.
Leavitt, W. G., 140, 584.
Lee, A. S., Jr., 131.
Lee, T. H., 61.
Lefschetz, Solomon, 58.
Lehmann, E. L., 351, 429.
Lehmer, D. H., 207, 353.
Lehner, G. R., 61.
Lehr, Marguerite, 350.
Leibler, R. A., 481.
Leighton, W., 663.
Leone, F. C., 276.
LeVeque, W. J., 128.
Levy, Harry, 201.

Levy, S. L., 567.
Lewis, Anne L., 276.
Lewis, F. A., 483.
Lewis, J. A., 61.
Linney, Margaret M., 353.
Lisman, Henry, 278.
Lister, W. G., 353.
Littlejohn, T. C., 277.
Litzinger, Marie, 277.
Livers, J. J., 61, 565.
Lloyd, F. H., 429.
Loewner, Charles, 131.
Loflin, Z. L., 487.
Long, Florence, 357.
Long, R. W., 353.
Long, Roy, 131.
Lorentz, G. G., 129.
Lowdenslager, D. B., 567.
Loweke, G. P., 203.
Ludford, G. S. S., 202.
Lufborrow, R. A., 567.
McClintock, W. J., 61.
McCune, D. C., 61.
MacDowell, Robert, 202.
McGar, F. H., Jr., 131.
MacLane, Saunders, 58, 565.
MacNerney, J. S., 277.
McPherson, J. R., 61.
Macon, Nathaniel, 353.
Maddox, J. L., Jr., 61.
Male, C. T., Sr., 351.
Mall, A. W., 61.
Maltenfort, Martin, 131.
Mancill, J. D., 201.
Manheim, J. H., 429.
Manning, F. L., 437.
March, H. W., 429.
Marm, Anna, 283.
Marriott, R. W., 480.
Marsh, R. W., 567.
Marth, E., 567.
Martin, A. V., 351.
Martin, M. H., 202.
Martin, W. T., 276.
Massey, W. S., 663.
Matheson, J. D., 61.
Mautner, F. I., 59.
Maxfield, J. E., 61.
May, Kenneth, 203.
Mayor, J. R., 61.
Mazurkiewicz, H. G., 481.
Meadows, P. E., 70, 429.
Mears, Florence, 209.
Meier, Paul, 354.
Mela, D. F., 354.
Menger, Karl, 276.
Merrill, A. S., 360.
Mewborn, A. B., 351.
Meyer, H. A., 565
Meyer, Jean-Pierre, 564.
Michel, R. J., 483, 582.
Michie, J. N., 429.
Mickle, E. J., 61, 512.
Miller, A. E., 131.
Miller, J. S., 479.
Mills, H. D., 277.
Milne, W. E., 207, 277.
Minker, Jack, 481.
Mitchell, B. E., 131, 663.
Moise, E. E., 128.
Molina, E. C., 661.
Moline, Florence, 481.
Moller, R. W., 59.
Montague, H., 588.
Montgomery, Robert E., 481.
Moore, W. K., 202, 481.
Moore, W. L., 504.
Morelock, J. C., 661.
Morgan, G. W., 61.
Morgan, Kathryn A., 128.
Morison, J. S., 131.
Morrill, D. E., 131.
Morris, C. R., 481.
Morrow, D. C., 203.
Morse, D. S., 351.
Morse, W. P., 565.
Mortola, A. J., 354.
Moser, Leo, 131.
Moskovitz, David, 131.
Mosteller, F. C., 354.
Motzkin, T. S., 429.
Moulton, E. J., 567.
Munroe, M. E., 202.
Munshower, C. W., 288, 483.
Murray, J. J., 350.
Myers, N. G., Jr., 62.
Myers, S. B., 354.
Nash, J. F., Jr., 201.
Neff, J. D., 567.
Neisius, M. V., 62.
Nesbeda, Paul, 481.
Nelson, A. C., 277.
Newsom, C. V., 58, 564.
Nickel, J. A., 132.
Nilson, E. N., 429.
Noether, G. E., 200.
Nordhaus, E. A., 277.
Norman, P. B., 354.
Norton, Frances A., 354.
Novak, Illse L., 429, 564.
Null, P. S., 203, 663.
Oberbeck, T. E., 351.
Oberg, E. N., 59.
O'Connor, D. G., 132.
Oldenburger, Rufus, 429.
Odland, Leola P., 62.
Olds, C. D., 440.
Olds, E. G., 62.
Olkin, Ingram, 277.
Ollivier, Arthur, 67.
Ollmann, L. F., 216, 578.
Olmsted, J. M. H., 211.
Onat, E. T., 276.
O'Neill, A. F., 663.
O'Neill, Barrett, 201.
Orloff, Daniel, 429.
Orton, W. R., 564.
Osserman, Robert, 565.
Ostrom, T. G., 62.
Overholtzer, G. K., 661.
Overman, J. R., 132.
Paine, S. T., 429, 663.
Palmquist, K. L., 128.
Paulson, E., 565.
Parker, Eugene, 203.
Parker, F. D., 354.
Parker, W. V., 492.
Paxton, W. D., Jr., 62.
Payne, A. H., 429.
Payne, G. W., 430.
Pearson, Patricia, 62.
Perlin, I. E., 661.
Perlis, Sam, 351.
Perry, M. V., 201.
Peterson, G. M., 62.
Peterson, J. K., 480.
Peterson, R. P., 565.
Pike, J. M., 430.
Pinzka, C. F., 430.
Playfoot, R., 661.
Plummer, L. G., 567.
Polak, E. J., 567.
Pollak, Barth, 430.
Pollard, Harry, 564.
Polley, J. C., 357.
Polya, George, 479.
Pomilla, Frank, 132.
Porter, D. H., 663.
Price, G. B., 202, 483.
Pulliam, F. M., 351.
Putnam, C. R., 351.
Randolph, J. F., 288, 588.
Rathbun, V. J., 62.
Rauch, H. E., 202.
Rayl, A., 663.
Reckzeh, John, 481.
Redheffer, R. M., 201.
Rees, M., 57.
Reeves, R. F., 132.
Reid, W. T., 203.
Renneman, Conrad, Jr., 62.
Restemeyer, W. E., 352.
Reynolds, C. N., 203.
Reynolds, T. L., 479, 487.
Richmond, D. E., 201.
Rickart, C. E., 278.
Rickey, F. A., 67, 487.
Ritchie, A. A., 132.
Ritchie, Gordon, 62, 663.
Robbins, Herbert, 479.
Roberson, R. E., 430.
Roberts, G. G., 213.
Roberts, V. E., 354.
Robertson, Fred, 145, 500.
Robinson, Abraham, 129.
Robinson, G. B., 354.
Robinson, H. A., 135, 492.
Robinson, Raphael M., 440.
Robison, G. B., 201.
Rohde, Virginia, 481.
Rose, I. H., 277.
Rosen, Saul, 430.
Rosenbaum, R. A., 592.
Rosenberg, Max, 430,
Rosenbloom, P. C., 62.
Ross, L. L., 354.
Rosskopf, M. F., 482.
Royden, H. L., 62.
Royster, W. C., 661.
Royston, R. W., 663.
Ruderman, R. D., 578.
Russell, I. D., 201.
Ruther, Joseph, 201.
Ryan, D. R., 350.
Ryan, J. A., 350.
Ryser, H. J., 512.
Saalfrank, C. W., 661.
Salkover, Meyer, 352.
Sammet, Jean E., 354.
Sams, C. S., 380.
Samuel, Pierre, 564.
Sanders, James, 202.
Sandy, J. M., 430.
Schafer, R. D., 437.
Schatten, R., 663.
Scheid, F. J., 200.
Schneckenburger, Edith R., 57, 482.
Schoderbek, J. J., 567.
Schoenberg, I. J., 202.
Schreiber, E. W., 285.
Schriro, G. R., 132.
Schuster, S., 567.
Schwartz, Abraham, 62.
Schwartz, Cecil, 482.
Schwartz, W. J., 277.
Schwid, Nathan, 62.
Scott, L. L., 202.
Scott, Elizabeth L., 351.
Seber, R. C., 354.
Seebald, H. A., 354.
Seebeck, C. L., Jr., 62, 430.
Segal, I. E., 352.
Service, J. H., 132.
Shapiro, A. S., 564.
Shapiro, Aaron, 216, 578.
Shapiro, E. I., 216.
Sheffield, R. D., 202, 352.
Shepherd, W. L., 482.
Sherer, C. R., 69, 518.
Sherwood, G. E. F., 201.
Shively, L. S., 203.
Shklov, Nathan, 202.
Shotwell, C. E., 132.
Shrikhande, S. S., 202.
Shuster, C. N., 661.
Silva, J. A., 132.
Silverman, Edward, 354.
Simester, J. H., 203.
Simoneau, G. M., 350.
Simpson, T. M., 59, 663.
Singer, James, 216.
Sledd, M. B., 200.
Sleight, E. R., 209.
Slotnick, M. M., 58, 204.
Small, W. A., 202.
Smith, C. B., 59.
Smith, D. J., 430.
Smith, Donald, 201.
Smith, F. C., 211, 580.
Smith, G. W., 202.
Smith, J. C., 482.
Smith, R. B., 278.
Smith, R. E., 62.
Smith, Robert, 201.

Snader, D. W., 58.
Snider, R. L., 132.
Soglin, Albert, 276.
Sousley, C. P., 576.
South, D. E., 565.
Spanier, E. H., 352.
Spears, O. S., 430.
Spencer, G. L., II, 202.
Spohn, R. H., 132, 430.
Spong, R. A., 132.
Springer, George, 277.
Stallard, Sam, 354.
Stark, L. W., 204.
Starr, D. W., 69.
Steimley, L. L., 201.
Steinbrenner, A. H., 128.
Stiefel, Edward, 201.
Stinetorf, Roscoe, 482.
Stipe, C. G., 132.
Stoddard, S., 661.
Stoll, R. R., 482.
Stone, M. H., 351, 565.
Stone, W. C., 351.
Stoner, Irwin, 354.
Stotz, H. C., 128.
Stovall, W. B., 663.
Strehler, A. F., 350.
Stulken, E. J., 132.
Sturm, R. G., 62.
Sumner, D. B., 132.
Swaffield, J. R., 430.
Swanson, A. G., 580.
Sweetland, P. C., 430.
Syer, H. W., 204.
Taliaferro, R. C., 567.
Talkington, A. D., 430.
Tarcisius, M., 204.
Tartler, A., 568.
Taylor, A. E., 498.
Taylor, Anna L., 132.
Taylor, H. E., 59.
Taylor, Mildred E., 508.
Tear, R. T., 354.
Teicher, H. E., 351.
Temte, Arnold, 129.
Terry, C. M., 204.
Thomas, R. E., 663.
Thompson, F. B., 204.
Thornton, W. G., 568.
Timon, W. E., Jr., 204.
Tolman, M. W., 663.
Torrance, C. C., 351.
Torrey, Marian M., 508.
Transue, W. R., 351, 512.
Trigg, C. W., 498.
Trjitzinsky, W. J., 201.
Tucker, Henry, 62.
Tuckerman, Bryant, 564.
Uhl, H. R., 132.
Vanaman, S. B., Jr., 352.
Vanderslice, J. L., 202.
Vaughn, H. E., 201.
Vezeau, W. A., 128.
Vlachos, N. D., 204, 482.
Waddell, M. C., 132.
Wade, B. T., 204.
Wadkins, O. L., Jr., 204.
Wahab, J. H., 62.
Walker, Lila P., 129.
Wall, F. J., 430.
Waller, Mary E., 564.
Wang, H. C., 201, 661.
Ward, J. A., 128, 213, 504.
Warwick, J. W., 663.
Wayne, Alan, 204.
Weber, Maria, 201.
Wechsler, Martin, 203.
Wedel, A. M., 133.
Weiner, L. M., 430.
Weintraub, H., 568.
Weisfeld, Morris, 278.
Weiss, E. H., 63.
Weiss, Lionel, 564.
Welch, Margaret, 201.
Wells, C. P., 277.
Wells, J. B., Jr., 352.
Welna, Cecilia T., 201.
Wendel, J. G., 354, 663.
Wermer, John, 278.
Wescott, M. E., 568.
Wesolowski, Edmund, 204.
Wessely, H. W., 354.
Wesson, J. R., 63.
Wester, Truman, 363, 564.
Wexler, Charles, 58.
Wheeling, R. F., 482.
White, F. D., 661.
Whitford, A. E., 482.
Whiteman, A. L., 59.
Whitney, Anne M., 133.
Whitney, H., 568.
Wiener, Norbert, 58.
Wilansky, A., 568.
Willard, D., 661.
Williams, Mary E., 480.
Willoughby, R. A., 200.
Wilson, J. A., 661.
Wilson, L. T., 128.
Wilson, R. H., 204.
Wing, G. M., 355.
Winn, W. F., 482.
Wolf, Louise, 70, 133, 590.
Wolf, Philip, 201.
Wolfe, Frances E., 129.
Wolfowitz, Jacob, 564.
Wolfsohn, N. Z., 202.
Wollan, G. N., 59.
Woodle, R. G., Jr., 355.
Woods, Dale, 63.
Worsing, R. A., 355.
Wright, B. F., 661.
Wright, C. B., 69, 518.
Wyler, Oswald, 277.
Wylie, C. R., Jr., 58.
Yagi, F., 565.
Yates, R. C., 662.
Yeardley, N. P., 277.
Yeaton, C. H., 564.
Young, D. M., Jr., 133.
Young, Elaine H., 276.
Young, F. H., 355.
Young, Gail, 128.
Young, J. W., 565.
Young, P. M., 283, 490
Young, R. L., 133.
Young, R. W., 431.
Zant, J. H., 564.
Zaring, W. M., 352.
Zatzkis, Henry, 201.
Zeichner, A. J., 202.
Zeigler, R. K., 431.
Zimmerberg, H. J., 133
Zorn, M. A., 576.
Zuckerman, H. S., 565.

NECROLOGY

Barnett, J., 482.
Bartz, R. D., 482.
Blakeley, Laura, 204.
Bull, Elsie O., 133.
Daugherty, R. D., 204.
Fite, W. B., 355.
Flowers, R. L., 63.
Foster, M. C., 482.
Graff, F. G., 355.
Hall, H. L., 204.
Hodge, F. H., 63.
Holyroyd, Ina E., 63.
Hubbard, J. F., 204.
Kimball, T. C., 278.
Lampland, C. O., 355.
Lawrence, V. S., Jr., 355.
Lennes, N. J., 204.
Litzinger, M., 482.
Mathewson, L. C., 133.
Meacham, E. D., 568.
Messick, J. F., 133.
Morris, Richard, 63.
Neményi, P. F., 278.
Newman, J. J., 482.
Nichols, I. C., 482.
Nicolet, Justin, 133.
O'Donnell, G. A., 278.
Poor, C. L., 355.
Richardson, A. V., 63.
Rosenbach, J. B., 133.
Schreiber, E. W., 278.
Shumway, R. R., 482.
Taylor, E. W. D., 482.
Van Horn, C. E., 63.
Vass, J. I., 133.
Wedel, E. B., 568.
Wilczewski, Joseph, 63.
Wolff, H. C., 133.
Zimmerman, B. C., 663.

REPORTS AND ANNOUNCEMENTS OF THE ASSOCIATION AND ITS SECTIONS

MEETINGS AND ANNOUNCEMENTS OF THE ASSOCIATION

Earle Raymond Hedrick Lectures, September 1, 2, 1952, 355–356.
New Sectional Governors of the Association, 483.
Preparation of Problem and Source Materials for the Mathematical Training of Social Scientists, 356–357.
Report of the Treasurer for the Year 1951, 281–282.
The Thirty-third Summer Meeting of the Association, H. M. GEHMAN, 664–668.
The Thirty-fifth Annual Meeting of the Association, H. M. GEHMAN, 205–208.
The Traveling Lecturer Program of the Southwestern Section, 486–487.
The Twelfth Annual William Lowell Putnam Mathematical Competition, 63–64.

THE AMERICAN
MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA, INC.

VOLUME 59 NUMBER 10

CONTENTS

Simplified Derivation of the Properties of Elementary Transcendentals . . . . . E. P. WIGNER 669

Note on a Paper of Bagchi and Chatterjee . . . . L. CARLITZ 683

A Note on Gambling Systems and Birth Statistics HERBERT ROBBINS 685

Mathematical Notes . . . . . J. M. DANSKIN, T. G. ROOM 687

Classroom Notes. . . . . D. C. LEWIS, D. E. RICHMOND 692

Elementary Problems and Solutions 696

Advanced Problems and Solutions 701

Recent Publications 708

Clubs and Allied Activities 710

Calendar of Future Meetings 712

Index to Volume 59 713

DECEMBER 1952

The AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

CARL B. ALLENDOERFER, *Editor*

ASSOCIATE EDITORS

R. V. CHURCHILL	J. W. GIVENS	GEORGE SPRINGER
PATRICK DU VAL	L. M. KELLY	E. P. STARKE
HOWARD EVES	H. D. LARSEN	G. B. THOMAS
F. A. FICKEN	IVAN NIVEN	E. P. VANCE

EDITH R. SCHNECKENBURGER

EDITORIAL CORRESPONDENCE should be addressed to the Editor, C. B. ALLENDOERFER, Department of Mathematics, University of Washington, Seattle 5, Washington.

ADVERTISING CORRESPONDENCE should be addressed to J. F. RANDOLPH, Morey Hall, University of Rochester, Rochester 3, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the MONTHLY should be sent to the Secretary-Treasurer, H. M. GEHMAN, University of Buffalo, Buffalo 14, N. Y.

THIS IS THE OFFICIAL JOURNAL OF THE
MATHEMATICAL ASSOCIATION OF AMERICA, INC.

(Devoted to the Interests of College Mathematics)

OFFICERS OF THE ASSOCIATION

President, SAUNDERS MACLANE, University of Chicago
Honorary President, W. D. CAIRNS, Oberlin College
First Vice-President, F. L. GRIFFIN, Wesleyan University
Second Vice-President, JEWELL H. BUSHEY, Hunter College
Secretary-Treasurer, H. M. GEHMAN, University of Buffalo
Associate Secretary, EDITH R. SCHNECKENBURGER, University of Buffalo
Editor, C. B. ALLENDOERFER, University of Washington

Additional Members of the Board of Governors: I. A. BARNETT, C. F. BARR, T. A. BICKERSTAFF, J. C. BRIXEY, B. H. BROWN, R. H. CAMERON, G. R. CLEMENTS, T. F. COPE, H. S. M. COXETER, W. M. DAVIS, J. M. EARL, L. R. FORD, J. S. FRAME, A. E. GAULT, R. F. GRAESSER, E. H. HANSON, M. R. HESTENES, A. S. HOWARD, C. G. JAEGER, R. D. JAMES, B. W. JONES, R. E. LANGER, D. H. LEHMER, F. A. LEWIS, C. C. MACDUFFEE, R. J. MICHEL, W. E. MILNE, C. W. MUNSHOWER, H. P. PETTIT, J. C. POLLEY, G. B. PRICE, G. E. RAYNOR, E. B. ROESSLER, F. H. STEEN.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION at Menasha, Wisconsin, and Seattle, Wash. during the months of January, February, March, April, May, June–July, August–September, October, November, December.

PRINTED IN THE UNITED STATES OF AMERICA

SIMPLIFIED DERIVATION OF THE PROPERTIES OF ELEMENTARY TRANSCENDENTALS

E. P. WIGNER, Princeton University

The purpose of the present paper is to give a derivation of the properties of the elementary transcendentals from a more unified point of view than is customary. It will be assumed that the general properties of analytic functions are established. In particular we shall use the theorem that the real (or imaginary) part of an analytic function cannot assume its maximum value in the inside of a closed domain in which it is regular. Similarly, use will be made of Weierstrass's theorem about a set of uniformly converging analytic functions, of Liouville's and Rolle's theorems.§ All of these theorems can be established without the use of the properties of the transcendental functions, using only elementary calculations and the concept of convergence. The present paper shows how the properties of the elementary transcendentals can then be derived from the aforementioned theorems by means of the methods of the theory of complex functions. From the point of view of this theory, this appears to be more appropriate than taking over the results concerning the properties of these functions, at least for real arguments, from other parts of analysis. An outline of our procedure follows.

As a first step, a special class of meromorphic functions, the so-called R functions, will be introduced and a number of properties of these functions, including two expansion theorems, established (Section 1). Some of these properties will not be used for the derivation of the properties of elementary transcendentals; the numbers of the corresponding theorems have a star attached to them.

The tangent then will be defined in Section 2 as the simplest periodic R function. The partial fraction and infinite product expansion of the tangent is a special case of the expansion theorems mentioned before while the addition theorem for this function will follow from its periodic nature. It is easy to obtain then the similar expansions for the other trigonometric and the exponential functions.

The writer wishes to admit at the outset that his definition of the exponential function is less direct than that of Caratheodory,† for instance. On the other hand, the derivation of the various expansion theorems which will be given below is less artificial and more general than the customary derivations.

§ *Cf. e.g.* A. Hurwitz and R. Courant, *Allgemeine Funktionentheorie etc.* Verlag Julius Springer, Berlin 1922, III 2, page 296; I 3, p. 63 and I 3, p. 56.

† C. Caratheodory, *Funktionentheorie*, Verlag Birkhauser, Basel 1950, Sec. 234, p. 233.

1. Properties of the R functions. R functions are defined ‡ as meromorphic functions of a complex variable z, the imaginary part of which is non-negative in the upper and non-positive in the lower half-plane. For the sake of the reader's convenience, some rather obvious properties of such functions are repeated here from an earlier article of the writer.‡ A few of these properties had been given previously by Fränz.‡

THEOREM 1.1. *If a set of R functions converges to a meromorphic function, this function itself is an R function.*

THEOREM 1.2. *An R function is real on the real axis and assumes real values only on the real axis.*

THEOREM 1.3. *The derivative of an R function is positive at every regular point on the real axis unless R is a constant.*

THEOREM 1.4. *All poles of an R function are on the real axis, they are simple and their residues negative.* The poles will be denoted henceforth by Z_μ, the corresponding residues by $-\gamma_\mu^2$; both the Z and γ are then real. A typical R function for real values of its argument is given in Figure 1.

---

‡ As far as the writer could determine, R functions appear in the mathematical literature first (though not under this name) in K. Loewner's article *Ueber monotone Matrixfunktionen*, Math. Zeits. 38, 177 (1933). In this article Loewner considers functions $R(m)$ of real symmetric or hermitean matrices m which satisfy the condition that $R(m_1)-R(m_2)$ be positive definite if m_1-m_2 is positive definite and if the characteristic values of m_1 and m_2 are in an interval (λ, λ'). He shows that the $R(m)$ are R functions which are regular in the (λ, λ') interval. The writer is much indebted for this quotation to S. Sherman. Certain generalizations of R functions also play an important role in the theory of moments. *Cf. e.g.* J. A. Shohat and J. D. Tamarkin, The Problem of Moments, Publication of the American Mathematical Society, New York 1943, particularly Chapters I and II. The R functions have also played a substantial if not always obvious role in electrical engineering, in particular in the theory of the reactance of circuits, ever since the pioneering investigations of O. J. Zobel, G. A. Campbell and R. M. Foster (Bell System Technical Journal, Volumes 1, 2 and 3, 1922–24). *Cf.* also W. Cauer, Preuss. Akad. Ber. 1931, p. 30. A more modern review of the theory has been given recently by H. W. Bode, Network Analysis and Feedback Amplifier Design (Van Nostrand, 1945). However, most of this work deals with rational R functions for which most of the contents of the present article are obvious. An exception is the article of K. Fränz, Elektrische Nachrichten Technik, 21, 8 (1944) who already considers transcendental R functions. The writer is much indebted for this quotation to Dr. T. Teichmann. For odd R functions, the expansion theorems 1.10 and 1.12 also follow from the work of P. I. Richards, Duke Math. Journ., 14, 777, 1947, which is based on W. Cauer's article in Bull. Amer. Math. Soc., 38, 713, 1932. This article anticipates, for odd R functions, many of the results of our Section 1. The connection between his *iPRF* function $Z(s)$ and the R functions is $Z(s)=-iR(is)$. *Cf.* also Richards' summary article, Quarterly Journ. of Math., 6, 21, 1948. The writer's interest in these functions was raised by his work on the quantum mechanical theory of reaction and scattering processes. *Cf.* in this connection E. P. Wigner and L. Eisenbud, Physical Rev. 72, 29 (1947) and the summary article Amer. Journ. of Physics, 17, 99 (1949). The first general investigation of the mathematical properties of these functions is given in Ann. of Math. 53, 36 (1951). It was not entirely possible to eliminate all repetition between this last article and the present one.

THEOREM 1.5. *A linear combination of R functions with positive coefficients is an R function.*

THEOREM 1.6. *The linear function $\alpha z+\beta$ is an R function if α is positive and β real.*

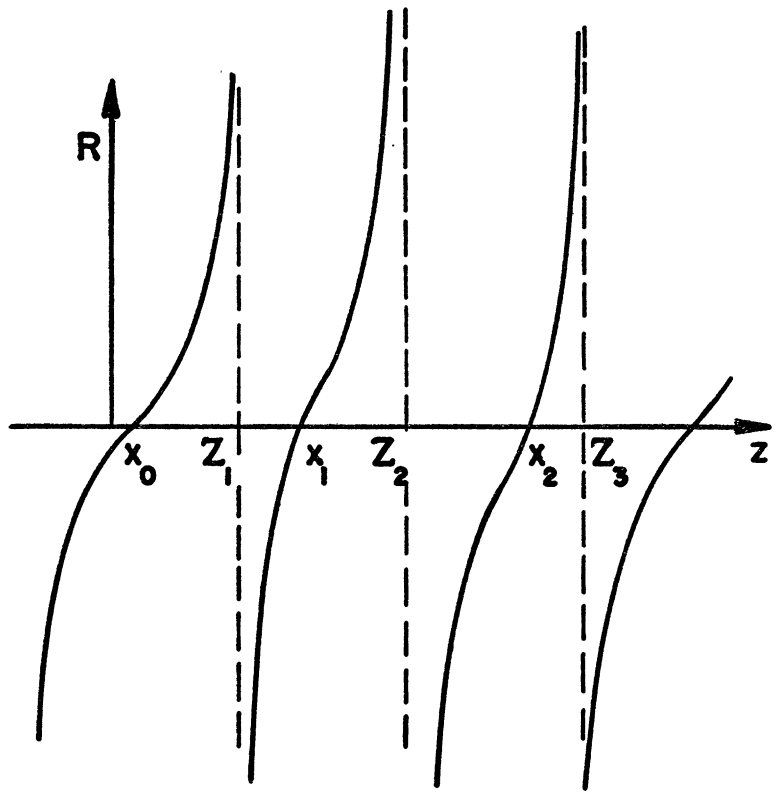


FIG. 1

THEOREM 1.7. *A linear fractional transform of an R function*

$$S = \frac{a_1R + b_1}{a_2R + b_2} \tag{1}$$

with real coefficients and positive determinant $a_1b_2-b_1a_2$ is again an R function.

THEOREM 1.8. *A real rational function is an R function if*

a) *the degree of its numerator does not exceed that of the denominator by more than one,*

b) *its value remains below a finite number as $z\to-\infty$,*

c) *its poles are real and their residues negative.*

The rational function in question can be written in the form

$$R(z) = \alpha z + \beta + \sum\left(\frac{\gamma_\mu^2}{Z_\mu - z} - \frac{\gamma_\mu^2}{Z_\mu}\right) \tag{2}$$

the number of terms in the sum being the degree of the denominator. Because of the second condition, $\alpha \geqq 0$, because of the last condition Z_μ and γ_μ are real; β is real also. Hence, by Theorem 1.6 and Theorem 1.7, all terms of (2) are R functions and it is an R function by Theorem 1.5.

THEOREM 1.9. *The series* (2), *finite or infinite, with positive or vanishing* α, *real* β, γ *and* Z *is an* R *function if the* Z_μ *have no finite condensation point and if the series on the right converges at least for one value of* $z \neq 0$. *In this case, it converges absolutely and uniformly in every closed domain of* z *which excludes the poles* Z.

In fact, the general term of the series in (2) has the form

$$\frac{\gamma^2}{Z - z} - \frac{\gamma^2}{Z} = \left(\frac{\gamma}{Z}\right)^2 z\left(1 + \frac{z}{Z - z}\right)$$

which shows: if the series converges for a $z \neq 0$, the series with the positive general term $(\gamma/Z)^2$ converges too, and so the series in (2) converges absolutely and uniformly in every closed domain excluding the poles Z, and represents, by Weierstrass' theorem, an analytic function. This function is clearly meromorphic and an R function by Theorems 1.8 and 1.1.

We now come to the expansion theorems for R functions.

THEOREM 1.10. *Every* R *function can be expanded in the form* (2).

In order to prove this without the use of the properties of the elementary transcendentals, we first establish the following Lemma due to M. Schiffer and V. Bargmann.§

LEMMA 1.10a. *Let* $R(z)$ *be an* R *function,* Z_1 *one of its poles and* $\gamma_1^2/(Z_1 - z)$ *its meromorphic part at* Z_1. *Then*

$$R_1(z) = R(z) - \gamma_1^2/(Z_1 - z) \tag{3}$$

is also an R *function.*

The proof of this theorem, due to Schiffer and Bargmann, first observes that R_1 is clearly meromorphic. If its imaginary part assumed a negative value $-\rho$ at a point z' of the upper half-plane, one could draw a circle about Z_1 with a radius larger than $2\gamma_1^2/\rho$ and large enough to contain z'. The imaginary part of R_1 is larger than $-\frac{1}{2}\rho$ along the part of this circle which lies in the upper half-

§ Cf. E. P. Wigner, Ann. of Math. 53, 36 (1951). Only the last part of the present proof, beginning with lemma 1.10b, is new.

plane because the imaginary part of R is positive and the absolute value of $\gamma_1^2/(Z_1-z)$ is smaller than $\frac{1}{2}\rho$ along this semicircle. Furthermore, R_1 is real at every point of regularity of the real axis, including Z_1. Finally, if Z_μ is a pole of R within the circle, one can draw a sufficiently small semicircle about Z_μ into the upper half-plane so that the imaginary part of R_1 becomes positive thereon. In order to demonstrate this, one observes that, by Theorem 1.4, Z_μ is a simple pole of R_1. Hence there is a neighborhood $|\zeta|<\epsilon$ of Z_μ with $\zeta=\zeta_1+i\zeta_2=Z_\mu-z$ where $R_{1\mu}=R_1-\gamma_\mu^2/\zeta$ is regular. Furthermore $R_{1\mu}$ remains real on the real axis so that, by Rolle's theorem, the imaginary part of $R_{1\mu}$ for $|\zeta|<\epsilon$, $\zeta_2>0$ is larger than $M\zeta_2$ where M is the minimum of the derivative of $R_{1\mu}$ in the direction of the imaginary axis. Hence the imaginary part of R_1 is, in the same region, larger than $(M+\gamma_\mu^2/|\zeta|^2)\zeta_2$. The bracket of this expression can however be made positive by choosing $|\zeta|^2$ small enough. Thus z' can be surrounded by a continuous path (*cf.* Figure 2) containing no singularities of R_1, on which the imaginary part of R_1 is everywhere larger than $-\frac{1}{2}\rho$. Hence the imaginary part of R_1 cannot be negative in the upper half-plane and a similar argument shows that it cannot be positive in the lower half-plane. Thus R_1 is an R function.

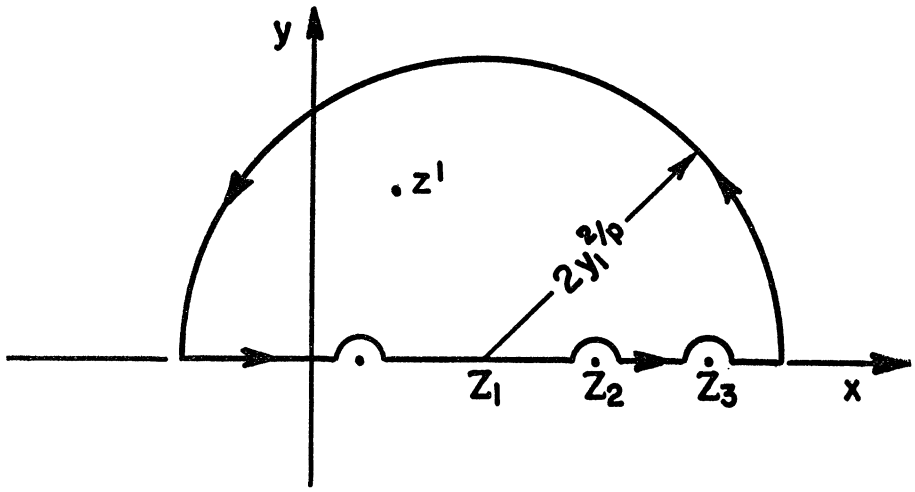


FIG. 2

We can subtract in a similar way any finite number of singularities from R, and the resulting function

$$R_n = R - \sum[\gamma_\mu^2/(Z_\mu - z) - \gamma_\mu^2/Z_\mu] \tag{4}$$

will remain an R function. Hence, its derivative at a real point of regularity z_0 will be positive by Theorem 1.3:

$$\sum\gamma_\mu^2/(Z_\mu - z_0)^2 < R'(z_0). \tag{5}$$

It follows that the whole series $\gamma_\mu^2/(Z_\mu - z_0)^2$ converges if z_0 is real. However, since $|\gamma_\mu^2/(Z_\mu - x - iy)^2| < \gamma_\mu^2/(Z_\mu - x)^2$, the series (5) converges absolutely also for complex z_0. The same is true of the series

$$\left|\frac{\gamma_\mu^2}{Z_\mu - z_0} - \frac{\gamma_\mu^2}{Z_\mu}\right| = |z_0| \sum \left|\frac{\gamma_\mu^2}{(Z_\mu - z_0)Z_\mu}\right|. \tag{6}$$

Except for a finite number of terms, its terms are smaller than $2|z_0|$ times the corresponding terms of (5). Hence, the conditions of Theorem 1.9 are fulfilled and R_n converges, as we subtract all the poles, by the second part of Theorem 1.9, to an analytic function $I(z)$. This $I(z)$ can have no finite singularity. As a limiting case of R functions, $I(z)$ is itself an R function. In order to prove Theorem 1.10, it remains to be shown only that such a function must be linear. This follows from Lemma 1.10b:

LEMMA 1.10b: *An entire R function $I(z)$ is a linear function $\alpha z + \beta$, with real β, $\alpha \geqq 0$.*

Because of Theorem 1.3, unless $I(z)$ is a constant, it is for real z a monotonically increasing function. Let us denote its value at $z = 0$ with β and consider the function

$$J(z) = 1/(\beta - I(z)). \tag{7}$$

This is, by Theorem 1.7, an R function, negative for positive z, positive for negative z. It has only one singularity, at $z = 0$. Let us subtract this singularity by Theorem 1.10a. The result

$$J_1(z) = J(z) + \gamma^2/z \tag{8}$$

is again an entire R function. It must be a constant unless it is monotonically increasing. Since $J(z)$ is negative for positive z, and conversely, $J_1(\infty) \leqq 0$ and $J_1(-\infty) \geqq 0$. Hence $J_1(z) = 0$ and $I(z) = \beta + z/\gamma^2$ which proves Theorem 1.10b and hence, Theorem 1.10.

THEOREM 1.11. *Given a set of real numbers x_μ, Z_μ without finite condensation points for which*

$$\cdots < Z_{-2} < x_{-2} < Z_{-1} < x_{-1} < Z_0 < x_0 < Z_1 < x_1 < Z_2 < x_2 < \cdots \tag{9}$$

and $Z_0 \leqq 0 < Z_1$, $c > 0$, the products (finite or infinite) in

$$R(z) = c\frac{x_0 - z}{Z_0 - z}\prod_\nu \frac{1 - z/x_\nu}{1 - z/Z_\nu}\prod_\nu \frac{1 - z/x_{-\nu}}{1 - z/Z_{-\nu}} \tag{9a}$$

converge absolutely and uniformly in every finite domain to an R function.

The ν in both products of (9a), as in all formulae below, runs over all positive integers. If the set (9) breaks off at a finite Z_l, the last factor in the first $\prod$ is $(1 - z/Z_l)^{-1}$; if the set starts with a finite x_s the last factor in the second

$\prod$ is $z-x_s$. The last two rules can be replaced by the convention of adding an $x_l=\infty$ to the set of x in the former and a $Z_s=-\infty$ to the set of Z in the latter case. Hence (9) always begins with a Z and ends with an x if it has a finite beginning or end.

To show Theorem 1.11 it will be necessary to establish the well-known theorem about the convergence of infinite products without the use of the exponential function.

LEMMA 1.11a. *The absolute convergence of* $\prod(1+a_\nu)$ *follows from the convergence of* $\sum|a_\nu|$.

Indeed, it can be shown that any partial product $\prod'(1+a_\nu)$ is closer to 1 than the corresponding sum $\sum|2a_\nu|$ as long as this sum does not exceed 1. We shall prove

$$|\prod'(1+a_\nu)-1|<2\sum'|a_\nu| \tag{10}$$

by induction. It is correct if the partial product contains only one factor. Let us assume that (10) is correct if the partial product contains only $n-1$ factors. It then follows that

$$\begin{aligned}|\prod'(1+a_\nu)(1+a_n)-1| &\leqq |\prod'(1+a_\nu)-1|+|a_n||\prod'(1+a_\nu)| \\ &<2\sum'|a_\nu|+|a_n|(1+2\sum'|a_\nu|) \\ &\leqq 2(|a_n|+\sum'|a_\nu|).\end{aligned} \tag{11}$$

The last part holds because $\sum'|a_\nu|\leqq\frac{1}{2}$ by assumption. This proves the inequality (10) for n factors and hence in general. The lemma 1.11a is an immediate consequence of (10).

Given now any closed domain, there is only a finite number of Z_μ the absolute value of which is smaller than the maximum value of $2|z|$ in the domain. Apart from the corresponding factors in the products in (9a), $|1-z/Z_\mu|>\frac{1}{2}$ holds for all z of the domain. The product of these factors will converge absolutely and uniformly by Lemma 1.11a if the series

$$\sum|a_\mu|=\sum\left|\frac{1-z/x_\mu}{1-z/Z_\mu}-1\right|=|z|\sum\left|\frac{Z_\mu^{-1}-x_\mu^{-1}}{1-z/Z_\mu}\right|\leqq 2|z|\sum|1/Z_\mu-1/x_\mu| \tag{12}$$

converges. The last series is, however, because of (9), smaller than $2|z/Z_1|$ for the first product and smaller than $2|z/x_{-1}|$ for the second product in (9a). Thus, the product of the factors considered represents, by Weierstrass' theorem, an analytic function in the domain considered and the right side of (9a) is a meromorphic function.

In order to prove that (9a) is an R function, it is, because of Theorem 1.1, necessary only to prove that every finite partial product of (9a) is an R function. These partial products are real rational functions and the degree of their numerators is equal to that of the denominator, unless the series (9) has a beginning

or end, in which case the degrees differ by ± 1. Second, all factors of (9a) tend to 1 as z goes to $\pm\infty$ save the exceptional factors $(1-z/Z_l)^{-1}$ and $z-x_s$, if such are present. At any rate $R(z)$ is bounded from above as $z\to-\infty$. Finally for $\mu\neq 0$, every factor is positive at $z=Z_\mu$, except

$$(1-z/x_\mu)/(1-z/Z_\mu)=Z_\mu(1-z/x_\mu)/(Z_\mu-z)$$

and the numerator of this expression is also positive at $z=Z_\mu$ both for $\mu<0$ and for $\mu>0$. Hence the residues of all poles Z_μ with $\mu\neq 0$ are negative and one can easily see that this holds also for the pole Z_0. It thus follows from Theorem 1.8 that the finite partial products of (9a) are R functions and from Theorems 1.11a and 1.1 that (9a) is also an R function.

THEOREM 1.12. *Every R function $R(z)$ can be expanded§ into a product of the form* (9a).

Because of Theorem 1.3, the poles Z_μ and zeros x_μ satisfy the conditions (9) whence (9a) represents an R function with the same poles and zeros as $R(z)$. Hence Theorem 1.12 will be proved if it can be shown that the ratio $r(z)$ of two R functions with the same poles and zeros is a constant. Since both zeros and poles of an R function must be, by Theorems 1.3 and 1.4, simple, $r(z)$ is an entire function. It cannot assume a negative real value on the real axis since both R functions are negative in the same (Z_μ, x_μ) intervals, positive elsewhere. It cannot assume a negative real value on the upper half-plane either since the imaginary parts of both R functions are positive there. For the same reason r cannot be negative on the lower half-plane. However, *an entire function $r(z)$ which does not assume any value of a finite interval (r_1, r_2) is necessarily a constant.* One sees this most easily by considering $\rho=((r-r_1)/(r-r_2))^{1/2}$. Since the radicand of this expression assumes negative real values only if r is in the interval (r_1, r_2), and since $r(z)$ does not assume values in this interval, ρ is a single-valued function which does not assume any purely imaginary value. Its real part is either positive for every z, or negative for every z. Application of Liouville's theorem to $1/(\rho+i)$ in the former and $1/(\rho-i)$ in the latter case shows that it is a constant. This then proves Theorem 1.12.

This result about the relation of two R functions which have the same poles and zeros can be generalized to the case when two other real values, a and $b>a$, are assumed by both R functions at the same two sets of values of z. By applying the above result to $(R-a)/(b-R)$ one finds:

THEOREM 1.13.* *If two R functions assume the real value a at the same values of z, and if this is true for another real number b, the two R functions can be obtained from each other by a fractional linear transformation which leaves a and b invariant.*

§ Under certain more restrictive assumptions, this theorem has been shown already by K. Fränz, *l.c.* footnote p. 670.

We now come to a couple of further theorems which are not necessary for the development of Section 2. They give the connection between differential equations and R functions. Let us consider the function $\phi(\xi, z)$ of the complex variable z and real variable ξ which obeys the differential equation

$$\frac{d^2\phi}{d\xi^2} = U(\xi)\frac{d\phi}{d\xi} + V(\xi)\phi - z\phi \tag{13}$$

in which U, V are real continuous functions of ξ in the interval $(0, \xi_0)$. Given z-independent initial conditions at $\xi=0$, it is well known that† ϕ and $d\phi/d\xi$, as functions of z, are entire functions.

THEOREM 1.14.* *The solution $R(\xi, z)$ of Riccati's equation*

$$\frac{dR(\xi, z)}{d\xi} = R^2 + UR - V + z \tag{14}$$

with the initial condition $R(0, z) = R_0$ and real R_0 is, for every value ξ of the interval $(0, \xi_0)$, an R function of z.

One obtains (14) from (13) by the substitution $R = -(d\phi/d\xi)/\phi$. As quotient of two entire functions, R is meromorphic. Let us separate $R = R_r + iR_s$ into real and imaginary parts. The equation for R_s is

$$dR_s/d\xi = 2R_rR_s + UR_s + y \tag{14a}$$

where y is the imaginary part of $z = x + iy$. It is necessary to show now that $R_s(\xi)$ has the same sign as y.

Because of $R_s(0, z) = 0$, this is evident for sufficiently small values of ξ. Furthermore, if y is positive, R_s has a positive slope at a point where $R_s = 0$. Hence, as ξ increases, R_s cannot become negative as long as R obeys the differential equation (14). Under the same condition, R_s cannot become positive if y is negative.

Naturally, R can have singularities and will not obey (14) at these. In order to prove that R_s remains positive for positive y as ξ passes through a singularity ξ_0, we note that $Q = -1/R$ vanishes at, and is regular in the neighborhood of, ξ_0. Hence, in the neighborhood of ξ_0, the imaginary part Q_s of $Q = Q_r + iQ_s$ vanishes more strongly than its derivative. It follows that we can set $Q_s = 0$ when calculating $dQ_s/d\xi$ in the neighborhood of ξ_0. The differential equation for Q_s then shows that $dQ_s/d\xi$ has, in the neighborhood of ξ_0, the same sign as yQ_r^2, *i.e.* as y. According to the preceding paragraph, this is true also for $Q_s = R_s|Q|^2$ as long as $\xi < \xi_0$ so that Q_s, and hence also R_s, retain their sign as ξ passes

† *Cf. e.g.* E. L. Ince, Ordinary Differential Equations. Dover Publications, New York, 1944. Section 3.31, p. 72.

through ξ_0. This then proves‡ Theorem 1.14.

Theorem 1.14 permits the definition of a wide variety of R functions and, of course, the R functions derive their significance from Theorems 1.14 and 1.15. However, the latter will not be proved here.§

In order to formulate Theorem 1.15, we consider an n dimensional real euclidean space E_n with rectangular coordinates $\xi_1, \xi_2, \cdots, \xi_n$. There are two regular closed surfaces S_1 and S in E_n, the former being entirely inside S. Given further a self-adjoint real operator§§ H defined for functions within S and for which

$$Hf = -\Delta f = -(\partial^2/\partial\xi_1^2 + \partial^2/\partial\xi_2^2 + \cdots + \partial^2/\partial\xi_n^2)f \tag{15}$$

holds if f vanishes within S_1 and belongs to the definition domain of the Laplace operator Δ. This operator is defined to be self-adjoint also and applicable to all twice continuously differentiable functions in S which have zero normal derivatives thereon. Let us consider then a solution of the equation

$$H\phi(\xi, z) = z\phi(\xi, z) \tag{16}$$

such that ϕ is, on the surface S, real and independent of z

$$\phi(\xi, z) = \psi(\xi) \quad \text{on } S. \tag{16a}$$

THEOREM 1.15.* *Under the conditions of the preceding paragraph the surface integral*

$$-\int \psi(\xi)\, \mathrm{grad}_n\, \phi(\xi, z)dS = R(z) \tag{17}$$

extended over the surface S is an R function of z; the grad_n *denotes the component of* grad ϕ *that is normal to the surface S.*

For the sake of completeness, three further and rather obvious criteria will be given here which suffice, in conjunction with Theorem 1.14, to prove that the logarithmic derivative of the Bessel functions and of a few other functions, including the Γ function, are R functions and hence permit partial fraction and product representations of the form (2) and (9a).

THEOREM 1.16.* *$aR(bz+c)$ is an R function of z if a, b, c are real and ab positive, and if $R(z)$ is an R function.*

---

‡ This theorem, although in a different form, has been known for a long time. *Cf. e.g.* L. Bieberbach, *Theorie der Differentialgleichungen*, Julius Springer, Berlin 1923, Section 2, Chapter III, p. 161.

§ For a plausibility argument cf. E. P. Wigner and L. Eisenbud, reference 1.

§§ For the definition of self-adjoint operators *cf. e.g.* B.v.Sz. Nagy, *Spektraldarstellung Linearer Transformationen des Hilbertschen Raumes.* Vol. 5 of *Ergebnisse der Mathematik und Grenzgebiete.* Springer Verlag, Berlin 1942 Section V4, p. 34.

THEOREM 1.17.* *Let $R_+(z)$ be an R function all the poles of which are positive, then*

$$R_0(z) = \frac{1}{z}\left[R_+(z^2) - R_+(0)\right] \tag{18}$$

is an odd R function. If $R_+(z) > 0$ for all negative z

$$R_0(z) = z\,R_+(z^2) \tag{18a}$$

is also an odd R function.

The first part follows from the expansion (2) for R_+ in which Z_μ can be replaced by $\zeta_\mu^2 = Z_\mu$ with positive ζ_μ. The identity

$$\frac{1}{z}\left(\frac{1}{\zeta_\mu^2 - z^2} - \frac{1}{\zeta_\mu^2}\right) = \frac{1}{2\zeta_\mu^2}\left(\frac{1}{\zeta_\mu - z} + \frac{1}{-\zeta_\mu - z}\right)$$

together with Theorem 1.9 then directly gives (18). In the second case, $-1/R_+(z)$ satisfies the conditions of the first case for R_+. Hence $-(1/z)(1/R_+(z^2) - 1/R_+(0))$ is an R function and remains one if $-R_+(0)/z$ is added to it. Hence (18a) is also an R function.

THEOREM 1.18.* *Conversely, if $R_0(z)$ is an odd R function which is regular at $z=0$, then $\sqrt{z}R_0(\sqrt{z})$ is also an R function of z.*

2. The elementary transcendentals. Let us consider the class t of R functions which (a) are periodic with period 1, (b) have only one singularity for $-\frac{1}{2} < z \leqq \frac{1}{2}$, (c) converge to i as z goes to $i\infty$. The existence of R functions satisfying these conditions will be apparent when the expansion (2) will be found for them. We now proceed to find this expansion.

Let us denote the position of the singularity in the $-\frac{1}{2} < z \leqq \frac{1}{2}$ interval by $\frac{1}{2} - x_0$, its residue by $-c < 0$. Because of the periodicity condition, there will be similar singularities at all points $n - \frac{1}{2} - x_0$, the n being any integer, and only at these points. Hence, if R functions of the class t exist, their expansion by (2) must be of the form

$$t(z) = \alpha z + \beta + c\sum\left(\frac{1}{n - \frac{1}{2} - x_0 - z} - \frac{1}{n - \frac{1}{2} - x_0}\right), \tag{19}$$

where n assumes every integral value from $-\infty$ to ∞.

It is worth while to remark that because of the absolute convergence of the series (19) one can unite the terms $\nu = n$ and $\nu = 1 - n$; ν then will run over all positive integers. Then

$$t(z) = \alpha z + \beta + \sum\left(\frac{c}{\nu - \frac{1}{2} - x_0 - z} - \frac{c}{\nu - \frac{1}{2} + x_0 + z}\right)$$

$$-\sum\left(\frac{c}{\nu - \frac{1}{2} - x_0} - \frac{c}{\nu - \frac{1}{2} + x_0}\right) \tag{19a}$$

$$= \alpha z + \beta' + \sum\left(\frac{c}{\nu - \frac{1}{2} - x_0 - z} - \frac{c}{\nu - \frac{1}{2} + x_0 + z}\right),$$

since the sums of both brackets of the first equation converge absolutely and the second of these is independent of z.

In order to impose the boundary condition $t(i\infty) = i$, we calculate the sum in (19a) for $z = x + iy$

$$\sum \frac{c}{\nu + \frac{1}{2} - x_0 - z} - \frac{c}{\nu + \frac{1}{2} + x_0 + z}$$

$$= \sum \frac{2c(x_0 + x + iy)}{(\nu + \frac{1}{2})^2 - (x_0 + x + iy)^2} \tag{19b}$$

$$= \frac{2ic}{y} \sum \frac{1 - i(x_0 + x)/y}{((\nu + \frac{1}{2})/y)^2 + (1 - i(x_0 + x)/y)^2}.$$

The factor by which the νth term of (19b) differs from $[1 + ((\nu + \frac{1}{2})/y)^2]^{-1}$ converges to 1 as y goes to infinity. Hence the sum (19b) converges to the sum of these expressions and, by the definition of the integral, (19b) converges to

$$\frac{2ic}{y} \sum \frac{1}{1 + (\nu + \frac{1}{2})^2/y^2} \to 2ic \int_0^\infty \frac{d\xi}{1 + \xi^2} = i\pi c \tag{19c}$$

if we denote the last integral by $\frac{1}{2}\pi$.

Since t converges to a fixed value i as $z \to i\infty$ and since this is true also of the last term in (19a), $\alpha z + \beta'$ must also converge to a fixed number. This is possible only if $\alpha = 0$ and since β' is real, it also must vanish. It then follows that $\pi c = 1$.

THEOREM 2.1. *The R functions of the class t have the form $t(x_0 + z)$ with real x_0 where*

$$t(z) = \frac{1}{\pi} \sum\left(\frac{1}{\nu - \frac{1}{2} - z} - \frac{1}{\nu - \frac{1}{2} + z}\right) = \frac{1}{\pi} \sum\left(\frac{1}{n - \frac{1}{2} - z} - \frac{1}{n - \frac{1}{2}}\right) \tag{20}$$

where ν runs over all positive integers, n over all integers.

It is easy to convince oneself that the t given by (20) fulfills the three conditions of the definition of the class. This is evident for (b) and the above calculation verifies (c). To verify (a) let N be so large that the first N terms of the series (20) represent both $t(z)$ and $t(z+1)$ with a smaller error than ϵ, then (20) shows that $t(z) - t(z+1)$ is within 2ϵ equal to

$$\frac{1}{N-\frac{1}{2}-z}+\frac{1}{N+3/2+z}$$

and this goes to 0 as N increases.

The function $t(z)$ has singularities at $\pm\frac{1}{2}$ and assumes in the interval $(-\frac{1}{2},\frac{1}{2})$ every real value. Let us consider therefore the R functions

$$\frac{\alpha t(z)+\beta}{-\beta t(z)+\alpha} \tag{21}$$

with real α, β (*cf.* Theorem 1.7). The function (21) is clearly periodic, has only one singularity for $-\frac{1}{2}<z\leqq\frac{1}{2}$, namely at the point where $t(z)=\alpha/\beta$, and one can verify directly that its value is i for $z=i\infty$. It therefore is in the class t and is equal to $t(x_0+z)$ with a definite x_0. In order to determine this x_0, one may insert $z=0$ in (21), because of $t(0)=0$, its value then is $\beta/\alpha=t(x_0)$. Introducing this into (21) gives the addition theorem

$$t(x_0+z)=\frac{t(z)+t(x_0)}{1-t(x_0)t(z)}\cdot \tag{21a}$$

The above derivation proves (21a) only for $-\frac{1}{2}<x_0\leqq\frac{1}{2}$. However, since both sides of (21a) are analytic functions of x_0, it must hold also in general.

The function $t(z)$ is odd, so vanishes at $z=0$. In order to calculate the value of its derivative t' at $z=0$, we note from (21a) with $x_0=1/2$ that

$$t(\tfrac{1}{2}+z)=-1/t(z). \tag{22}$$

The residue of the left side of (22) at $z=0$ is $-1/\pi$, hence the derivative of $t(z)$ at $z=0$ is

$$t'(0)=\pi. \tag{22a}$$

From the odd and real nature of t it follows that it is imaginary on the imaginary axis and, because of (22), it is also imaginary if z has the form $\frac{1}{2}+iy$.

The poles of $-1/t(z)$ are all the integers. From this and its odd nature follows the expansion

$$-\frac{1}{t(z)}=-\frac{1}{\pi z}+\frac{1}{\pi}\sum\left(\frac{1}{\nu-z}-\frac{1}{\nu+z}\right). \tag{23}$$

Let us calculate, finally, the addition formula for an arbitrary fractional linear transform of t

$$e(z)=\frac{a_1t(z)+b_1}{a_2t(z)+b_2}\cdot$$

For $e(x_0+z)$ one obtains a fractional linear function of $e(z)$ with the matrix

$$\left\| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix} \right\| \left\| \begin{matrix} 1 & t(x_0) \\ -t(x_0) & 1 \end{matrix} \right\| \left\| \begin{matrix} a_1 & b_1 \\ a_2 & b_2 \end{matrix} \right\|^{-1}.$$

This will be a diagonal matrix if $a_1 = -a_2 = i$; $b_1 = b_2 = 1$. Hence the addition theorem for

$$e(z) = \frac{it(z) + 1}{-it(z) + 1} \tag{24}$$

has the form $e(x_0+z) = f(x_0)e(z)$. Writing $z=0$ herein one finds $f(x_0) = e(x_0)$ and

$$e(z + z') = e(z)e(z'). \tag{24a}$$

Evidently, $e(z)$ is periodic with period 1. From (24) one finds $e(0)=1$, $e(i\infty)=0$, $e(-i\infty)=\infty$, that $|e(z)|=1$ for real z and from the remark after (22a) that it is real for imaginary z: it assumes every positive value as z drops from $i\infty$ to $-i\infty$.

The value of the derivative e' of e at $z=0$ can be obtained from (24) and (22a): $e'(0) = 2it'(0) = 2\pi i$. The differential equation

$$e'(z) = 2\pi i e(z) \tag{25}$$

then easily follows from (24a). It shows that $e(z)$ can have no poles and is an entire function. The various formulae for the trigonometric and hyperbolic functions follow either from (21a) or (24a) by means of

$$\begin{aligned} t(z) &= \tan \pi z, \\ e(z) &= e^{2\pi i z}. \end{aligned} \tag{6}$$

The infinite product representation for the tangent is a direct consequence of Theorem 1.12:

$$\begin{aligned} t(z) &= \frac{\pi}{2} \frac{z}{z + \frac{1}{2}} \prod \frac{1 - z/\nu}{1 - z/(\nu - \frac{1}{2})} \prod \frac{1 + z/\nu}{1 + z/(\nu + \frac{1}{2})} \\ &= \frac{\cdots (1 - z/2)(1 - z)z(1 + z)(1 + z/2) \cdots}{\cdots (1 - 2z/3)(1 - 2z)(1 + 2z)(1 + 2z/3)(1 + 2z/5) \cdots}. \end{aligned} \tag{27}$$

Calculating $t'(z)/t(z)$, one obtains the partial fraction representation of $s(2z)$ and, hence, of $s(z)$ and $c(z)$. The most obvious way to obtain the infinite product representation for $c(z) = \cos \pi z$ is to show that it satisfies the differential equation $c'(z)/c(z) = -\pi t(z)$. The infinite product for the sine can be deduced either in the same fashion or by multiplying the infinite products for $c(z)$ and $t(z)$.

It is worth while yet to point out that every complex number $x+iy$ can be expressed in the form $re(\phi)$ with real r and ϕ. Set $r = (x^2+y^2)^{1/2}$ and $t(\phi) = (r-x)/y = y/(r+x)$; there is one and only one ϕ between $-\frac{1}{2}$ and $\frac{1}{2}$ which satisfies this last equation. This then shows that $\pi\phi$ is the angle in the Gauss plane. The remark after (24a) shows, furthermore, that there is an imaginary

number $i\rho$ such that $r=e(i\rho)$ if r is positive. Hence, unless $x+iy=0$, there is a complex $\phi+i\rho$ such that $e(\phi+i\rho)=x+iy$. This defines the logarithm; its properties, including its derivative, follow from the properties of the exponential function.

One can also easily see that the π defined in (19c) is the ratio of the circumference of a circle to its diameter. For this purpose one extends the integral from $-\infty$ to ∞ and deforms the path of integration to a small circle about i; the integrand $dz/(z-i)(z+i)$ is easily seen to approach ds/r as the radius of the circle contracts.

NOTE ON A PAPER OF BAGCHI AND CHATTERJEE

L. CARLITZ, Duke University

In a recent paper [1], Bagchi and Chatterjee showed that if $f(u)$ is analytic except for poles in the finite plane and satisfies

$$f(u+v)-f(u-v)=-\frac{f'(u)f'(v)}{(f(u)-f(v))^2},$$

then $f(u)=\wp(u)+C$, where $\wp(u)$ is the Weierstrass $\wp$-function. In the present note we obtain a somewhat similar result for the equation

$$f(u+v)-f(u-v)=\frac{2f'(u)f(v)}{1-k^2f^2(u)f^2(v)}, \tag{1}$$

which is satisfied by the Jacobi elliptic function $sn\ u$.

It is clear first by taking $u\to v$, that $f(0)$ is finite, and by taking $v\to 0$ it is seen that either $f(u)\equiv$constant or $f(0)=0$. Moreover if we divide both sides of (1) by $2v$ and let $v\to 0$ we infer $f'(0)=1$. Then for $u=0$, (1) becomes $f(v)-f(-v)=2f(v)$ so that $f(v)$ is an odd function. If we next interchange u and v, (1) becomes

$$f(u+v)+f(u-v)=\frac{2f(u)f'(v)}{1-k^2f^2(u)f^2(v)}. \tag{2}$$

Comparison of (1) and (2) yields the addition formula

$$f(u+v)=\frac{f(u)f'(v)+f'(u)f(v)}{1-k^2f^2(u)f^2(v)}.$$

Now following [1], put

$$f(u+v)+f(u-v)=2\sum_{r=0}^{\infty}\frac{v^{2r}}{(2r)!}f^{(2r)}(u);$$

and let

$$f(v) = v + \frac{1}{3!} av^3 + \cdots. \tag{3}$$

Substituting in (2) we get

$$f(u) + \tfrac{1}{2}v^2 f''(u) + \cdots = \frac{f(u)(1 + \tfrac{1}{2}av^2 + \cdots)}{1 - k^2 f^2(u)(v^2 + \cdots)}.$$

Comparing coefficients of v^2 in both members we get

$$\tfrac{1}{2}f''(u) = k^2 f^3(u) + \tfrac{1}{2}af(u).$$

Multiplying by $f'(u)$ and integrating, we have

$$f'^2(u) = k^2 f^4(u) + af^2(u) + C; \tag{4}$$

since $f'(0) = 1$, it follows that $C = 1$.

The coefficient a occurring in (3) is arbitrary. However by means of a linear transformation, (4) can be reduced to the familiar Legendre form (see, for example [2, Ch. 22]). Indeed if we put $g(x) = \lambda f(x/\lambda)$, $x = \lambda u$, then (4) becomes

$$g'^2(x) = \frac{k^2}{\lambda^4} g^4(x) + \frac{a}{\lambda} g^2(x) + 1. \tag{5}$$

Now if we choose λ so that

$$\frac{k^2}{\lambda^4} + \frac{a}{\lambda^2} + 1 = 0, \tag{6}$$

then (5) is in normal form and it is evident that

$$g(x) = sn(x, k^2/\lambda^4).$$

If we prefer we can think of λ as arbitrary and then a is determined by (6).

Thus we have proved the following

THEOREM. *If $f(u)$ is analytic except for poles in the finite plane and satisfies* (1), *then either $f(u) \equiv$ constant or*

$$f(u) = \frac{1}{\lambda} sn\,(\lambda u, k^2/\lambda^4),$$

where λ is arbitrary.

References

1. Hari das Bagchi and Phatik Chand Chatterjee, Note on a second functional equation connected with the function $\wp(z)$, this MONTHLY, vol. 59 (1952) pp. 91–92.

2. E. T. Whittaker and G. N. Watson, Modern Analysis, 4th edition, Cambridge, 1927.

A NOTE ON GAMBLING SYSTEMS AND BIRTH STATISTICS

HERBERT ROBBINS, University of North Carolina

Let X_n denote the gain of a gambler at the nth of a series of games. We assume that the random variables X_n are independent and that

$$EX_n = 0, \qquad E\,|X_n| \leqq C < \infty \qquad (n = 1, 2, \cdots). \tag{1}$$

We also suppose that the gambler's opponent has unlimited capital and is willing to play indefinitely, and that the gambler has the option of ending the series of games at any time. He uses a "system" to decide when to do this, defined by a sequence of functions $f_n(x_1, \cdots, x_n)$, $(n=1, 2, \cdots)$, such that

$$f_n(x_1, \cdots, x_n) = 1 \text{ or } 0, \tag{2}$$

$$\text{if } f_n(x_1, \cdots, x_n) = 0 \text{ then for every } m > 0, \tag{3}$$

$$f_{n+m}(x_1, \cdots, x_n, x_{n+1}, \cdots, x_{n+m}) = 0,$$

and plays or does not play the $(n+1)$th game according as $f_n(X_1, \cdots, X_n)=1$ or 0. By (3) the series of games ends with the first n such that $f_n(X_1, \cdots, X_n) = 0$. We assume that the first game is played always, and set $f_0 \equiv 1$. The total gain of a gambler using this system is the random variable

$$W = \sum_{n=0}^{\infty} f_n(X_1, \cdots, X_n) \cdot X_{n+1}, \tag{4}$$

assuming that this series converges with probability 1, while the duration of play is the random variable

$$D = \sum_{n=0}^{\infty} f_n(X_1, \cdots, X_n). \tag{5}$$

THEOREM. *If $ED < \infty$ then $EW = 0$.*

Proof. If $ED < \infty$ then by (1),

$$\begin{aligned} \sum_{n=0}^{\infty} E\,|f_n(X_1, \cdots, X_n) \cdot X_{n+1}| &= \sum_{n=0}^{\infty} Ef_n(X_1, \cdots, X_n) \cdot E\,|X_{n+1}| \\ &\leqq C \cdot \sum_{n=0}^{\infty} Ef_n(X_1, \cdots, X_n) = C \cdot ED < \infty. \end{aligned} \tag{6}$$

It follows [1, page 34] that the series (4) converges with probability 1 and that

$$EW = \sum_{n=0}^{\infty} Ef_n(X_1, \cdots, X_n) \cdot EX_{n+1} = 0. \tag{7}$$

If the system is such that $ED = \infty$ then it may happen that $EW \neq 0$. For example, let $\Pr(X_n = -1) = \Pr(X_n = 1) = \frac{1}{2}$ for all n and let $f_n(x_1, \cdots, x_n) = 0$ if and only if for some integer p, $1 \leqq p \leqq n$, the relation $x_1 + \cdots + x_p = a$ holds,

where a is a fixed integer. The series of games stops as soon as the total gain of the gambler equals a. It is well known from the theory of random walk [2] that the probability is 1 that the equation $X_1 + \cdots + X_n = a$ will hold for some n. It follows that Pr $(W=a)=1$ and therefore $EW=a$. Of course, from this fact and the theorem just proved it follows that $ED=\infty$ if $a \neq 0$, which is also known. (In fact $ED=\infty$ even if $a=0$, and this follows from the fact that $ED=\infty$ for $a=\pm 1$.)

The foregoing considerations were suggested by an attempt [3] to explain an observed preponderance of male children in a certain group of families by the hypothesis that whereas the probability of a male birth may be approximately 1/2, male children are regarded as more desirable, so that parents are more likely to stop having children when a male child is born. Writing -1 for a female child and 1 for a male child, the excess W of male over female births for a given family will have the form (4), where again Pr $(X_n=-1)=\Pr(X_n=1)=\frac{1}{2}$. In this case, however, we might have $f_n(x_1, \cdots, x_n)=0$ if and only if at least one of the numbers $x_1, \cdots, x_n=1$, which would amount to stopping as soon as the first male child is born. For this particular system $ED=\sum_{n=1}^{\infty} n/2^n=2$ and hence, as for any system for which $ED<\infty$, $EW=0$. If the system described in the previous paragraph could be used for some $a \geqq 1$, then $EW=a>0$, but there are practical difficulties consequent on the fact that $ED=\infty$ for such a system.

References

1. A. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrechnung, Berlin, 1933.

2. G. Pólya, Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Strassennetz, Mathematische Annalen 84, 1921, pp. 149–160.

3. S. R. Winston, Birth control and the sex-ratio at birth, American Journal of Sociology 38, 1932, pp. 225–231.

MATHEMATICAL NOTES

EDITED BY F. A. FICKEN, University of Tennessee

Material for this department should be sent to F. A. Ficken, University of Tennessee, Knoxville 16, Tenn.

DRESHER'S INEQUALITY

J. M. DANSKIN, The Rand Corporation

Melvin Dresher has proved by an ingenious method, using moment-space theory,* the inequality which I shall call Dresher's inequality:

If $p \geqq 1 \geqq r \geqq 0$, $f(x) \geqq 0$, $g(x) \geqq 0$, *and* $\phi(x)$ *is a distribution function*§ *then*†

$$\left[\frac{\int [f(x)+g(x)]^p d\phi(x)}{\int [f(x)+g(x)]^r d\phi(x)}\right]^{1/(p-r)} \leqq \left[\frac{\int [f(x)]^p d\phi(x)}{\int [f(x)]^r d\phi(x)}\right]^{1/(p-r)} + \left[\frac{\int [g(x)]^p d\phi(x)}{\int [g(x)]^r d\phi(x)}\right]^{1/(p-r)}. \tag{1}$$

It is the purpose of this note to give an elementary proof of this inequality, based on the Minkowski inequality and an inequality due to Radon.‡ The Radon inequality is gotten easily by transforming the Hölder inequality;§§ thus Dresher's inequality is seen to be a mélange of the Hölder and Minkowski inequalities. It is appropriate to note here that Dresher's inequality is a generalization of an inequality due to Beckenbach.††

The Radon inequality is as follows:

If $\lambda > 0$, $\alpha_i \geqq 0$, $\beta_i > 0$, $i = 1, \cdots, n$, *then*

$$\frac{(\sum \alpha_i)^{\lambda+1}}{(\sum \beta_i)^{\lambda}} \leqq \sum \frac{\alpha_i^{\lambda+1}}{\beta_i^{\lambda}}. \tag{2}$$

In using this to prove Dresher's inequality, we shall need only two terms. The inequality is then stated as follows:

If $\lambda > 0$, $\alpha_1 \geqq 0$, $\alpha_2 \geqq 0$, $\beta_1 > 0$, *and* $\beta_2 > 0$, *then*

$$\frac{(\alpha_1+\alpha_2)^{\lambda+1}}{(\beta_1+\beta_2)^{\lambda}} \leqq \frac{\alpha_1^{\lambda+1}}{\beta_1^{\lambda}} + \frac{\alpha_2^{\lambda+1}}{\beta_2^{\lambda}}. \tag{3}$$

---

* Forthcoming.

§ In particular, we may take $\phi(x) \equiv x$, and read each $d\phi(x)$ as dx.

† We assume that neither f nor g vanishes identically.

‡ J. Radon, Über die absolut additiven Mengenfunktionen, Wiener Sitzungsber. (II a), vol. 122, 1913, p. 1351. See also Hardy, Littlewood, and Polya, Inequalities, Cambridge, 1934, p. 61, problem 65, and C. B. Morrey, A class of representations of manifolds, Amer. J. Math. vol. 55, 1933, p. 692.

§§ Hardy, Littlewood, Polya, loc. cit.

†† E. F. Beckenbach, A class of mean value functions, this MONTHLY vol. 57, 1950, pp. 1–6.

We make in (3) the substitutions

$$\alpha_1 = \left[\int f^p d\phi\right]^{1/p} \qquad \beta_1 = \left[\int f^r d\phi\right]^{1/r}$$

$$\lambda = \frac{r}{p-r}.$$

$$\alpha_2 = \left[\int g^p d\phi\right]^{1/p} \qquad \beta_2 = \left[\int g^r d\phi\right]^{1/r}$$

Here, for the moment, we have assumed $p>r>0$.

Then the right side of (3) becomes the right side of (1), and the left side of (3) becomes

$$\frac{\left\{\left[\int f^p d\phi\right]^{1/p} + \left[\int g^p d\phi\right]^{1/p}\right\}^{p/(p-r)}}{\left\{\left[\int f^r d\phi\right]^{1/r} + \left[\int g^r d\phi\right]^{1/r}\right\}^{r/(p-r)}}. \tag{4}$$

But by Minkowski's inequality with $p \geqq 1$ and $0<r\leqq 1$ respectively, both

$$\left\{\left[\int f^p d\phi\right]^{1/p} + \left[\int g^p d\phi\right]^{1/p}\right\}^{p} \geqq \int (f+g)^p d\phi \tag{5}$$

and

$$\left\{\left[\int f^r d\phi\right]^{1/r} + \left[\int g^r d\phi\right]^{1/r}\right\}^{r} \leqq \int (f+g)^r d\phi, \tag{6}$$

so that the quantity (4) is not less than the expression on the left side of (1), and Dresher's inequality is proved for $p \geqq 1$ and $p>r>0$. Extension to the remaining cases is trivial, and thus the proof is complete.

THE COMPOSITION OF ROTATIONS IN EUCLIDEAN THREE-SPACE

T. G. Room, University of Sydney

In the usual treatment by vectors of rotations about an axis in Euclidean space of three dimensions, it is proved that:

THEOREM 1. *If $\mathcal{R}_{\mathbf{u},\delta}$ represents the rotation about an axis through the origin parallel to the unit vector* $\mathbf{u}$, *through an angle* 2δ, *and* $\mathbf{x}$ *is the position vector of a general point, then*

$$\mathcal{R}_{\mathbf{u},\delta}\mathbf{x} = \mathbf{x} + 2[\mathbf{u} \times (\mathbf{u} \times \mathbf{x}) \sin\delta + \mathbf{u} \times \mathbf{x} \cos\delta] \sin\delta.$$

If $\delta \neq (2r+1)\pi/2$, the rotation is determined by the single vector $\boldsymbol{\lambda} = \mathbf{u} \tan\delta$, and, if we write $\mathcal{R}_{\boldsymbol{\lambda}} = \mathcal{R}_{\mathbf{u},\delta}$, Theorem 1 becomes:

THEOREM 2. $\mathcal{R}_{\boldsymbol{\lambda}}\mathbf{x} = \mathbf{x} + 2[\boldsymbol{\lambda} \times (\boldsymbol{\lambda} \times \mathbf{x}) + \boldsymbol{\lambda} \times \mathbf{x}]/(1+\boldsymbol{\lambda}^2)$.

The resultant of two rotations about concurrent axes is a rotation about an axis concurrent with them, so that we may write

$$\mathcal{R}_{\lambda_2}\mathcal{R}_{\lambda_1} = \mathcal{R}_{\lambda},$$

and this note is concerned principally with the vector and matrix forms of the expression for λ in terms of λ_1 and λ_2.

For the vector form I prove:

THEOREM 3. *If* $\mathcal{R}_{\lambda_2}\mathcal{R}_{\lambda_1}=\mathcal{R}_{\lambda}$, *then*

$$\lambda = (\lambda_1 + \lambda_2 - \lambda_1 \times \lambda_2)/(1 - \lambda_1 \cdot \lambda_2).$$

If the denominator vanishes, then the resultant rotation is through an angle π about an axis parallel to $\lambda_1+\lambda_2-\lambda_1\times\lambda_2$. The exceptional case of a rotation through π should be allowed for in all theorems which follow, but since it may always be dealt with as a limiting case, I do not make further reference to it.

Theorem 3 is a consequence of the theorems numbered 4 and 6 below. Theorem 4 might be called *the fundamental vector relations for a spherical triangle*, namely:

THEOREM 4. *ABC is a right-handed spherical triangle on a unit sphere center O. If the sensed segments OA, OB, OC represent the unit vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, *and* $\alpha=\mathbf{u}\tan A$, $\beta=\mathbf{v}\tan B$, $\gamma=\mathbf{w}\tan C$, *then*

$$\begin{aligned}\alpha + \beta + \gamma &= \beta\cdot\gamma\alpha - \beta\times\gamma\\ &= \gamma\cdot\alpha\beta - \gamma\times\alpha\\ &= \alpha\cdot\beta\gamma - \alpha\times\beta.\end{aligned}$$

Let $\mathbf{u}'$, $\mathbf{v}'$, $\mathbf{w}'$ be the position vectors of the vertices of the polar triangle of ABC, so that we have the four sets, each of three relations:

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} &= \cos a,\\ \mathbf{v}\times\mathbf{w} &= \mathbf{u}'\sin a,\\ \mathbf{v}'\cdot\mathbf{w}' &= -\cos A,\\ \mathbf{v}'\times\mathbf{w}' &= \mathbf{u}\sin A.\end{aligned}$$

If we write $D_{\mathbf{x},\mathbf{y},\mathbf{z}}$ for $\mathbf{x}\cdot\mathbf{y}\times\mathbf{z}$, the identical relation connecting the four vectors $\mathbf{u}'$, $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is

$$D_{\mathbf{u},\mathbf{v},\mathbf{w}}\mathbf{u}' = D_{\mathbf{u}',\mathbf{v},\mathbf{w}}\mathbf{u} - D_{\mathbf{u}',\mathbf{u},\mathbf{w}}\mathbf{v} + D_{\mathbf{u}',\mathbf{u},\mathbf{v}}\mathbf{w},$$

where $D_{\mathbf{u},\mathbf{v},\mathbf{w}}=\sin A\sin b\sin c=\sin B\sin c\sin a=\sin C\sin a\sin b$, $D_{\mathbf{u}',\mathbf{v},\mathbf{w}}=\mathbf{u}'\cdot\mathbf{u}'\sin a=\sin a$, $D_{\mathbf{u}',\mathbf{u},\mathbf{w}}=-\mathbf{u}'\cdot\mathbf{v}'\sin b=\cos C\sin b$, and $D_{\mathbf{u}',\mathbf{u},\mathbf{v}}=\mathbf{u}'\cdot\mathbf{w}'\sin c=-\cos B\sin c$; so that

$$\mathbf{v}\times\mathbf{w}\sin B\sin c = \mathbf{u}\sin a - \mathbf{v}\cos C\sin b - \mathbf{w}\cos B\sin c,$$

i.e.,

$$\beta \times \gamma \cot B \cot C \sin B \sin c$$
$$= \alpha \cot A \sin a - \beta \cot B \cos C \sin b - \gamma \cot C \cos B \sin c,$$

whence the result follows, since

$$\frac{\cos A}{\cos B \cos C} = \cos a \tan B \tan C - 1 = \beta \cdot \gamma - 1.$$

A consequence of theorem 4 is

THEOREM 5. *If H is the orthocenter of the spherical triangle, then OH is parallel to $\alpha+\beta+\gamma$.*

For: $\alpha+\beta+\gamma$ is parallel to the plane containing α and $\beta\times\gamma$, that is, to that containing $\mathbf{u}$, and $\mathbf{u}'$, and this is the plane of the great circle through A and the pole of the great circle BC. Thus $\alpha+\beta+\gamma$ is parallel to the line of intersection of the three planes perpendicular to the side-arcs of the triangle and through the opposite vertices.

The other theorem required in the proof of theorem 3 is Donkin's theorem, namely

THEOREM 6. *ABC is a spherical triangle on a sphere center O. The resultant of a rotation through an angle $-2A$ about OA, followed by $-2B$ about OB, followed by $-2C$ about OC is the identity transformation.*

Denote by $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$ the reflection transformations in the three planes BOC, COA, AOB. Then the rotation through $-2A$ about OA is the resultant of $\mathfrak{S}_2$ followed by $\mathfrak{S}_3$, and similarly for the others, so that the resultant of the three rotations is

$$(\mathfrak{S}_3\mathfrak{S}_2)(\mathfrak{S}_1\mathfrak{S}_3)(\mathfrak{S}_3\mathfrak{S}_2) = \mathfrak{S}_2\mathfrak{S}_1^2\mathfrak{S}_3^2\mathfrak{S}_2 = \mathfrak{I}.$$

THEOREM 7. *If* $\mathbf{f}(\alpha, \beta) = (\alpha+\beta-\alpha\times\beta)/(1-\alpha\cdot\beta)$ *then* $\mathbf{f}[\mathbf{f}(\alpha, \beta), \gamma] = \mathbf{f}[\alpha, \mathbf{f}(\beta, \gamma)]$,

a result which may be readily verified by direct evaluation.

If an orthogonal 3-by-3 matrix represents a rotation (not through an odd multiple of π) about a line through the origin in Cartesian three-space, then it may be expressed in the standard Cayley form $(\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}$, where $\mathbf{A}$ is skew symmetric, say

$$\mathbf{A} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

It is to be proved that

THEOREM 8. *The rotation corresponding to* $\mathbf{A}$ *is about an axis parallel to the vector* (a_1, a_2, a_3) *through an angle* 2δ *given by* $\tan \delta = \sqrt{(a_1^2+a_2^2+a_3^2)}$.

In other words, (a_1, a_2, a_3) is precisely the vector $\mathfrak{a}$ used in Theorem 2. For convenience write $\mathbf{a}$ for $\mathfrak{a}$, and $t=\sqrt{(a_1^2+a_2^2+a_3^2)}$. In vector form, the transform of the point of position vector $\mathbf{x}$ by the rotation represented by $\mathbf{a}$ is

$$\bar{\mathbf{x}} = \mathbf{x} + 2[\mathbf{a} \times (\mathbf{a} \times \mathbf{x}) + (\mathbf{a} \times \mathbf{x})]/(1 + t^2).$$

Now $\mathbf{a}\times\mathbf{x} = \mathbf{A}\mathbf{x}$ (treating both as column-vectors) so that in matrix form this equation may be written

$$\bar{\mathbf{x}} = \left[\mathbf{I} + \frac{2}{1 + t^2}(\mathbf{A}^2 + \mathbf{A})\right]\mathbf{x};$$

also

$$\mathbf{A}^2 = \mathbf{a}\mathbf{a}^T - \mathbf{a}^T\mathbf{a}\mathbf{I},$$

where $\mathbf{a}\mathbf{a}^T$ is the dyadic product of the column vector $\mathbf{a}$ and its transpose, the row vector $\mathbf{a}^T$, *i.e.*, the 3-by-3 matrix with elements a_ia_j, and $\mathbf{a}^T\mathbf{a}$ is the scalar, or inner, product, *i.e.*, $\mathbf{a}^T\mathbf{a} = t^2$. To prove that this form is identical with the Cayley form, we have to prove that

$$\mathbf{I} + \mathbf{A} = \frac{1}{1 + t^2}[(1 + t^2)\mathbf{I} + 2(\mathbf{a}\mathbf{a}^T - t^2\mathbf{I} + \mathbf{A})](\mathbf{I} - \mathbf{A}).$$

The identity of the two expressions follows from the relations $\mathbf{a}^T\mathbf{A} = \mathbf{o}$, and $\mathbf{A}^2 = \mathbf{a}\mathbf{a}^T - t^2\mathbf{I}$.

Corresponding to Theorem 3 we have

THEOREM 9. *If* $\mathbf{L}$ *and* $\mathbf{M}$ *are two orthogonal 3-by-3 matrices, given by* $\mathbf{L} = (\mathbf{I}+\mathbf{A})(\mathbf{I}-\mathbf{A})^{-1}$, $\mathbf{M} = (\mathbf{I}+\mathbf{B})(\mathbf{I}-\mathbf{B})^{-1}$, *and their product* $\mathbf{N} = \mathbf{M}\mathbf{L}$, *is given by* $\mathbf{N} = (\mathbf{I}+\mathbf{C})(\mathbf{I}-\mathbf{C})^{-1}$, *then* $\mathbf{C} = (\mathbf{A}+\mathbf{B}+\mathbf{A}\mathbf{B}-\mathbf{B}\mathbf{A})/[\det(\mathbf{I}+\mathbf{A}\mathbf{B})]^{1/2}$.

This theorem may be proved as a relation among skew-symmetric 3-by-3 matrices, but only by making use of their associated vectors, that is, not in such a way as would make possible any application of this result to matrices of other sizes. Posed in this way it constitutes quite a nice "Elementary Problem." More easily we may prove it directly from Theorem 3, by using the following relations:

(i) as a matrix, $-\mathbf{a}\times\mathbf{b}$ may be expressed as $\mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T$,

(ii) $\mathbf{A}\mathbf{B} = \mathbf{b}\mathbf{a}^T - \mathbf{a}^T\mathbf{b}\mathbf{I}$, so that

(iii) $\mathbf{b}\mathbf{a}^T - \mathbf{a}\mathbf{b}^T = \mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}$,

(iv) $\det(\mathbf{I}+\mathbf{A}\mathbf{B}) = \det[(1-\mathbf{a}^T\mathbf{b})\mathbf{I}+\mathbf{b}\mathbf{a}^T] = (1-\mathbf{a}^T\mathbf{b})^3+(1-\mathbf{a}^T\mathbf{b})^2\mathbf{a}^T\mathbf{b} = (1-\mathbf{a}^T\mathbf{b})^2$,

(v) from Theorem 3, for the vectors associated with $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, we have

$$\mathbf{c} = (\mathbf{a} + \mathbf{b} - \mathbf{a} \times \mathbf{b})/(1 - \mathbf{a}\cdot\mathbf{b}).$$

At the referee's suggestion I add a note of how Theorems 3 and 8 appear at a slightly higher level of sophistication.

For Theorem 3 I use again $\mathfrak{\lambda}$ for the vector representing a rotation, taking it now to have rectangular components λ, μ, ν. The rotation may also be represented by the quaternion $q=(1/\sqrt{1+t^2})\,(1+\lambda i+\mu j+\nu k)$. It is well-known that: *The resultant of the two rotations represented by the two quaternions q_1, q_2 is represented by the product of the quaternions*, namely, by

$$q_1q_2=\frac{1-\lambda_1\lambda_2-\mu_1\mu_2-\nu_1\nu_2}{\sqrt{\{(1+t_1^2)(1+t_2^2)\}}}\left\{1+i\,\frac{\lambda_1+\lambda_2+\mu_1\nu_2-\mu_2\nu_1}{1-\lambda_1\lambda_2-\mu_1\mu_2-\nu_1\nu_2}+j\cdots+k\cdots\right\}.$$

The vector corresponding to this quaternion is that given in Theorem 3.

Theorem 8 is another aspect of the well-known theorem (in the notation of Theorem 8): *The orthogonal matrix corresponding to the rotation represented by the vector* **a** *is* exp $((2\delta/t)\mathrm{A})$. It is easily verified that

$$\exp\left(\frac{2\delta}{t}\,\mathrm{A}\right)=\mathrm{I}+\frac{1}{1+t^2}\,(\mathrm{A}+\mathrm{A}^2).$$

CLASSROOM NOTES

EDITED BY G. B. THOMAS, Massachusetts Institute of Technology

All material for this department should be sent to G. B. Thomas, Department of Mathematics, Massachusetts Institute of Technology, Cambridge 39, Mass.

A CONNECTION BETWEEN TAYLOR'S THEOREM AND LINEAR DIFFERENTIAL EQUATIONS

D. C. LEWIS, Johns Hopkins University

The following amusing elementary derivation and generalization of Taylor's theorem (with the integral form for the remainder) was accidentally encountered while teaching a calculus course in which, for the convenience of the engineering students, the introduction of Taylor's theorem was postponed until after they had had some elementary theory of differential equations.

We consider the general homogeneous linear differential equation of the nth order with coefficients $a_1(x), \cdots, a_n(x)$ continuous on a certain interval I of the x-axis,

$$y^{(n)}+a_1(x)y^{(n-1)}+\cdots+a_{n-1}(x)y'+a_n(x)y=0, \tag{1}$$

and suppose we consider the system of independent solutions $y=y_k(x,\ t)$, $k=0,\ 1,\ 2,\ \cdots,\ n-1$, determined by the conditions

$$(d^k/dx^k)y_k(x,\ t)\,\big|_{x=t}=1, \qquad (d^h/dx^h)y_k(x,\ t)\,\big|_{x=t}=0, \qquad h\neq k. \tag{2}$$

Here the parameter t may take on any value in I. Then the solution of the corresponding non-homogeneous equation,

$$Y^{(n)} + a_1(x)Y^{(n-1)} + \cdots + a_n(x)Y = F(x) \tag{3}$$

(where $F(x)$ is continuous on I), satisfying the initial conditions

$$Y^{(k)}(c) = b_k, \quad k = 0, 1, 2, \cdots, n-1, c \in I, \tag{4}$$

can be written in the form

$$Y(x) = \sum_{k=0}^{n-1} b_k y_k(x, c) + \int_c^x F(t) y_{n-1}(x, t)dt. \tag{5}$$

This essentially well known result can be verified *a posteriori*, by differentiation of (5) and substitution in (3), using, among the other elements in the definition of the $y_k(x, t)$, the fact that $y_{n-1}(x, t)$ satisfies (1) for every value of t. It is also possible to deduce (5) as the end result of the Lagrange method of variation of parameters.

But *every* function $f(x)$ of class C^n on I can be regarded as being a solution of (3) no matter how the coefficients $a_1(x), \cdots, a_n(x)$ may have been chosen, so long as we take

$$F(x) = f^{(n)}(x) + \sum_{k=1}^{n} a_k(x) f^{(n-k)}(x). \tag{6}$$

Hence, with this definition of $F(x)$, we find that every function $f(x)$ of class C^n on I satisfies the identity,

$$f(x) = \sum_{k=0}^{n-1} f^{(k)}(c) y_k(x, c) + \int_c^x F(t) y_{n-1}(x, t)dt, \tag{7}$$

which we of course deduce from (5) by taking appropriate values for the b's.

Formula (7) may be regarded as a generalization of Taylor's theorem. In fact, if $a_1 \equiv a_2 \equiv \cdots \equiv a_n \equiv 0$, we must have, from (6), $F(x) = f^{(n)}(x)$, and, from (1) and (2), we have

$$y_k(x, t) = (x - t)^k/k! \qquad k = 0, 1, \cdots, n-1.$$

Hence (7) becomes in this special case

$$f(x) = \sum_{k=0}^{n-1} f^{(k)}(c) \frac{(x-c)^k}{k!} + \frac{1}{(n-1)!} \int_c^x f^{(n)}(t)(x-t)^{n-1}dt,$$

which is Taylor's theorem with integral form of the remainder, usually proved directly by an inductive process employing integration by parts.

Obviously there are many other special cases of (7) which can be obtained by special choices of the a's.

INVERSE SQUARE ORBITS. A SIMPLE TREATMENT

D. E. RICHMOND, Williams College

The determination of the orbit of a body attracted to a center by an inverse square force is of great historical importance. Its rôle in astronomy and atomic physics makes the derivation of continuing interest. However, it is usually accessible to the student only after a fairly extensive knowledge of differential equations. There are available a number of vector treatments but they too require a considerable degree of mathematical maturity for their understanding.

This paper presents an extremely simple vector derivation of the equation of the orbit and of the relation between velocity and radial distance. It also includes a simple construction for the velocity. This treatment presupposes a knowledge of vector differentiation but uses neither scalar nor vector products. It is believed that the approach is well within the understanding of a college sophomore.

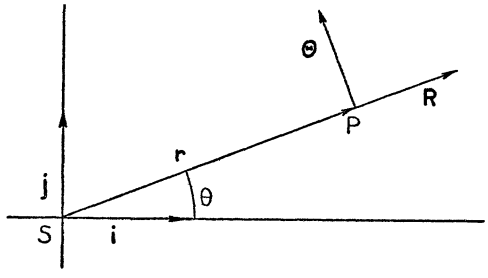


FIG. 1

Let a planet P be located at any time t by the vector $\mathbf{r}(t)$ from the sun S to P. Let $r(t)$ be the length of $\mathbf{r}(t)$ and let $\theta(t)$ specify its direction. Hereafter, the argument t will be omitted. In terms of the unit vectors $\mathbf{i}$ and $\mathbf{j}$,

$$\mathbf{r} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) = r\mathbf{R},$$

where $\mathbf{R}$ is a unit vector in the direction of $\mathbf{r}$.

The velocity $\mathbf{v}=d\mathbf{r}/dt$, hence

$$\mathbf{v} = r'(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + r(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j})\theta' = r'\mathbf{R} + r\theta'\mathbf{\Theta} \tag{1}$$

where $\mathbf{\Theta}$ is a unit vector perpendicular to $\mathbf{R}$ in the direction of increasing θ, and where primes denote differentiation with respect to t.

Implicit in (1) is the relation $d\mathbf{R}/dt=\theta'\mathbf{\Theta}$. A companion relation is $d\mathbf{\Theta}/dt = -\theta'\mathbf{R}$, which is immediately verified by differentiating $\mathbf{\Theta}=-\sin\theta\mathbf{i}+\cos\theta\mathbf{j}$.

One other preliminary is necessary. For central forces, $r^2\theta'=h$, a constant. The derivation is familiar and will be be omitted here.

The inverse square law is expressed by

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{-\mu}{r^2}\mathbf{R} \tag{2}$$

where μ is a constant. To integrate (2), it is desirable to express the right member as a time derivative. This suggests substituting $\mathbf{R} = -(1/\theta')d\mathbf{\Theta}/dt$. Using $r^2\theta' = h$,

$$\frac{d\mathbf{v}}{dt} = \frac{-\mu}{r^2}\left(-\frac{1}{\theta'}\frac{d\mathbf{\Theta}}{dt}\right) = \frac{\mu}{r^2\theta'}\frac{d\mathbf{\Theta}}{dt} = \frac{\mu}{h}\frac{d\mathbf{\Theta}}{dt}. \tag{3}$$

Then

$$\frac{h}{\mu}\mathbf{v} = \mathbf{\Theta} + \mathbf{C} \tag{4}$$

where $\mathbf{C}$ is a constant vector.

The value of $\mathbf{C}$ depends upon the initial conditions. Let us assume that at $t=0$, P is closest to S, that is, at perihelion, and that $\mathbf{i}$ is drawn in this direction. Since $r'=0$ at $t=0$, $\mathbf{v}_0$ is in the direction of $\mathbf{\Theta}_0=\mathbf{j}$. Hence $\mathbf{C}=\epsilon\mathbf{j}$ where ϵ is constant. Thus

$$\frac{h}{\mu}\mathbf{v} = \mathbf{\Theta} + \epsilon\mathbf{j}. \tag{5}$$

Figure 2 gives the corresponding geometrical construction.

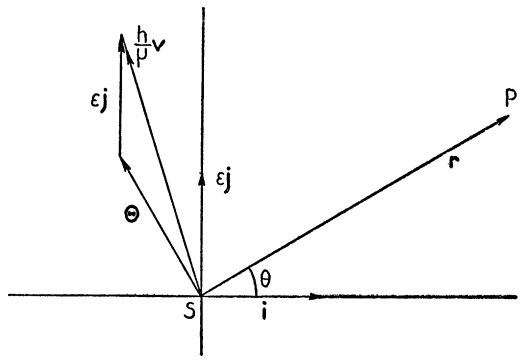


FIG. 2

Let us now take components of the vector $(h/\mu)\mathbf{v}$ in the direction of $\mathbf{\Theta}$. The result is $(h/\mu)r\theta'$ or, since $r^2\theta' = h$, $h^2/(\mu r)$. By (5), this must equal the projection of $\mathbf{\Theta}+\epsilon\mathbf{j}$ on $\mathbf{\Theta}$, namely, $1+\epsilon\cos\theta$. Hence

$$\frac{h^2}{\mu r} = 1 + \epsilon\cos\theta \tag{6}$$

or

$$r = \frac{p}{1 + \epsilon \cos \theta}, \qquad p = \frac{h^2}{\mu}, \tag{7}$$

which is the polar equation of a conic. We have assumed that r is a minimum at $\theta = 0$. This implies that $\epsilon > 0$, excluding the trivial case of the circular orbit ($\epsilon = 0$).

From the law of cosines applied to the triangle of sides Θ, $\epsilon\mathbf{j}$, and $(h/\mu)\mathbf{v}$ (figure 2),

$$\frac{h^2 v^2}{\mu^2} = 1 + \epsilon^2 + 2\epsilon \cos \theta.$$

Substituting for $\epsilon \cos \theta$ from (6)

$$v^2 = \frac{2\mu}{r} - \frac{\mu^2(1 - \epsilon^2)}{h^2}.$$

In the important case of an elliptical orbit ($\epsilon < 1$), the semi-major axis

$$a = \frac{p}{1 - \epsilon^2} = \frac{h^2}{\mu(1 - \epsilon^2)}.$$

Then

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right], \tag{8}$$

a widely used relation.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separated, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1041. *Proposed by SHE SLID and O PARISH, Massachusetts Institute of Technology*

Surely the days of anagrams are not dead. Here are four well known mathe-

maticians: (1) A DONUT SHIP, (2) SHE IS A NUT, (3) SEWER STAIRS, (4) FIRE ON SUB.

E 1042. *Proposed by P. L. Chessin, Cooper Union*

A sequence $\{a_n\}$, where a_0 and a_1 are given numbers, is defined recursively by $a_{n+2}=(a_{n+1}+a_n)/2$. Find $\lim_{n\to\infty} a_n$.

E 1043. *Proposed by O. J. Ramler, Catholic University, Washington, D.C.*

Prove that the sum of the ratios in which a point within a triangle divides the Cevians of this point is never less than 6 and that the product of the ratios is never less than 8.

E 1044. *Proposed by J. E. Wilkins, Jr., Nuclear Development Associates, Inc.*

Find

$$\prod_{r=1}^{m-1}\left(2\sin\frac{\pi r}{m}\right)^r.$$

E 1045. *Proposed by S. W. Golomb, Harvard University*

Let n^{ϕ} (pronounced "n-phi-torial") denote the product of all integers up to n which are prime to n. Then $n^{\phi}\equiv\pm 1 \pmod{n}$. Prove this, and determine when the value is $+1$ and when it is -1.

SOLUTIONS

Rectification of the Loop of the Folium of Descartes

E 1009 [1952, 181]. *Proposed by Arthur Danzl, St. John's University*

Rectify the loop of the folium of Descartes by means of elementary functions either finitely, if possible, or approximately.

I. *Solution by Julian Braun, Washington, D.C.* Parametric equations of the folium of Descartes are

$$x = 3at/(1+t^3), \qquad y = 3at^2/(1+t^3).$$

Then

$$s = \int_A^B (dx^2+dy^2)^{1/2} = 3a\int_0^\infty \frac{[(1-2t^3)^2+(2t-t^4)^2]^{1/2}}{(1+t^3)^2}\,dt$$

$$= 3a\int_0^\infty f(t)dt = 6a\int_0^1 f(t)dt.$$

We now construct the following table:

t	$f(t)$	t	$f(t)$	t	$f(t)$
0	1.00000000	9/24	1.04194530	17/24	0.65336168
1/24	1.00317665	10/24	1.02036581	18/24	0.59051585
2/24	1.01147315	11/24	0.98916542	19/24	0.53185302
3/24	1.02292811	12/24	0.94861100	20/24	0.47940652
4/24	1.03545699	13/24	0.89957859	21/24	0.43477191
5/24	1.04690976	14/24	0.84349983	22/24	0.39892511
6/24	1.05515547	15/24	0.78226362	23/24	0.37206317
7/24	1.05818861	16/24	0.71808575	24/24	0.35355339
8/24	1.05424911				

Applying Simpson's rule, Weddle's rule, and Newton's sixth degree integration formula to the data in the above table we obtain

$$s \doteq 4.9174888a, \quad 4.9174889a, \quad 4.9174887a$$

respectively. Hence we conclude that, to six decimal places,

$$s = 4.917489^{-}a.$$

II. *Remarks by O. E. Stanaitis, St. Olaf College.* We observe that

$$s = 6a\int_0^1 \frac{\sqrt{1+tf(t)}}{(1+t^3)^2}\,dt,$$

where $f(t) = 4t - 4t^2 - 4t^4 + 4t^5 + t^7$. It is easily proved that $f(t) < 1$ for $0 \leqq t < 1$ and $f(1) = 1$. Expansion by the binomial formula for $0 \leqq t < 1$ gives

$$\sqrt{1+tf(t)} = \sum_{n=0}^{\infty} \binom{1/2}{n} t^n [f(t)]^n.$$

Since $\binom{1/2}{n} = O(n^{-3/2})$ it follows that the series is absolutely convergent in the closed interval $0 \leqq t \leqq 1$, and therefore

$$s = 6a\sum_{n=0}^{\infty} g_n, \qquad g_n = \binom{1/2}{n}\int_0^1 \frac{t^n[f(t)]^n}{(1+t^3)^2}\,dt.$$

The integrals are of a well known type (*cf.*, Rinehart Math. Tables, Nos. 235, 247, 248) and may be easily calculated. If we notice that the expression for s is an alternating series and therefore the absolute value of the remainder r_n is less than the absolute value of the first neglected term, we have

$$|r_n| < |g_{n+1}| < \left|\binom{1/2}{n+1}\right| \int_0^1 t^{n+1}dt = \left|\binom{1/2}{n+1}\right| \frac{1}{n+2}.$$

Thus if 3 terms of the series are taken $|r_3| < 2^{-7}$.

Also solved by the proposer.

Covering by Dissected Squares

E 1010 [1952, 181]. *Proposed by Michael Goldberg, Washington, D. C.*

Given n equal unit squares, dissect each of these in exactly the same way with straight cuts into $p(n)$ parts such that the n $p(n)$ pieces may be assembled to form a square of edge $\sqrt{n}$. Investigate $k(n)$, the minimal values of $p(n)$, and show that $k(n) \leqq 5$ for all n.

Solution by the Proposer. Let $[\sqrt{n}]$ be the integral part of $\sqrt{n}$. Then if $\sqrt{n}-[\sqrt{n}]>1/2$, take $[\sqrt{n}]$ as the edge of a basic square. If $\sqrt{n}-[\sqrt{n}]<1/2$, take $[\sqrt{n}]-1$ as the edge of a basic square. Then if the basic square is placed in the corner of a square of edge $\sqrt{n}$, as shown in Fig. 1, it will be bordered by an L-shaped region of width w. This width is restricted by the relation $3/2>w>1/2$. By a single cut, this L-shaped region can be changed into a rectangle of width w.

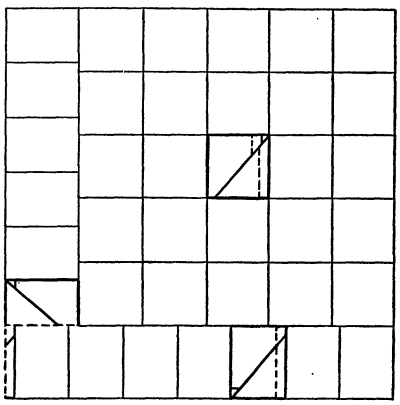


FIG. 1

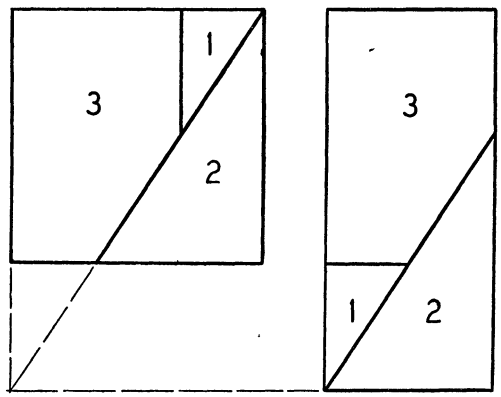


FIG. 2

Fill the basic square with the required number of given unit squares. The remainder of the unit squares must now be transformed by dissection into rectangles of width w. Since $3/2>w>1/2$, this transformation never requires more than three pieces, as shown by the solid lines. (See separate demonstration of this construction.) The dotted lines forming the required cut in the rectangle of width w to transform it back into the L-shaped region will, with proper orientation of the division of a fundamental rectangle, divide one of the fundamental rectangles into, at most, five pieces. All other fundamental rectangles and squares should then be cut into the same five pieces.

Dissection of the unit square: A unit square can be cut into three pieces to form a rectangle of width w, where $3/2>w\geqq 1$, as shown in Fig. 2. The top edge of piece 1 is never more than half the horizontal dimension of the rectangle. Therefore, a vertical cut dividing this rectangle into any desired ratio can always be selected so that it cuts pieces 2 and 3 without cutting piece 1. Therefore, only five pieces will be formed.

If $1 \geqq w > 1/2$, the same dissection is used, except that w is now the short edge of the rectangle. A horizontal cut dividing the rectangle into any desired ratio can always be selected so that piece 1 is not cut. Therefore, at most, only five pieces will be formed. If only piece 3 is cut, then only four pieces will be formed.

Note: In a paper by the proposer and Dr. B. M. Stewart, now awaiting publication, the bound of this problem has been improved by showing that $k(n) \leqq 4$ for all n. In particular, $k(ab) \leqq 3$, where a and b are integers satisfying $b < a < 4b$; $k(4a) \leqq 3$; $k(4a+1) \leqq 3$; $k(a^2+b^2) \leqq 3$; $k[a^2+(a-b)^2] \leqq 2$ for $a \geqq 4b-1$; $k(2) = k(5) = k(10) = 2$.

Traffic Signals

E 1011 [1952, 249]. *Proposed by D. D. Wall, Los Angeles, California*

If traffic signals are "set for 30," at what other constant speeds may one travel? Assume signals evenly spaced, all change at the same time, and that they alternate in color spacewise as well as timewise.

Solution by C. A. Swanson, University of British Columbia. Let the distance between successive lights be d, so that the time between two changes of a signal is $T = d/30$. Suppose a car passes the first light at time $t = 0$; it must then pass the second light between times $(2k-1)T$ and $2kT$, $\cdots$, the nth light between times $(n-1)(2k-1)T$ and $[(n-1)(2k-1)+1]T$, where $k = 1, 2, \cdots$. Let the car pass the second light at time $(2k-1)T+\epsilon$, so that the constant velocity of the car is $v = d/[(2k-1)T+\epsilon]$. Since v is constant, the car passes the third light at time $2d/v = 2[(2k-1)T+\epsilon]$, $\cdots$, the nth light at time $(n-1)[(2k-1)T+\epsilon]$. In order that the car not stop at the nth light, or anywhere before the nth light,

$$(n-1)[(2k-1)T+\epsilon] < [(n-1)(2k-1)+1]T,$$

or

$$\epsilon < T/(n-1).$$

Thus

$$v = d/[(2k-1)T+\epsilon] \geqq 30/[(2k-1)+1/(n-1)],$$

equality being attained in the limit as $n \to \infty$. We therefore have

$$30/[(2k-1)+1/(n-1)] \leqq v \leqq 30/(2k-1). \tag{1}$$

Now suppose the car passes the first light at time T, the second light at time $2kT-\epsilon$, $\cdots$, the nth light at time $nkT-(n-1)\epsilon$, so that we must have

$$nkT-(n-1)\epsilon > (n-1)T,$$

or

$$\epsilon < T/(n-1).$$

Then

$$30/(2k-1) \leqq v \leqq 30/[(2k-1) - 1/(n-1)]. \tag{2}$$

Combining (1) and (2) we have, for the car's possible constant velocities,

$$30/[(2k-1) + 1/(n-1)] \leqq v \leqq 30/[(2k-1) - 1/(n-1)]. \tag{3}$$

If n is taken as infinite then result (3) reduces to

$$v = 30/(2k-1) = 30, 10, 6, \cdots .$$

Also solved by Julian Braun, J. L. Hildebrand, Vern Hoggatt, A. R. Hyde, David Mandelbaum, W. H. Mead, Jr., D. V. Newton, C. S. Ogilvy, R. Z. Vause, Jr., and the proposer. Most of these solutions assumed an infinite number of signals.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey, All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4513. *Proposed by Leonard Carlitz, Duke University*

Put $\{m\}!=(x^m-1)(x^{m-1}-1)\cdots(x-1)$, $\{0\}!=1$.

A. Show that

$$\frac{\{2m\}!\{2n\}!}{\{m+n\}!\{m\}!\{n\}!} \qquad (m, n \geqq 0) \tag{1}$$

is a polynomial with rational coefficients.

B. The quotient

$$\frac{\{mn\}!}{\{m\}!(\{n\}!)^m}(x-1)^m \tag{2}$$

is not integral for all $m, n \geqq 0$. How can it be modified to be integral and at the same time reduce to $(mn)!/m!(n!)^m$ for $x=1$?

4514. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

$\psi(t)=$constant and $\psi(t)=|t|$ are solutions of the integral equation

$$2\psi(t) = \int_{-1}^{1} \psi(2tx)dx,$$

as is easily verified. Are there any other solutions?

4515. *Proposed by Karl Goldberg, Brooklyn, N.Y.*

Prove

$$\sum x \equiv 2^{-4} + (2^{p-1} - 1)/p \pmod{p},$$

where p is a prime and the summation extends over all integers $x<p$ which are greater than $(p-1)/2$ and whose inverses mod p are also greater than $(p-1)/2$.

4516. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In a tetrahedron $ABCD$ let L be the second Lemoine point (*i.e.* the point whose distances from the planes of the faces are proportional to the circumradii of these faces) and let L', L'', L''' be the harmonic conjugates of L with respect to the points where lines through L cut the edges BC and DA, CA and DB, AB and DC. Show (1) the tetrahedron $LL'L''L'''$ is self-conjugate with respect to the sphere $ABCD$; (2) the polar planes of L, L', L'', L''' with respect to the sphere $ABCD$ coincide with the polar planes of these points with respect to the tetrahedron $ABCD$.

4517. *Proposed by George Lorentz, University of Toronto*

A sequence a_n, $n=0, 1, \cdots$, is called logarithmically concave if

$$a_n^2 - a_{n-1}a_{n+1} \geqq 0, \qquad n = 1, 2, \cdots.$$

Prove that if the sequences a_n, b_n are positive and logarithmically concave, then their convolution $c_n=a_0b_n+a_1b_{n-1}+\cdots+a_nb_0$ has the same properties.

SOLUTIONS

Integral Reducing to an Elementary Function

4450 [1951, 495]. *Proposed by R. M. Redheffer, University of California, Los Angeles*

Let $g(x)$ be a polynomial of degree $\leqq 2n+1$. Show that a necessary and sufficient condition that the integral

$$\int e^{x^2} g(x)dx$$

is an elementary function is the vanishing of the determinant

$$\begin{vmatrix} g(0) & 1 & 0 & 0 \cdots & 0 & 0 \\ g^{(2)}(0) & 4 & 1 & 0 \cdots & 0 & 0 \\ g^{(4)}(0) & 0 & 8 & 1 \cdots & 0 & 0 \\ g^{(6)}(0) & 0 & 0 & 12 \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g^{(2n-2)}(0) & 0 & 0 & 0 \cdots & 4n-4 & 1 \\ g^{(2n)}(0) & 0 & 0 & 0 \cdots & 0 & 4n \end{vmatrix}.$$

Solution by P. Somanatham, M. R. College, Vizianagram, South India. Let

$$g(x) = \sum_{j=0}^{2n+1} A_j x^j.$$

It is obvious that

$$\int xe^{x^2}dx, \quad \int x^3e^{x^2}dx, \cdots, \quad \int x^{2r+1}e^{x^2}dx$$

are elementary functions. So it suffices to prove that the necessary and sufficient condition that the integral

$$\int (A_0 + A_2x^2 + \cdots + A_{2n}x^{2n})e^{x^2}dx \tag{1}$$

may be an elementary function is the vanishing of the given determinant. Now it is well known that $\int e^{x^2}dx$ is not elementary. Further

$$\int A_2x^2e^{x^2}dx = A_2(x/2)e^{x^2} - \int \tfrac{1}{2}A_2e^{x^2}dx,$$

$$\int A_4x^4e^{x^2}dx = A_4(x^3/2)e^{x^2} - \int (3/2)A_4x^2e^{x^2}dx$$

$$= A_4(x^3/2)e^{x^2} - 3A_4(x/4)e^{x^2} + \int (3/4)A_4e^{x^2}dx;$$

and so on. Hence the necessary and sufficient condition that (1) may be an elementary function is that

$$\left(A_0 - \frac{A_2}{2} + \frac{3A_4}{4} - \frac{3\cdot 5A_6}{8} + \cdots\right)e^{x^2}$$

should vanish, i.e., the expression in the brackets should be zero. Since $g(0) = A_0$, $g^{(2)}(0) = 2!\,A_2$, $g^{(4)}(0) = 4!\,A_4$, $\cdots$, the necessary and sufficient condition is the

vanishing of

$$g(0) - \frac{g^{(2)}(0)}{2\cdot 2!} + \frac{3g^{(4)}(0)}{4\cdot 4!} - \frac{3\cdot 5g^{(6)}(0)}{8\cdot 6!} + \cdots$$

or (upon multiplying by $4\cdot 8\cdot 12\cdots 4n$) the vanishing of

$$4\cdot 8\cdot 12\cdots 4n\cdot g(0) - 8\cdot 12\cdots 4n\ g^{(2)}(0) + 12\cdot 16\cdots 4n\cdot g^{(4)}(0) - \cdots$$

which is exactly the expanded form of the given determinant.

Also solved by A. S. Hendler, M. S. Klamkin, M. S. Robertson, and the Proposer.

Modulus of an Analytic Function

4451 [1951, 496]. *Proposed by Paul Erdös, University College, London, England*

Let

$$f(z) = \prod_{j=1}^{n} (z - z_j), \qquad |z_j| = 1.$$

Prove that one can connect the origin with a point of the unit circle, so that everywhere on this path, except at the origin, $|f(z)| < 1$.

Solution by Paul Cohen, Brooklyn College, New York. We assume merely that $f(z)$ is non-constant and analytic for $|z| \leqq 1$, that $|f(0)| = 1$ and that $f(z) \neq 0$ for $|z| < 1$.

Let $r < 1$. $1/f(z)$ is regular in $|z| \leqq r$ and attains its maximum modulus for $|z| \leqq r$ at some point ζ, $|\zeta| = r$. Thus $|f(\zeta)| < 1$. Let $z = re^{i\theta}$. By the continuity of $|f(z)|$ it follows that for each $r < 1$ there is an interval of θ, $\theta_1(r) < \theta < \theta_2(r)$ throughout which $|f(z)| < 1$.

Now the maximum modulus theorem applied to $f(z)$ in $|z| \leqq r$ assures that $|f(z)| > 1$ for some point on $|z| = r$. Hence we deduce the existence of θ_1 and θ_2 such that

$$|f(re^{i\theta_1})| = |f(re^{i\theta_2})| = 1.$$

The curves defined by $z = re^{i\theta_1}$ and $z = re^{i\theta_2}$, as r varies, extend all the way from the origin to the unit circle, and they do not meet. Otherwise they would bound a region having no zeros and on whose boundary $|f(z)| = 1$, but this is seen to be impossible by the maximum modulus theorem applied to $1/f(z)$. It is now evident that the required path can be traced from origin to unit circle with $\theta(r)$ always between $\theta_1(r)$ and $\theta_2(r)$.

Comments by the Proposer. Similarly one can connect the origin with a point of the unit circle by a path on which everywhere, except at the origin, $|f(z)| > 1$. To the conjecture that there exists a radius on which $|f(z)| < 1$, Charles Loewner has found a counter-example. The counter-example is not simple and it is not

given explicitly. The interesting problem remains: find an elementary counter-example.

The question then rises: Does there exist a constant c such that one can connect the origin with the unit circle by a path of length $<c$ so that on this path $|f(z)|<1$ and c is independent of $f(z)$?

A Number Theory Identity

4452. [1951, 496]. *Proposed by Leo Moser, University of Alberta, Canada*

Prove that

$$\sum_{r=1}^{n} \mu(r)[n/r]\{3-2\omega(r)\}^2 = 9 + 8\{\Pi_1^*(n) + \Pi_2^*(n)\},$$

where $\mu(r)$ is the Möbius function, $[n/r]$ denotes the largest integer not exceeding n/r. $\omega(r)$ is the number of distinct prime divisors of r (i.e., $\omega(1)=0$, $\omega(p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k})=k$), and $\Pi_k^*(n)$ is the number of integers t, not exceeding n, for which $\omega(t)=k$.

Solution by E. M. Wright, University of Aberdeen, Scotland. Let

$$m = p_1^{a_1}\cdots p_k^{a_k} \qquad (p_1 < p_2 < \cdots < p_k)$$

so that $\omega(m)=k$, and write $m'=p_1\cdots p_k$. Now $\mu(r)=0$ or $=(-1)^{\omega(r)}$ as r has, or has not, a squared factor. Again, the number of divisors r of m' for which $\omega(r)=j$ is $\binom{k}{j}$. Hence

$$\sum_{r\mid m} \mu(r)x^{\omega(r)} = \sum_{r\mid m'} \mu(r)x^{\omega(r)} = \sum_{j=0}^{k} (-1)^k \binom{k}{j} x^j = (1-x)^k.$$

Differentiating s times with respect to x and putting $x=1$, we have

$$\sum_{r\mid m} \mu(r)T_s(r) = \begin{cases} (-1)^k k! & (s=k), \\ 0 & (s\neq k), \end{cases}$$

where

$$T_s(r) = \omega(r)\{\omega(r)-1\}\cdots\{\omega(r)-s+1\}.$$

Hence

$$(-1)^k k!\Pi_k^*(n) = \sum_{m=1}^{n}\sum_{r\mid m} \mu(r)T_k(r) = \sum_{r=1}^{n} \mu(r)[n/r]T_k(r),$$

where $\Pi_0^*(n)=1$. The required result follows at once, since

$$9T_0 - 8T_1 + 4T_2 = (3-2\omega)^2.$$

Also solved by Paul Cohen, Emil Grosswald, H. T. LaBorde and the Proposer.

A Summation Suggested by Certain Results of Ramanujan

4453 [1951, 569]. *Proposed by H. F. Sandham, Trinity College, Ireland*

Prove that

$$\frac{1^5}{e^{\pi}+1}+\frac{3^5}{e^{3\pi}+1}+\frac{5^5}{e^{5\pi}+1}+\cdots=\frac{31}{504}.$$

Solution by Leonard Carlitz, Duke University. This solution was suggested by the Proposer's paper, *Three summations due to Ramanujan*, Quarterly Journal of Mathematics, Oxford, (2), I(1950), pp. 238–240. We have the known formula

$$\sum_{m=0}^{\infty} e^{-(2m+1)\pi x}=\frac{1}{e^{\pi x}-e^{-\pi x}}=\frac{1}{2\pi x}+\frac{1}{2\pi}\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{x+ni}+\frac{1}{x-ni}\right\}.$$

Differentiating five times we get

$$\pi^5\sum_{m=0}^{\infty}(2m+1)^5e^{-(2m+1)\pi x}=\frac{1}{2\pi}\frac{5!}{x^6}+\frac{5!}{2\pi}\sum_{n=1}^{\infty}(-1)^n\left\{\frac{1}{(x+ni)^6}+\frac{1}{(x-ni)^6}\right\}.$$

Next multiply both members by $(-1)^{x-1}$ and sum for x integral from $x=1$ to $x=\infty$; thus

$$\pi^5\sum_{m=0}^{\infty}\frac{(2m+1)^5}{e^{(2m+1)\pi}+1}=\frac{5!}{2\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^6}-\frac{5!}{2\pi}\sum_{m,n=1}^{\infty}(-1)^{m+n}\left\{\frac{1}{(m+ni)^6}+\frac{1}{(m-ni)^6}\right\}.$$

Note that interchange of m and n in the double sum changes the sign of each sum; hence the double sum vanishes. Finally we use the familiar formula

$$\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^{2k}}=(-1)^{k-1}\frac{2^{2k-1}-1}{(2k)!}B_{2k}\pi^{2k},$$

where $B_2=1/6$, $B_4=1/30$, $B_6=1/42$, $\cdots$ are the Bernoulli numbers. Hence

$$\sum_{m=0}^{\infty}\frac{(2m+1)^5}{e^{(2m+1)\pi}+1}=\frac{5!}{2\pi^6}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m^6}=\frac{5!}{2\pi^6}\frac{31}{6!}B_6\pi^6=\frac{31}{504}.$$

The same method, using $4k+1$ differentiations, yields

$$\sum_{m=0}^{\infty}\frac{(2m+1)^{4k+1}}{e^{(2m+1)\pi}+1}=\frac{2^{4k+1}-1}{8k+4}B_{4k+2}.$$

Also solved by T. M. Apostol, Joshua Barlaz, L. Carlitz (a second solution), J. S. Frame, Stanley Katz and A. M. Peiser, O. E. Stanaitis, Ernest Trost, R. E. Wild, and the Proposer.

Functions such that the Derivative of the Quotient is the Quotient of the Derivatives

4455 [1951, 569]. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn*

If

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)}{g'(x)},$$

solve for $f(x)$ in terms of $g(x)$, and for $g(x)$ in terms of $f(x)$. (This is an extension of one of the problems proposed for the 1951 William Lowell Putnam Prize Competition.)

Solution by E. J. Scott, University of Illinois. Using the abbreviations f, g, f', g', for $f(x)$, $g(x)$, *etc.*, we have

$$(gf' - fg')/g^2 = f'/g', \tag{1}$$

from which

$$f'/f - g'/g = [1 - g/g']^{-1}.$$

Integrating, we have

$$\ln f/g = \int [1 - g/g']^{-1}dx,$$

whence

$$f = g \exp\left\{\int [1 - g/g']^{-1}dx\right\}.$$

Similarly, from (1) we obtain

$$(g'/g)^2 - (f'/f)(g'/g) + f'/f = 0,$$

whence

$$g'/g = \tfrac{1}{2}[f'/f \pm \sqrt{(f'/f)^2 - 4(f'/f)}\,].$$

Integrating and solving for g, we get

$$g = \sqrt{f} \exp\left[\pm \frac{1}{2}\int \sqrt{(f'/f)^2 - 4(f'/f)}\,dx\right].$$

Also solved by Emil Grosswald, Hyman Kaufman, R. A. Miller, George Millman, C. S. Ogilvy, Azriel Rosenfeld, O. E. Stanaitis, and the Proposer.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 80 Waterman Street, Providence 6, Rhode Island, and not to any of the other editors or officers of the Association.

Elementary Analysis. By K. O. May. New York, John Wiley and Sons, Inc. 1952. xviii+635 pages (including tables). $5.00.

This book is a unified treatment of algebra, trigonometry, analytic geometry and some of the concepts of calculus. It is by no means just another book in this field but is a thoroughly wrought piece of text-book writing. At least one year of high school algebra and plane geometry are supposed. The book is adaptable to students wishing a thorough grounding for calculus and beyond and to students wishing a terminal course. This last class of students can get some of the important ideas of mathematics which have given the subject its use and beauty. The author gives the teacher useful suggestions in making up a course.

The first three chapters are called: (1) Introduction, (2) Logic, (3) Numbers and Elementary Operations. They present topics such as the basic concept of a set and the construction of the number system from the positive integers. With a class of interested and well prepared students discussion of this material would be excellent and very profitable. If student preparation warrants, this material could be used otherwise or omitted and a course begun with chapter four. A good small bibliography of further readings for interested students is included.

The more traditional part of the book discusses the linear, quadratic, and power functions as well as the exponential, logarithmic, and circular functions. There are also chapters on analytic geometry of two and three dimensions, polynomials, the conics, complex numbers and algebraic functions with graphs of irrational functions. Parametric equations and polar coordinates are handled to give the student the necessary ideas for later courses. Tables occur at the end of the book.

It is clear that the author has spent both time and care in getting together a great many problems chosen to gain manipulative practice and illustrate the text material. The book is a vast store house of problems well chosen from the uses of mathematics in the physical sciences, economics and business. A teacher using the book should be on the lookout for the many exercises extending the theory. Included among such are important loci on polar coordinates and problems on various properties of the conics. The problems are graded in difficulty and more difficult problems are specially indicated. The reviewer would like to see more answers to problems given.

The author's style is lively and clear and his exposition excellent. The set-up of the book and the typography are very good. The inclusion in problem 14 on page 185 of information not needed to solve the problem may have been intentional.

Each college and university offering a unified Freshman course has its own particular problems. The reviewer believes that careful consideration of various courses designed from this well-written book may help to find a solution of such problems and to provide a first course in college mathematics which is beneficial to students and enjoyable to teachers.

W. N. HUFF
University of Oklahoma

Teaching Mathematics in the Secondary School by L. B. Kinney and C. R. Purdy. New York, Rinehart and Company, Inc. 1952. xviii+381 pages. $5.00.

This book is designed to furnish the teacher of mathematics information about what mathematics should be taught in the secondary school and how it should be taught. The authors survey our modern society and the nature of the secondary school program as a setting for the determination of what we should teach. Modern principles of learning are applied in illustrating the techniques, materials, and activities that have been found to be effective in teaching mathematics.

The authors begin appropriately by showing the role of mathematics in modern society. This concise chapter leaves the reader with an appreciation of the power of mathematics and a desire for more extensive information about the applications of mathematics. A review of dominant events in the development of secondary education furnishes a background for considering curriculum problems in mathematics. A chapter on how to direct learning in mathematics applies learning principles to the problem of teaching mathematics. Frequent illustrations of the techniques of specific master teachers of the subject are used to make the discussion realistic. In this way the book is unique in giving master teachers recognition for superior work.

A major portion of the book is given to a discussion of the aims, content, problems, and methods of teaching various fields of secondary and junior college mathematics. This discussion is made concrete by describing the current practices of good teachers. This manner of telling how to teach a topic is vivid and will be effective if the reader does not conclude that the technique described is the one and only method to be used. Although examples and materials are illustrated, no attempt is made to treat comprehensively all the topics of each area. It is the area of general mathematics which is receiving emphasis in the mathematics program of the secondary school today. Consequently the authors devote four chapters to the function, organization, administration and teaching of general mathematics from junior high school through junior college. Flow charts are used comparing the teacher's responsibilities with the order of pupil experiences in the development of concepts.

The remainder of the book deals with the use of teaching aids, planning units of instruction, constructing and using tests, and recreational mathematics. The chapter on teaching aids is devoted to methods of using teaching aids rather than a description of the great variety of materials which are available

to the teacher. The discussion of the long-unit assignment outlines the organization of a unit and illustrates them with parts of typical units. The description of tests includes a variety of types of test items to illustrate how to construct and use tests. It is unfortunate that this chapter is limited to achievement testing and, thus, does not treat the larger problem of evaluation which involves the measurement of objectives such as appreciations, attitudes, or work habits.

This book presents a wealth of practical hints for improving the teaching of mathematics. It is written in a brief, concise, readable form which is a distinct contrast to much literature in the field. However, the reviewer frequently had the feeling that the treatment of many topics was too brief. This briefness may be necessary in a text which treats a topic of such a wide scope as the teaching of mathematics from junior high school through junior college. Every teacher of mathematics will learn new ways of teaching old topics by reading this book.

DONOVAN A. JOHNSON
University of Minnesota

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB TOPICS

A recent pamphlet by C. B. Read, *Articles on the History of Mathematics*, contains an extensive bibliography which should prove very useful in planning club programs. The bibliography covers historical articles which have appeared in five periodicals prior to January 1, 1952: *The American Mathematical Monthly*, *The Mathematics Teacher*, *National Mathematics Magazine*, *Scripta Mathematica*, and *School Science and Mathematics*.

The material is divided into the following major categories, each of which suggests a possible club topic:

The history of mathematics as a whole
Mathematics in the ancient world
Mathematics in other countries
Mathematics in America
Numerals and number systems
Calculating methods and devices
Mathematical terms and symbols

Algebra and algebraic problems
Logarithms and exponentials
The numbers *e*, *pi*, and *i*
Complex numbers and the theory of numbers
Geometry and geometrical problems (analytic geometry, geometrical constructions, non-euclidean geometry, special geometrical curves, geometrical articles not otherwise specified)
Astronomy, trigonometry, surveying, and navigation
The calculus and related topics
Mathematical books and manuscripts
Mathematical periodicals
Mathematicians of ancient times
Mathematicians of the middle ages
Mathematicians of the sixteenth century
Mathematicians of the seventeenth century
Mathematicians of the eighteenth century
Mathematicians of the nineteenth and twentieth centuries
Miscellaneous.

The pamphlet was published as a Bulletin (University Studies No. 26) by the Municipal University of Wichita, Wichita, Kansas.

CLUB NEWS

The University of Virginia chapter of Pi Mu Epsilon in 1951–52 again sponsored prize examinations: one for students taking freshman mathematics, the other for students taking sophomore mathematics.

The Mathematics Club of the University of Massachusetts has awarded Allan Dickenson the fee for membership in the Mathematical Association of America for his outstanding work in the Mathematics Club during 1951–52.

During 1951–52, pledges of the Mount St. Scholastica College chapter of Kappa Mu Epsilon sponsored an open house, at which time the mathematics department received visitors for the entire college. Various mathematical concessions were arranged to amuse the guests. These consisted of optical illusions; brain teasers; identification of mathematical tools, figures, solids; and famous men of mathematics. Prizes were awarded to the most successful contestants.

PI MU EPSILON NATIONAL MEETING

A national meeting of Pi Mu Epsilon Fraternity was held at Michigan State College, East Lansing, Michigan, on Sunday evening, August 31, and Monday, September 1, 1952, in conjunction with meetings of the Mathematical Association of America, the American Mathematical Society, the Institute of Mathematical Statistics, and the Econometric Society.

On Sunday evening there was an informal discussion meeting followed by a meeting of the National Council. Members met for luncheon and business meeting in Phillips Hall.

The following papers were presented by Pi Mu Epsilon members on Monday:

So you want to be a graduate student, by Director General C. C. MacDuffee, Wisconsin Beta

Modification of infinite series, by Bevan K. Youse, Georgia Alpha

Rapid square roots, by Charles D. Parker, Michigan Alpha

Almost periodic functions, by John E. Hoffman, Oklahoma Alpha

Matrix inversion, by Verna Lair, Delaware Alpha

Metric extension, by Alan J. Goldman, New York Gamma

The G.C.D. algorithm, by Wilson M. Zaring, Kentucky Alpha.

CALENDAR OF FUTURE MEETINGS

Thirty-sixth Annual Meeting, Washington University, St. Louis, Missouri, December 30, 1952.

Thirty-fourth Summer Meeting, Laval University, Quebec, Canada, August 31–September 1, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Carnegie Institute of Technology, Pittsburgh, Pennsylvania, May, 1953.

ILLINOIS, University of Illinois, Navy Pier, Chicago, May 8–9, 1953.

INDIANA, Ball State Teachers College, Muncie, May 2, 1953.

IOWA, Cornell College, Mount Vernon, April 17–18, 1953.

KANSAS

KENTUCKY, University of Louisville, Spring, 1953.

LOUISIANA-MISSISSIPPI, Millsaps College, Jackson, Mississippi, February 13–14, 1953.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Howard University, Washington, D. C., December 6, 1952.

METROPOLITAN NEW YORK, Spring, 1953.

MICHIGAN, Western Michigan College of Education, Kalamazoo, April 11, 1953.

MINNESOTA, St. Olaf College, Northfield, May 9, 1953.

MISSOURI, William Jewell College, Liberty, Spring, 1953.

NEBRASKA

NORTHERN CALIFORNIA, San Francisco State College, January 31, 1953.

OHIO

OKLAHOMA

PACIFIC NORTHWEST, Montana State University, Missoula, June, 1953.

PHILADELPHIA

ROCKY MOUNTAIN, University of Colorado, Boulder, April, 1953.

SOUTHEASTERN, Alabama Polytechnic Institute, Auburn, March 13–14, 1953.

SOUTHERN CALIFORNIA, Los Angeles City College, March 14, 1953.

SOUTHWESTERN, New Mexico College of Agriculture and Mechanic Arts, State College, January 2–3, 1953.

TEXAS

UPPER NEW YORK STATE, United States Military Academy, West Point, Spring, 1953.

WISCONSIN, Mount Mary College, Milwaukee, May, 1953.



BASIC MATHEMATICS FOR ENGINEERING AND SCIENCE

By Walter R. Van Voorhis and Elmer E. Haskins, Fenn College

The subjects in this text are unified under the function concept, so basic to scientific mathematics. Within this framework most of the traditional material of algebra, trigonometry and analytic geometry, plus a few additional topics selected with a view to strengthening the students' preparation for calculus, is presented in an integrated pattern designed to stimulate a high degree of interest throughout.

619 Pages 6" × 8½" Published 1952

MODERN ELEMENTARY STATISTICS

By John E. Freund—Alfred University

Designed for students in the social and natural sciences who have very little background in mathematics. It emphasizes the *meaning* of statistics rather than the acquisition of mathematical skills. Theoretical distributions are introduced as early as Chapter 3 on a more or less intuitive level. Chapter 7 has a discussion and repeated emphasis on the meaning of probability statements. For the first time, the modern theory of the testing of hypotheses is presented on the non-technical level.

Approx. 418 Pages 6" × 8½" Published 1952

PLANE TRIGONOMETRY—Third Edition

By Fred W. Sparks—Texas Technological College and Paul K. Rees—Louisiana State University

PLANE TRIGONOMETRY, Third Edition, like earlier editions, covers all essentials—including logarithms, graphs of trigonometric functions, and trigonometric equations.

The method used by the U. S. Air Force for designating directions is explained and problems in elementary air navigation have been added to most exercises dealing with the solution of triangles. The United States Naval Academy used the Revised Edition in their classes, and has also adopted the new Third Edition.

275 Pages with tables; 199 Pages without tables 6¼" × 9¼" Published 1952

See these texts and other new and helpful Prentice-Hall books at our exhibit at the convention



HOLT

BRIEF TRIGONOMETRY Revised Edition

Edward A. Cameron, *University of North Carolina*

"This revised edition seems to me to maintain the high standard set by the original edition. Also, I believe the new edition contains a few notable improvements from the standpoint of its teachability." H. P. EVANS, *University of Wisconsin*

". . . the essence of brevity and at the same time possessing flexibility."—D. L. HOLL, *Iowa State College.*

1952, 154 pages, $2.30

BRIEF COURSE IN ANALYTICS

Revised Edition

M. A. Hill, Jr., and J. B. Linker, *University of North Carolina*

"The revised edition is in several ways an improvement over the first edition, especially in regard to the addition of the reference material and tables and the new diagrams done on coördinate paper. The text has the virtue of being simply written."— P. M. CURRAN, *Fordham University.* *1951, 224 pages, $2.60*

COLLEGE ALGEBRA Fifth Edition

H. L. Rietz and A. R. Crathorne
Revised by J. William Peters, *University of Illinois*

". . . a good 'teachable' book. The presentation of material is good —for example, I am impressed by the brevity and clarity of the treatment of the method of mathematical induction."—H. M. BACON, *Stanford University.* *1951, 387 pages, $3.50*

MATHEMATICS ESSENTIAL FOR ELEMENTARY STATISTICS

Revised Edition

Helen M. Walker, *Teachers College, Columbia University*

"Dr. Walker has done an excellent job in assembling the mathematics essential for work in elementary statistics. Every student who plans to study statistics should first master the material in this book."—MILES A. TINKER, *University of Minnesota.*

1951, 382 pages, $3.25

HENRY HOLT AND COMPANY 383 Madison Avenue, New York 17

M. Wiles Keller *Purdue University*

BASIC ALGEBRA

covers the essentials which must be mastered by students (1) who require some understanding and manipulative ability in algebra for use in elementary courses in the sciences or (2) who must have this basic training as background for later work in mathematics, engineering, and the sciences. A large number of drill exercises are provided, each graded and carefully chosen for suitability to the rule and variations involved.

COLLEGE ALGEBRA

provides a thorough review on a mature level of the fundamentals of algebra. Topics and exercises are arranged in order of difficulty, and give carefully patterned review and repetitive drill on basic algebraic techniques.

Houghton Mifflin Company

BOSTON NEW YORK CHICAGO DALLAS SAN FRANCISCO

FOUNDATIONS OF ALGEBRAIC TOPOLOGY

By Samuel Eilenberg and Norman Steenrod

The need for an axiomatic treatment of homology and cohomology theory has long been felt by topologists. Professors Eilenberg and Steenrod, of Columbia and Princeton Universities respectively, present here for the first time an axiomatization of the complete transition from topology to algebra.

356 pages *$7.50*

PRINCETON UNIVERSITY PRESS
PRINCETON, NEW JERSEY

MATHEMATICS OF FINANCE
Mouzon—Rees



A practical new course requiring no more than one semester of college algebra. Numerous well-chosen examples are worked out in detail in the text and there are 1150 problems. Terms are defined and clearly illustrated. Each chapter has a summary and review section.

BASIC SKILLS IN MATHEMATICS
Price—Knowler

For Freshmen with insufficient mathematical background. It provides skills necessary for an intelligent understanding of the quantitative aspects of everyday living, and places considerable emphasis on the meaning of concepts. Skills are constantly repeated; new ideas built on previously developed ones.

ANALYTIC GEOMETRY AND CALCULUS
Longley—Smith—Wilson

Plane and solid analytic geometry, differential and integral calculus, including a chapter on differential equations, in a combined course. Science and engineering majors can be introduced to calculus before completing a course in analytic geometry. Clear style; many examples, problems.

New York 11
Chicago 16
Atlanta 3
Dallas 1
Columbus 16
San Francisco 3
Toronto 5

Home Office:
Boston

GINN AND COMPANY

HEATH College Texts

COMING!

EXTENSIVELY REVISED
COMPLETELY RESET

WILLIAM L. HART'S

Fourth Edition

COLLEGE ALGEBRA

THE TEXT—extensively rewritten

THE PROBLEMS—largely augmented and entirely new

Maintains the successful pedagogy of earlier editions and adds these new features:

A NOVEL INTRODUCTION TO SIGNED NUMBERS
A UNIQUE CHAPTER ON DISCRETE PROBABILITY
APPENDIX NOTES ON SETS WITH PROBABILITY
CONTACTS

PUBLICATION
SPRING 1953

D. C. HEATH
AND COMPANY

SALES OFFICES: NEW YORK CHICAGO SAN FRANCISCO ATLANTA DALLAS
HOME OFFICE: BOSTON

NEW BOOKS FOR SECOND SEMESTER CLASSES!

Ready January 1st —

AN INTRODUCTION TO MATHEMATICAL THOUGHT

By E. R. Stabler, *Hofstra College*

An elementary presentation of the foundations of mathematics, the chief aim of which is to provide a unified and substantial approach to the logical structure of mathematics, and to develop a corresponding philosophical point of view toward mathematical knowledge. Suitable for general education courses or for introductory foundations courses.

Cloth — 5⅝ x 8⅜ — c. 280 pages *Probable price $4.50*

Ready January 15th —

INTRODUCTION TO MEASURE AND INTEGRATION

By M. E. Munroe, *U. of Illinois*

A *textbook* for courses in measure theory on the graduate level. Purpose of the book is to present measure theory from the abstract or postulational point of view and yet to do this in such a way that the student as well as the specialist will find the work helpful. Theory developed fully. Some applications to other topics such as probability, Banach spaces, orthogonal expansion, etc.

Cloth — 5⅝ x 8⅜ — c. 352 pages *Probable price $7.50*

Published —

ELEMENTS OF MATHEMATICS

By H. M. Roberts & D. S. Stockton, *U. of Connecticut*

A combination text and workbook for remedial mathematics courses for college freshmen whose mathematical background is deficient. Complete review of secondary school arithmetic, algebra, and those topics in geometry which are desirable for students who may go on to study trigonometry and analytic geometry. Generous assortment of factual and interesting problems for drill. Perforated work sheets and tests for student assignments.

Paper — 7¼ x 9½ — 250 pages *$3.00*

Published —

ADVANCED CALCULUS

By W. Kaplan, *University of Michigan*

Published in May and already adopted at eighty leading colleges and universities this fall, this remarkable text has already taken its place as a leader in the field. Adaptable not only for courses in *advanced calculus*, but also for courses in *vector analysis, complex variables,* and *partial differential equations.* Emphasis on applications. High level of rigor maintained throughout.

Cloth — 6 x 9 — 679 pages. *$8.50*

Published —

THEORY OF MATRICES

By S. Perlis, *Purdue University*

Another new text already being adopted at many colleges, suitable for courses in matrix theory and linear algebra. Self-contained and simple, yet presents a complete theory. Canonical sets are the main theme. Role of determinants is reduced to a minimum.

Cloth — 5⅝ x 8⅜ — 237 pages *$5.50*

Complete details available upon request

ADDISON-WESLEY REPRESENTATIVES

Western United States
J. W. STACEY, INC.
551 Market St., San Francisco, Calif.

Europe
W. S. HALL & CO.
696 Keizersgracht, Amsterdam C, Holland
457 Madison Ave., New York 22, N. Y.

Canada
THOMAS ALLEN, LTD.
266 King St. West, Toronto

Latin America, Orient, & South Africa
HENRY M. SNYDER & CO.
440 Fourth Ave., New York 16, N. Y.
391 Young Hotel Bldg., Honolulu 9, T. H.
428-432 George St., Sydney, Australia

ADDISON-WESLEY PRESS, INC. Cambridge 42, Mass.

Timely McGRAW-HILL Books

an established leader . . .

TRIGONOMETRY, PLANE AND SPHERICAL

By LLOYD L. SMAIL, Lehigh University, 405 pages, (with tables) $3.75

A basic text for a standard college course in trigonometry, with due attention to both the numerical and theoretical aspects of the subject. The arrangement has been planned to give utmost flexibility, so as to make the book adaptable to courses of varying lengths and needs.

a new contribution in business administration . . .

MATHEMATICS OF FINANCE

By LLOYD L. SMAIL, Lehigh University. In press

This book was written as a text for college classes in the mathematical methods of finance and investment, particularly for students in business administration or commerce. All important definitions and results are carefully formulated and displayed in separate italicized statements, and all important formulas have been stated in the form of theorems giving the meanings of the symbols involved. Special emphasis is laid on a minimum number of fundamental formulas.

a popular success . . .

COLLEGE ALGEBRA

By ROSS R. MIDDLEMISS, Washington University, 344 pages, $3.50

The text covers all the topics usually taught in a standard course. The principal aim is to make the algebra course more valuable and more stimulating by emphasizing *reasoning and clear thinking*—and by thus combating the student's tendency toward mechanical operations unaccompanied by real thought.

an important new interpretation . . .

ALGEBRA FOR COLLEGE STUDENTS

By Ross R. MIDDLEMISS, Washington University. In press

A new treatment of the author's *College Algebra,* this text is designed for the less advanced student. For a slower, more detailed study, the fundamental material—through quadratic equations with one unknown—has been expanded. The lessons have been shortened and geared in treatment to a somewhat less mature student with a background of only one year of high school algebra. Emphasis is upon a real understanding.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, Inc.

330 WEST 42nd STREET • NEW YORK 18, N. Y.

MACMILLAN

December is the time for holiday thoughts

December also means—

. . . it's time to order this text for your second semester classes

ELEMENTARY DIFFERENTIAL EQUATIONS

by Earl D. Rainville

Providing a complete introduction to elementary differential equations, the material in this text is arranged so that it can be used as effectively for a semester course as for a year course. For the most part, topics which some instructors may wish to omit are set off in special chapters. The book is intended for students who have completed the standard calculus course and is designed to instill in the student an interest in continuing his studies beyond the scope of this book. Emphasis is placed on the careful development and execution of methods of solution. $5.00

. . . and to consider this one for your future calculus courses

CALCULUS

by John F. Randolph

Dr. Randolph has written his important introductory calculus text on the principle that a modern text on calculus must be flexible enough to provide routine knowledge of the subject and its applications for all members of the class, yet includes in the appendix many proofs and more advanced aspects of the subject. The author anticipates each topic gradually so that the student is primed for its eventual definition and ready to master it. $5.00

THE MACMILLAN COMPANY

60 FIFTH AVE., NEW YORK 11, N.Y.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN